ARC CLUSTER SETS OF HOLOMORPHIC FUNCTIONS

By

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Let f be a complex-valued function defined in the unit disk D, let C denote the unit circle, and let W denote the Riemann sphere. For each point $p \in C$, let $\mathfrak{T}(p)$ denote the set of all Jordan arcs contained in $D \cup \{p\}$ and having one end point at p. For each $t \in \mathfrak{T}(p)$, define the *cluster set of f at p relative to the arc t* by

$$C_t(f, p) = \bigcap_{r>0} \overline{f(t \cap \{z: |z-p|<r\})} .$$

We remark that if f is a continuous function then for each $t \in \mathfrak{T}(p)$ the set $C_t(f, p)$ is a non-empty closed connected subset of W. We shall refer to a non-empty closed connected subset of W as a continuum, even if the set is a singleton.

If A and B are two non-empty closed subsets of W, define

$$M(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\} ,$$

where $d(a, b)$ denotes the chordal distance between the points a and b on W. The distance $M(A, B)$ is a metric on the set of all non-empty closed subsets in W. In particular, for a fixed $p \in C$, M imposes a topology on $\mathfrak{C}_f(p)$, where

$$\mathfrak{C}_f(p) = \{C_t(f, p): t \in \mathfrak{T}(p)\} ,$$

that is, $\mathfrak{C}_f(p)$ is the set whose elements are the sets $C_t(f, p)$. We call this topology the M-topology on $\mathfrak{C}_f(p)$. It is easy to verify that the M-topology on $\mathfrak{C}_f(p)$ is separable.

Belna and Lappan [1, Theorem 1, p. 211] have proved that if f is a continuous function in D and p is a point of C, and if $\mathfrak{C}_f(p)$ is not compact in the M-topology, then p is an ambiguous point for f, that is, there exist two arcs t_1 and t_2 in $\mathfrak{T}(p)$ such that $C_{t_1}(f, p) \cap C_{t_2}(f, p) = \phi$. They have also given an example of a function f holomorphic in D and a point $p \in C$ for which $\mathfrak{C}_f(p)$ is not compact in the M-topology [1, Remark 2, p. 212]. In this paper we will show that

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if f is a holomorphic function in D which behaves nicely enough near the point $p\in C$, then $\mathfrak{C}_f(p)$ is a compact set in the M-topology.

We begin with some definitions:

Definition 1. *Let t_1, t_2, and t_3 be Jordan arcs in $\mathfrak{T}(p)$. If there exist Jordan arcs t_4, t_5, and t_6 in $\mathfrak{T}(p)$ such that $t_1\subset t_4$, $t_2\subset t_5$ and $t_6\subset t_3$, where $t_4\cup t_5$ is a Jordan curve and $t_6-\{p\}$ is contained in the bounded region whose boundary is $t_4\cup t_5$, then we say that t_3 is between t_1 and t_2.*

Let Z^+ denote the set of positive integers.

Definition 2. *The sequence $\{t_n\}$ of Jordan arcs in $\mathfrak{T}(p)$ is said to be a directed sequence if for each $n\in Z^+$, the arc t_{n+1} is between t_n and t_{n+2}.*

Lemma 1. *Let f be a continuous function in D and let $p\in C$. If $\{t_n\}$ is a directed sequence of arcs in $\mathfrak{T}(p)$ such that $C_{t_n}(f, p)=K_n$ and if K is a continuum such that $M(K_n, K)\to 0$ but $K\notin\mathfrak{C}_f(p)$, then there exists a directed sequence of arcs $\{s_k\}$ in $\mathfrak{T}(p)$ and a positive number $\varepsilon>0$ such that for each $k\in Z^+$ there exists $n_k\in Z^+$ such that s_k is between t_{n_k} and $t_{n_{k+1}}$ and $d(C_{s_k}(f, p), K)>\varepsilon$.*

Proof. The conclusion of this lemma can be reformulated as follows: there exists $\varepsilon>0$ such that for each $n\in Z^+$ and for each real number $\delta>0$ there exists $m\in Z^+$ where $m>n$ such that $t_n\cap\{z\in D:|z-p|<\delta\}$ and $t_m\cap\{z\in D:|z-p|<\delta\}$ are contained in different components of $\{z\in D:d(f(z), K)<\varepsilon, |z-p|<\delta\}$, where we assume that $M(f(t_n), K)<\varepsilon$ for each $n\in Z^+$. Let us assume that the lemma is false. Then for each $k\in Z^+$ there exists an integer N_k such that for each $\delta>0$ we have that $n>N_k$ implies that all of the sets $t_n\cap\{z\in D:|z-p|<\delta\}$ lie in the same component of $\{z\in D:d(f(z), K)<1/k, |z-p|<\delta\}$. Thus, for each $n>N_k$ and each $\delta>0$ there exists a Jordan arc q_n leading from a point of t_n to a point of t_{n+1} such that

$$q_n\subset\{z\in D:|z-p|<\delta, d(f(z), K)<1/k\}\ .$$

Thus we may choose a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ such that $n_k>N_k$ for each $k\in Z^+$, and using the same reasoning as before on the sequence $\{t_{n_k}\}$ we may conclude that for each $k\in Z^+$ there exists a Jordan arc p_k leading from a point on t_{n_k} to a point on $t_{n_{k+1}}$ such that for each $k\in Z^+$,

$$p_k\subset\{z\in D:|z-p|<1/k, d(f(z), K)<1/k\}\ ,$$

and the portion t'_k of t_{n_k} between the terminal point of p_{k-1} and the starting point of p_k satisfies the relationship $M(f(t'_k), K)<1/k$. It is no loss of generality to assume that p_k meets t_{n_k} and $t_{n_{k+1}}$ in exactly one point each. Then letting t be the Jordan arc obtained by splicing together all of the arcs t'_k and p_k, we have that $C_t(f, p)=K$, in violation of the hypothesis that $K\notin\mathfrak{C}_f(p)$. Thus the lemma is proved.

Lemma 2. *If f is a continuous function in D and if $p\in C$ such that $\mathfrak{C}_f(p)$ is not compact in the M-topology, then there exist directed sequences $\{t_n\}$ and $\{s_n\}$ of arcs in $\mathfrak{T}(p)$, a number $\varepsilon>0$, and a continuum K such that, letting $K_n=C_{t_n}(f, p)$ and $L_n=C_{s_n}(f, p)$, we have for each $n\in Z^+$ that $M(K_n, K)<1/n$, $d(L_n, K)>\varepsilon$, and the arc s_n is between t_n and t_{n+1}.*

Proof. The result will follow from Lemma 1 by showing that for a sequence of arcs $\{t_n\}$ in $\mathfrak{T}(p)$ satisfying $C_{t_n}(f, p)=K_n$ and $M(K_n, K)<1/n$, where $K\notin\mathfrak{C}_f(p)$, there is a subsequence of $\{t_n\}$ which is a directed sequence. If the arcs t_n are not mutually disjoint then they can be shortened individually so that an infinite subset of the shortened arcs are mutually disjoint, since otherwise there would exist an arc $t\in\mathfrak{T}(p)$ where t is contained in the union of the t_n's and $C_t(f, p)=K$, is violation of the assumption on K. But now a directed subsequence of the shortened arcs t_n can be selected. Thus the validity of the lemma depends on the existence of an appropriate continuum K and on Lemma 1. But the existence of an appropriate continuum K follows from the hypothesis that $\mathfrak{C}_f(p)$ is not compact in the M-topology, and the lemma is established.

Theorem 1. *If $p\in C$ and if f is a holomorphic function in D which is bounded in a neighborhood (relative to D) of p, then $\mathfrak{C}_f(p)$ is compact in the M-topology.*

Proof. Suppose that $\mathfrak{C}_f(p)$ is not compact in the M-topology. Let $\{t_n\}$, $\{s_n\}$, K_n, L_n, K, and ε all be as in Lemma 2. It is no loss of generality to assume that all of the arcs s_n and t_n originate at the origin, terminate at p, and that no pair of these arcs have any point in common outside the set $\{0, p\}$. We may further assume that $M(K_n, K)<\varepsilon/2$ for each $n\in Z^+$. Let Δ_n be the bounded region whose boundary is the Jordan curve $t_n\cup t_{n+1}$, and let Δ'_n be the bounded region whose boundary is the Jordan curve $s_n\cup s_{n+1}$. Since $s_n-\{0, p\}\subset\Delta_n$ and $t_{n+1}-\{0, p\}\subset\Delta'_n$, we obtain the cluster set relationships $L_n\subset C_{\Delta_n}(f, p)$ and $K_{n+1}\subset C_{\Delta_n'}(f, p)$, where if G is a subset of D we define $C_G(f, p)$ by

$$C_G(f, p)=\bigcap_{r>0}\overline{f(G\cap\{z:|z-p|<r\})}\,.$$

Also, by [2, Theorem 5. 2. 1, p. 91], we have the relationship

$$BdC_{\Delta_n}(f, p)\subset C_{t_n}(f, p)\cup C_{t_{n+1}}(f, p)=K_n\cup K_{n+1}$$

and

$$BdC_{\Delta_n'}(f, p)\subset C_{s_n}(f, p)\cup C_{s_{n+1}}(f, p)=L_n\cup L_{n+1}\,,$$

where $Bd\ E$ is used to denote the boundary of the set E.

Let n be a fixed integer greater than 1. Since f is bounded in a neighborhood of p, each of the cluster sets mentioned above is a bounded set. Since $M(K_k, K)<\varepsilon/2$ and $d(L_k, K)>\varepsilon$ for each $k\in Z^+$, and by the boundary cluster set relationship given above, there exists a point $w_0\in L_n\cup L_{n+1}$ such that

$$|w_0|>\sup\{|w|:d(w, K)<\varepsilon/2\}\,.$$

If $w_0\in L_n$, then the fact that L_n is contained in a bounded set whose boundary is $K_n\cup K_{n+1}$ leads to the existence of a point $w_1\in K_n\cup K_{n+1}$ such that $|w_1|>|w_0|$. But $d(w_1, K)<\varepsilon/2$, is violation of the choice of w_0. Similarly, if $w_0\in L_{n+1}$, then the fact that L_{n+1} is contained in a bounded set whose boundary is $K_{n+1}\cup K_{n+2}$ leads to the existence of a point $w_2\in K_{n+1}\cup K_{n+2}$ such that $|w_2|>|w_0|$. But $d(w_2, K)<\varepsilon/2$, also is violation of the choice of w_0. Thus the assumption that $\mathfrak{C}_f(p)$ is not compact in the M-topology is untenable, and this proves the theorem.

We remark that the proof of Theorem 1 used only the assumption that f is bounded on a union of three consecutive regions $\varDelta_n$. We will make use of this in the proof of Lemma 3 below.

Theorem 2. *If f is holomorphic and bounded in D, then $\mathfrak{C}_f(p)$ is compact in the M-topology for each point $p\in C$.*

Theorem 2 follows immediately from Theorem 1.

Lemma 3. *Let f be holomorphic in D and let $p\in C$. If $\{t_n\}$ is a directed sequence of arcs in $\mathfrak{T}(p)$, if $K_n=C_{t_n}(f, p)$ for each $n\in Z^+$, and if K is a continuum such that $M(K_n, K)\to 0$, then either $K\in\mathfrak{C}_f(p)$, $\infty\in K$, or there exist three arcs q_1, q_2, and q_3 in $\mathfrak{T}(p)$ such that $f(z)\to\infty$ on q_1 and on q_3 as $z\to p$, f is bounded on q_2, and q_2 is between q_1 and q_3.*

Proof. Assume that $K\notin\mathfrak{C}_f(p)$ that $\infty\notin K$, and that each K_n is a bounded set. By defining $\varDelta_n$ as in the proof of Theorem 1, we have that if there exists an integer N such that f is bounded in each of the regions $\varDelta_n$ for $n>N$, then by the remark following the proof of Theorem 1 we must have that $K\in\mathfrak{C}_f(p)$, in violation of our assumption. Thus there exist n_1, $n_2\in Z^+$, $n_2>n_1$, such that f is unbounded in each of $\varDelta_{n_1}$ and $\varDelta_{n_2}$. There exist paths q_1, q_3 in $\mathfrak{T}(p)$ such that $q_1-\{p\}\subset\varDelta_{n_1}$, $q_3-\{p\}\subset\varDelta_{n_2}$, and $f(z)\to\infty$ as $z\to p$ along q_1 and q_3. Letting $q_2=t_{n_2}$, we have that $C_{q_2}(f, p)=K_{n_2}$ is a bounded set, so that f is bounded on q_2, and further q_2 is between q_1 and q_3. This completes the proof of the lemma.

Lemma 3 suggests the following definition.

Definition 3. *Let p be a point in C. We say that the function f is in the class I_p if f is holomorphic in D and if for each pair of arcs t_1, t_2 in $\mathfrak{T}(p)$ for* *which* $f(z)\to\infty$ *as* $z\to p$ *along* $t_j (j=1, 2)$ *we have that* f *is unbounded on each path* t *in* $\mathfrak{T}(p)$ *for which* t *is between* t_1 *and* t_2.

For each $p\in C$ the class I_p includes all holomorphic functions in D for which ∞ is not an asymptotic value at p. We will relate the class I_p to the class of all normal holomorphic functions (see [2, p. 86] and [3] for a discussion of normal holomorphic functions).

Theorem 3. *If* f *is a normal holomorphic function in* D, *then* f *is in the class* I_p *for each* $p\in C$.

Proof. If $p\in C$ and if ∞ is an asymptotic value of f at p along two disjoint paths t_1 and t_2 in $\mathfrak{T}(p)$, and if t is any path in $\mathfrak{T}(p)$ between t_1 and t_2, then by a remark of Lehto and Virtanen [3, p. 53] $f(z)\to\infty$ as $z\to p$ along t. Thus f is in the class I_p and the theorem is proved.

We next show that if f is in the class I_p then f cannot have very many distinct asymptotic values at p.

Theorem 4. *If* p *is a point in* C *and if* f *is a holomorphic function in the class* I_p, *then* f *may have at most two finite asymptotic values at* p.

Proof. If f is a holomorphic function in D having three distinct finite asymptotic values a_1, a_2, a_3 at p, then there exist three disjoint arcs t_1, t_2, t_3 in $\mathfrak{T}(p)$ such that $f(z)\to a_j$ as $z\to p$ along $t_j (j=1, 2, 3)$. But then there exist paths q_1 q_2 in $\mathfrak{T}(p)$ such that q_j is between t_j and t_{j+1} and $f(z)\to\infty$ as $z\to p$ along $q_j (j=1, 2)$. This means that t_2 is between q_1 and q_2 and f is bounded on t_2. It follows that f cannot be in the class I_p, and the theorem is proved.

We now return to a consideration of the compactness of $\mathfrak{C}_f(p)$ in the M-topology and we relate this to the class I_p.

Theorem 5. *If* $p\in C$ *and if* f *is a function in the class* I_p, *then* $\mathfrak{C}_f(p)$ *is compact in the* M*-topology.*

Proof. If $\mathfrak{C}_f(p)$ is not compact in the M-topology, then there exist sequences of continua $\{K_n\}$ and $\{L_n\}$, a continuum K, and a real number $\varepsilon>0$ as described in Lemma 2. We may assume that $M(K_n, K)<\varepsilon/2$ for each $n\in Z^+$. Since f is in the class I_p, then $\infty\in K$ by Lemma 3. Then there exists a bounded set L such that $L_n\subset L$ for each $n\in Z^+$ and $d(L, K)>\varepsilon$. Letting Δ'_n be the region defined in the proof of Theorem 1, and using the facts that $K_{n+1}\subset C_{\Delta_n'}(f, p)$, $BdC_{\Delta_n'}(f, p)\subset L_n\cup L_{n+1}$, and that ∞ is in the same component of the complement of $L_n\cup L_{n+1}$ as is K_{n+1}, we obtain that f must be unbounded in Δ'_n for each $n\in Z^+$. Thus for each $n\in Z^+$ f has ∞ as an asymptotic value at p along a path q_n such that $q_n-\{p\}\subset\Delta'_n$. Letting $\{s_n\}$ be the sequence of arcs described in Lemma 2 (namely, s_n is one part of the boundary of Δ'_n) we have that s_{n+1} is between q_n and q_{n+1} for each $n \in Z^+$ but f is bounded on s_{n+1}, is violation of the assumption that f is in the class I_p. Thus the theorem is proved.

We note that Theorem 5 gives a sufficient, but not a necessary condition for $\mathfrak{C}_f(p)$ to be compact in the M-topology. For example, if $f(z)=\exp\{-1/(1-z)^2\}$, then it is easy to verify that $\mathfrak{C}_f(1)$ is compact in the M-topology but f is not in the class I_1.

The proof of Theorem 5 actually tells us much more than the statement of the theorem, for the proof reveals that the behavior forbidden to a function in class I_p must be repeated infinitely often in order that $\mathfrak{C}_f(p)$ not be compact in the M-topology. We formulate this result as our final theorem.

Theorem 6. *Let f be a holomorphic function in D. If $p \in C$ and if $\mathfrak{C}_f(p)$ is not compact in the M-topology, then there exist two directed sequences of arcs $\{p_n\}$ and $\{q_n\}$ in $\mathfrak{T}(p)$ such that for each $n \in Z^+$, $f(z) \to \infty$ as $z \to p$ on p_n, f is bounded on q_n, and q_n is between p_n and p_{n+1}.*

The converse of Theorem 6 is not valid, as the function $f(z)=\exp\{\exp((1+z)/(1-z))\}$ shows. By using basic properties of the exponential function and the fact that $w=(1+z)/(1-z)$ maps D onto the right half plane, it can easily be shown that f satisfies the conclusion of Theorem 6 but $\mathfrak{C}_f(1)$ is compact in the M-topology.

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