SOME OSCILLATION THEOREMS FOR THIRD ORDER NON-LINEAR DIFFERENTIAL EQUATIONS

By

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1. This paper is a study of the oscillation properties of solutions of the differential equation

$$y'''+p(t)y'+f(t,y)=0\,. \tag{1.1}$$

Throughout we shall assume that $p(t)$ is continuous and does not change sign on $[a,\infty)$, $a\geq 0$ and $f(t,y)\in C[[a,\infty)\times(-\infty,\infty)=S]$ with $a(t)\varphi(y)\geq f(t,y)\geq h(t)\Psi(y)$ for $(t,y)\in S$, where $a(t)$ and $h(t)$ are locally integrable functions and

$$\frac{\varphi(y)}{y}\,,\ \frac{\Psi(y)}{y}\geq\alpha>0\,.$$

A non-trivial solution of a differential equation is said to be oscillatory if it has zeros for arbitrarily large values of the independent variable. Motivation for the study of oscillation properties of the solutions of (1.1) comes from two directions. The equation

$$y'''+p(t)y'+q(t)y=0$$

has been studies extensively and some recent papers are those of *Greguš* [3], *Hanan* [4], *Zlámal* [14], *Lazer* [7] and *Švec* [10]. On the other hand the non-linear second order differential equations have been studies by *Atkinson* [1], *Bhatia* [2], *Nehari* [11] and *Waltman* [12] and third order by *Waltman* [13] and *Heidel* [5]. Two cases $a(t)$, $h(t)$ nonnegative and $a(t)$, $h(t)$ nonpositive are discussed in this paper. The technics used to prove theorems in this paper are not new.

2. The case a(t) and $h(t)$ are nonnegative is considered in this section and two theorems are provided.

Theorem 2.1. *Let $p(t)$ be nonpositive and $a(t)$, $h(t)$ nonnegative. If*

$$\int_a^\infty\left[\alpha\, a(t)-\frac{2}{3\sqrt{3}}(-p(t))^{3/2}\right]dt=\infty$$

and

$$\int_a^\infty\left[\alpha\, a(t)-\frac{2}{3\sqrt{3}}(-p(t))^{3/2}\right]dt=\infty$$

then every continuable solution of (1.1), *which has a zero is oscillatory.*

Proof. Let $y(t)$ be any solution of (1.1) which has a zero and is nonoscillatory. Let its last zero be t_0 and $y(t)>0$ for $t>t_0$. (Similar proof follows when $y(t)<0$, for $t>t_0$). Now we assert that $y'(t)$ cannot change signs more than twice in $[t_0, \infty)$. Let us assume that T_1 and T_2 are two consecutive points in $[t_0, \infty)$ where $y'(t)$ changes sign. Multiplying (1.1) by $y'(t)$ and integrating by parts between T_1 and T_2, we get

$$-\int_{T_1}^{T_2} y''^2(t)dt + \int_{T_1}^{T_2} p(t)y'^2(t)\,dt + \int_{T_1}^{T_2} f(t, y(t))y'(t)dt = 0$$

or

$$-\int_{T_1}^{T_2} y''^2(t)dt + \int_{T_1}^{T_2} p(t)y'^2(t)\,dt + \int_{T_1}^{T_2} a(t)\,\frac{\varphi(y(t))}{y(t)}\,y'(t)y(t)dt \geq 0 \tag{1}$$

(If $y'(t)$ is negative within (T_1, T_2), replace $a(t)\varphi(y(t))$ by $h(t)\Psi(y(t))$). Since $p(t)\leq 0$, $a(t)\geq 0$, $\frac{\varphi(y(t))}{y(t)}>0$, it follows from (1) that $y(t)y'(t)$ is nonnegative in (T_1, T_2). Thus $y'(t)$ cannot change its sign more than twice within $[t_0, \infty)$ and there exists a number $t_1\geq t_0$ such that either $y(t)y'(t)\leq 0$ or

$$y(t)y'(t)\geq 0 \quad \text{for} \quad t\geq t_1 .$$

Let $y(t)y'(t)\leq 0$. Since $y(t)>0$ for $t>t_0$ and $y(t_0)=0$, there exists $t_1>t_0$ such that $y'(t_1)=0$, $y'(t)\leq 0$ for $t>t_1$. Multiplying (1.1) by $y'(t)$ and integrating by parts between t_1 and t, we have

$$\begin{aligned} y''(t)y'(t) &= \int_{t_1}^{t} y''^2(t)dt - \int_{t_1}^{t} p(t)y'^2(t)dt - \int_{t_1}^{t} f(t, y(t))y'(t)dt \\ &\geq \int_{t_1}^{t} y''^2(t)dt - \int_{t_1}^{t} p(t)y'^2(t)dt - \int_{t_1}^{t} h(t)\frac{\Psi(y(t))}{y(t)}y(t)y'(t)dt \\ &\geq 0 . \end{aligned}$$

Thus $y''(t)\leq 0$ and this contradicts that t_0 is the last zero of $y(t)$ and thus we have $y(t)y'(t)\geq 0$ for $t\geq t_1$.

Let
$$x(t)=\frac{y'(t)}{y(t)}\geq 0 \quad \text{for} \quad t>t_1 .$$

$$\begin{aligned} x''(t)-3x'(t)x(t) &= -x^3(t)-p(t)x(t)-\frac{f(t, y(t))}{y(t)} \\ &\leq -\left[x^3(t)+p(t)x(t)+h(t)\frac{\Psi(y(t))}{y(t)}\right] \\ &< -[x^3(t)+p(t)x(t)+\alpha h(t)] . \end{aligned}$$

The minimum of the function

$$x^3+p(t)x+\alpha h(t)\ , \quad x>0$$

is when $x=\frac{-p(t)}{3}$. Thus we have

$$\frac{d}{dt}\left[x'(t)+\frac{3}{2}x^2(t)\right]<-\left[\alpha h(t)-\frac{2}{3\sqrt{3}}(-p(t))^{3/2}\right]. \tag{2}$$

Integrating both the sides of (2) from t_1 to t, we get

$$x'(t)<x'(t_1)+\frac{3}{2}x^2(t_1)-\frac{3}{2}x^2(t)-\int_{t_1}^{t}\left[\alpha h(t)-\frac{2}{3\sqrt{3}}(-p(t))^{3/2}\right]dt\ .$$

$\to-\infty$ as $t\to\infty$, consequently $x(t)$ would become negative which is contradictory. Hence $y(t)$ is oscillatory solution.

Theorem 2.2. *Let $p(t)$, $a(t)$ and $h(t)$ be nonnegative. If $[\alpha a(t)-p'(t)]$ and $[\alpha b(t)-p'(t)]$ are positive and*

$$\int_a^\infty t[\alpha a(t)-p'(t)]dt=\infty\ , \quad \int_a^\infty t[\alpha h(t)-p'(t)]dt=\infty$$

then every continuable solution of (1.1) *which has a zero is oscillatory.*

Proof. Suppose $y(t)$ is a solution of (1.1) which has a zero but does not oscillate. Let t_0 be its last zero and $y(t)>0$, for $t>t_0$. (Similar proof follows if $y(t)<0$, $t>t_0$). Now there are two possibilities. Either $y'(t)$ has a zero after t_0 or $y'(t)>0$ for $t>t_0$.

In first possibility, let $t_1>t_0$ be the first zero of $y'(t)$ after t_0. Multiplying (1.1) by $y(t)$ and integrating from t_0 to t, we get

$$y''(t)\,y(t)+\frac{1}{2}y'^2(t_0)-\frac{1}{2}y'^2(t)+\frac{1}{2}p(t)y^2(t)$$
$$+\int_{t_0}^{t}y^2(s)\left[\alpha h(s)-\frac{1}{2}p'(s)\right]ds<0\ . \tag{3}$$

From (3) it follows that at every zero of $y'(t)$, $y(t)y''(t)<0$ and thus t_1 is the only zero of $y'(t)$ and $y'(t)<0$ for $t>t_1$. Now $y''(t)$ can not remain negative, otherwise t_0 is not the last zero of $y(t)$. If $y''(t)$ remains positive then $\lim_{t\to\infty} y'(t)$ exists but from (3), we have

$$\frac{1}{2}y'^2(t)>y''(t)y(t)+\frac{1}{2}y'^2(t_0)+\frac{1}{2}p(t)y^2(t)$$
$$+\int_{t_0}^{t}y^2(s)\left[\alpha h(s)-\frac{p'(s)}{2}\right]ds\ . \tag{4}$$

Here all the terms on the right are positive and the last term is increasing. Since $\lim_{t\to\infty} y'(t)$ exists and is finite, it follows at once that $y(t)$ has a zero within (t_1, ∞) which is contradictory.

If $y''(t)$ changes its sign for arbitrarily large t, $y'(t)$ has maxima for arbitrarily large t. Since $\lim_{t\to\infty} y(t)$ exists and is finite and $y'(t)<0$ $\limsup_{t\to\infty} y'(t)=0$ and thus the set of maxima of $y'(t)$ must contain a subsequence $\{t_n\}$ such that $\lim_{n\to\infty} y'(t_n)=0$. But putting t_n in (4), we get $\lim_{n\to\infty} [y'^2(t_n)]>0$, which is contradiction. Thus we proved that if $y(t_0)=y'(t_1)=0$, then t_0 is not the last zero of $y(t)$ which contradicts $y(t)>0$, $t>t_0$.

In case of second possibility, since $y'(t)>0$, $t>t_0$, we have $y'''(t)=-p(t)y'(t) -f(t, y)\leq 0$ and thus $y''(t)$ is a decreasing function for $t>t_0$. Suppose there exists $t_1>t_0$ such that $y''(t)\leq 0$, $t>t_1$. Then there exists $t_2>t_1$ such that $y''(t)<0$, $t>t_2$. Hence $y'(t)$ is positive and monotone decreasing and $\lim_{t\to\infty} y'(t)$ exists and is finite and nonnegative. But

$$y'(t)=y'(t_2)+\int_{t_2}^{t} y''(s)ds<y''(t_2)(t-t_2) .$$

Since $y''(t_2)$ is negative it follows that $y'(t)\to-\infty$ as $t\to\infty$, which is contradiction. Thus we have $y''(t)$ is a decreasing function and nonnegative for $t>t_1$. Now integrating (1.1) from t_1 to t, we get

$$y''(t)-y''(t_1)+p(t)y'(t)-p(t_1)y(t_1)+\int_{t_1}^{t} y(s)\left[\frac{f(s, y(s))}{y(s)}-p'(s)\right]ds=0$$

or

$$y''(t_1)+p(t_1)y(t_1)>\int_{t_1}^{t} y(s)[\alpha h(s)-p'(s)]ds .$$

Since $y(t)>y'(t_1)(t-t_1)$, we get

$$y''(t_1)+p(t_1)y(t_1)>y'(t_1)\int_{t_1}^{t} (s-t_1)[\alpha h(s)-p'(s)]ds .$$

The left hand side is independent of t, but right hand side has t and tends to ∞ as $t\to\infty$, which is contradictory. Thus $y(t)$ is not nonoscillatory solution.

3. Now we shall deal with the case when $p(t)$, $a(t)$ and $h(t)$ are nonpositive. If we denote

$$F[y(t)]=2y(t)y''(t)-y'^2(t)+p(t)y^2(t) , \tag{5}$$

where $y(t)$ is a solution of (1.1), then

$$F[y(t)]=F[y(a)]+\int_a^t\left[p'(t)-2\frac{f(t,y(t))}{y(t)}\right]y^2(t)dt\ .$$

This can be verified by direct differentiation and we shall use this identity in this section several times. The first two lemmas can be proved in a same way as proved by Lazer [6] for the linear case.

Lemma 3.1. *Let $p(t)$, $a(t)$ and $h(t)$ be nonpositive and $y(t)$ any solution of* (1.1). *If $y(t_0)\geq 0$, $y'(t_0)\leq 0$ and $y''(t_0)>0$($t_0\in[a,\infty)$, arbitrary), then*

$$y(t)>0,\ y'(t)>0,\ y''(t)>0,\ y'''(t)\geq 0\quad for\quad t>t_0$$

and $\lim_{t\to\infty} y(t)=\lim_{t\to\infty} y'(t)=\infty$. *Again if $y(t_0)\leq 0$, $y'(t_0)\leq 0$, and $y''(t_0)<0$, then $y(t)<0$, $y'(t)<0$, $y''(t)<0$, $y'''(t)\leq 0$ for $t>t_0$ and* $\lim_{t\to\infty} y(t)=\lim_{t\to\infty} y'(t)=-\infty$.

Lemma 3.2. *Let $p(t)$, $a(t)$ and $h(t)$ be nonpositive. If $y(t)$ be a nonoscillatory solution of* (1.1), *then there exists a number $C\in[a,\infty)$ such that either $y(t)y'(t)>0$ or $y(t)y'(t)\leq 0$ for $t>C$.*

Theorem 3.3. *Let $p(t)$, $a(t)$, $h(t)$ be nonpositive and $p(t)$ bounded. If*

$$\int_a^\infty [p'(t)-2\alpha a(t)]dt=\infty\ ,$$

$$\int_a^\infty [p'(t)-2\alpha h(t)]dt=\infty$$

and $y(t)$ is any continuable nonoscillatory solution of (1.1) *then either*

$$\lim_{t\to\infty}|y(t)|=\lim_{t\to\infty}|y'(t)|=\infty$$

or

$$\lim_{t\to\infty}|y(t)|=\lim_{t\to\infty}\inf|y'(t)|=\lim_{t\to\infty}\inf|y''(t)|=0\ .$$

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1) such that $y(t)>0$ for $t>t_0$. (Similar proof follows for $y(t)<0$ for $t>t_0$). By lemma 3.2, there exists a number $t_1\geq t_0$ such that either

$$y(t)y'(t)>0$$

or

$$y(t)y'(t)\leq 0$$

for $t>t_1$. When $y(t)y'(t)>0$, we have $y'(t)>0$ for $t>t_1$ and $y(t)$ is an increasing function and thus we can take $y(t)\geq c$ for $t>t_1$. Now

$$F[y(t)]=F[y(t_1)]+\int_{t_1}^{t}\left[p'(t)-2\frac{f(t,y)}{y(t)}\right]y^2(t)dt$$
$$>F[y(t_1)]+c^2\int_{t_1}^{t}[p'(t)-2\alpha a(t)]dt\,.$$

Since last term is increasing, there exists $t_2\geq t_1$ such that $F[y(t)]>0$ which gives from (5) that $y(t)y''(t)>0$ and thus $y''(t)>0$ for $t>t_2$. Also

$$y'''(t)=-p(t)y'(t)-f(t,y)\geq -p(t)y'(t)-\alpha a(t)y(t)\geq 0\,.$$

Thus we have

$$\lim_{t\to\infty}|y(t)|=\lim_{t\to\infty}|y'(t)|=\infty\,.$$

When $y(t)y'(t)\leq 0$, then $y(t)>0$, $y'(t)\leq 0$ for $t>t_1$ and $y(t)$ is bounded and $\lim_{t\to\infty}y(t)$ exists. It can easily be proved by the use of mean value theorem that

$$\lim_{t\to\infty}\inf|y'(t)|=\lim_{t\to\infty}\inf|y''(t)|=0$$

We assert that $\lim|y(t)|=0$. Suppose this is not true, i.e.

$$\lim_{t\to\infty}|y(t)|=A\neq 0\,.$$

Integrating (1.1) by parts from t_1 to t, we get

$$y''(t)+p(t)y(t)-y''(t_1)-p(t_1)y(t_1)$$
$$=\int_{t_1}^{t}\left[p'(t)-\frac{f(t,y)}{y(t)}\right]y(t)dt$$
$$\geq A\int_{t_1}^{t}[p'(t)-\alpha a(t)]dt\,.$$

Here right hand side tends to ∞ as $t\to\infty$ and left hand side is bounded, which shows that $\lim_{t\to\infty}y(t)=0$ and proves the theorem.

Lemma 3.4. *Let $p(t)$, $a(t)$ and $h(t)$ be nonpositive. Let $y(t)$ be a solution of* (1.1) *such that $F[y(t_0)]\geq 0$ and $\left[p'(t)-2\alpha\left(\frac{a(t)}{h(t)}\right)\right]\geq 0$. If $y(t)$ has a constant sign in certain right hand neighborhood of t_0, then it retains the same sign for all $t>t_0$.*

Proof. Let us assume that t_1 be the first number greater than t_0 such that $y(t_1)=0$. Now

$$F[y(t_1)]=F[y(t_0)]+\int_{t_0}^{t_1}\left[p'(t)-2\frac{f(t,y)}{y(t_0)}\right]y^2(t)dt$$
$$\geq F[y(t_0)]+\int_{t_0}^{t}[p'(t)-2\alpha a(t)]y^2(t)dt$$
$$>0\,.$$

(When $y(t)$ is negative, change $a(t)$ by $h(t)$). But $F[y(t_1)]=-y'^2(t_1)$, which is contradiction and thus the lemma follows.

Theorem 3.5. *Let $p(t)$, $a(t)$ and $h(t)$ be nonpositive. If*

$$\int_a^\infty [p'(t)-2\alpha a(t)]dt=\infty ,$$

$$\int_a^\infty [p'(t)-2\alpha h(t)]dt=\infty$$

and

$$\int_a^\infty tp(t)dt>-\infty ,$$

then every continuable nontrivial solution $y(t)$ of (1.1) *is either nonoscillatory such that*

$$\lim_{t\to\infty} |y(t)|=\lim_{t\to\infty} |y'(t)|=\infty$$

or oscillatory if and only if $F[y(t)]<0$, for all $t\in[a,\infty)$.

Proof. Let $y(t)$ be a nonoscillatory solution of (1.1) such that $y(t)>0$ for $t>t_0$ (Similar proof follows for $y(t)<0$, $t>t_0$). By lemma 3.2, there exists a number $t_1>t_0$ such that either $y(t)y'(t)\leq 0$ or $y(t)y'(t)>0$ for $t>t_1$. We assert that $y(t)y'(t)\nleq 0$ for $t>t_1$. Suppose $y(t)y'(t)\leq 0$ for $t>t_1$. Since

$$\int_a^\infty tp(t)dt>-\infty ,$$

there exists $t_2>t_1$ such that $\int_{t_2}^\infty tp(t)dt\geq -1$. Multiplying (1.1) by t and integrating from t_2 to t, $t_2<t$, we obtain

$$ty''(t)-t_2y''(t_2)-y'(t)+y'(t_2)+y'(t)\int_{t_2}^t tp(t)dt$$
$$-\int_{t_2}^t y''(s)\int_{t_2}^s up(u)du\,ds=-\int_{t_2}^t tf(t,y)dt$$

or

$$ty''(t)-2y'(t)+y'(t_2)-\int_{t_2}^t y''(s)\int_{t_2}^s up(u)duds$$
$$\geq t_2y''(t_2)-\int_{t_2}^t tf(t,y)dt \tag{6}$$
$$\geq t_2y''(t_2)-\alpha\int_{t_2}^t t\,a(t)y(t)dt \tag{7}$$

Either $y''(t)\geq 0$ for $t\geq t_2$ (change t_2 if necessary) or $y''(t)$ has positive and negative values for arbitrarily large t. In first case

$$-\int_{t_2}^{t} y''(s)\int_{t_2}^{t} up(u)duds \leq \int_{t_2}^{t} y''(s)ds = y'(t)-y'(t_2),$$

therefore (7) becomes

$$ty''(t)-y'(t)\geq t_2 y''(t_2)-\alpha\int_{t_2}^{t} ta(t)y(t)dt \tag{8}$$

Since $\lim_{t\to\infty} y(t)$ exists and is finite, it follows from mean value theorem that

$$\liminf_{t\to\infty} y'(t)=\liminf_{t\to\infty} ty''(t)=0 .$$

But this contradicts the fact that right hand side of (8) is positive and increasing. Thus our assertion follows for the case $y''(t)\geq 0$.

When $y''(t)$ has positive and negative values for arbitrarily large t, then there exists a sequence of points $\{t_n\}$, $n\geq 3$, $\lim_{n\to\infty} t_n=\infty$ with the following properties

(i) $t_i<t_{i+1}$, $i=3, 4, 5, \cdots$

(ii) $y''(t_i)=0$, $i=3, 4, 5, \cdots$

(iii) $\lim_{i\to\infty} y'(t_i)=0$.

The existence of such a sequence $\{t_i\}$ is clear since $y'(t)\leq 0$ and $\limsup_{t\to\infty} y'(t)=0$. Let

$$A=\int_{t_3}^{\infty} up(u)du,\ A>-1,\ t_3>t_2 .$$

Thus

$$\begin{aligned}
&-\int_{t_3}^{t} y''(s)\int_{t_3}^{t} up(u)duds \\
&=\int_{t_3}^{t} y''(s)\left[\int_{s}^{\infty} up(u)du-A\right]ds \\
&=\int_{t_3}^{t} y''(s)\int_{s}^{\infty} up(u)duds-A\int_{t_3}^{t} y''(s)ds \\
&\leq\int_{t_3}^{t} y''(s)\int_{s}^{\infty} up(u)duds-y'(t_3) .
\end{aligned}$$

Putting this in (6) (replacing t_2 by t_3) gives

$$ty''(t)-2y'(t)+\int_{t_3}^{t} y''(s)\int_{s}^{\infty} up(u)du\,ds \geq -\int_{t_3}^{t} t f(t, y)dt \tag{9}$$

Let $Q(s)=\int_{s}^{\infty} up(u)du$, then

$$\begin{aligned}\int_{t_3}^{t} y''(s)Q(s)ds &= y''(t)\int_{t_3}^{t} Q(s)ds-\int_{t_3}^{t} y'''(s)\int_{t_3}^{s} Q(u)duds\\ &= y''(t)\int_{t_3}^{t} Q(s)ds+\int_{t_3}^{t} p(s)y'(s)\int_{t_3}^{t} Q(u)duds\\ &\quad +\int_{t_3}^{t} f(sy(s))\int_{t_3}^{t} Q(u)duds\\ &\leq y''(t)\int_{t_3}^{t} Q(s)ds+\int_{t_3}^{t} f(s, y(s))\int_{t_3}^{s} Q(u)duds\end{aligned}$$

or

$$\int_{t_3}^{t} y''(s)Q(s)ds \leq y''(t)\int_{t_3}^{t} Q(s)ds-\int_{t_3}^{t} (s-t_3)f(s, y(s))ds\,.$$

Substituting this into (9), we get

$$\begin{aligned}&ty''(t)-2y'(t)+y''(t)\int_{t_3}^{t} Q(s)ds-\int_{t_3}^{t} (s-t_3)f(s, y(s))ds\\ &\geq -\int_{t_3}^{t} sf(s, y(s))ds\end{aligned}$$

or

$$ty''(t)-2y'(t)+y''(t)\int_{t_3}^{t} Q(s)\,ds \geq -\int_{t_3}^{t} t_3 f(s, y(s))ds\,.$$

Replacing t by t_i in (10) (where $\{t_i\}$ is the sequence defined before), we get

$$-2y'(t_i)\geq -\alpha t_3\int_{t_3}^{t_i} a(t)y(t)dt \tag{11}$$

The right hand side of (11) is positive and increasing in t_i, while the left hand side converges to zero as $i\to\infty$. This contradiction proves our assertion and thus we have only $y(t)y'(t)>0$ and now it can be proved in a similar way as of theorem 3.3 that

$$\lim_{t\to\infty}|y(t)|=\lim_{t\to\infty}|y'(t)|=\infty\,.$$

Now remains to prove that a solution $y(t)$ of (1.1) is oscillatory if and only if $F[y(t)]<0$. Suppose the above condition holds but $y(t)$ is nonoscillatory. Then $|y(t)|\to\infty$ as $t\to\infty$ and from a certain t_0, $|y(t)|\geq 1$. Therefore,

$$\int_{t_0}^{t}\left[p'(t)-\frac{2f(t,y)}{y(t)}\right]y^2(t)dt \geq \int_{t_0}^{t}\left[p'(t)-2a(t)\frac{\varphi(y)}{y}\right]dt$$
$$\geq \int_{t_0}^{t}[p'(t)-2\alpha a(t)]dt$$
$$\to\infty \text{ as } t\to\infty\ . \qquad (12)$$

But we have

$$\int_{t_0}^{t}\left[p'(t)-\frac{2f(t,y)}{y(t)}\right]y^2(t)dt=F[y(t)]-F[y(t_0)]<-F[y(t_0)]$$

which is contradiction with (12).

Now let $F[y(t_0)]\geq 0$, then $y(t)>0$ or $y(t)<0$ holds in certain right hand neighborhood and thus by lemma (3.3), $y(t)$ is not oscillatory.

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