# C-CONTINUOUS FUNCTIONS

By

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## 1. Introduction

The idea of c-continuous functions was conceived by the authors in an entirely unrelated setting. These functions have a rather nice relationship to the classical theorem "*Every one-to-one onto continuous function from a compact space onto a Hausdorff space is a homeomorphism*" and have many basic properties of their own similar to properties possessed by continuous functions.

In Section 2, we study basic properties of c-continuos functions and give equivalent definitions of c-continuous functions.

In Sections 3 and 4, our main theorems are proved.

Throughout this paper *compactness* is taken to mean *every open cover has a finite subcover* and subsets of a space are compact provided they are compact considered as subspaces. The reader is referred to [1] and [2] for definitions not defined in this paper.

## 2. Basic properties of c-continuous functions

**Definition 1.** Let $X$ and $Y$ be topological spaces, let $f: X \to Y$ be a function, and let $p \in X$. Then $f$ is said to be *c-continuous at* $p$ provided if $U$ is an open subset of $Y$ containing $f(p)$ such that $Y-U$ is compact, then there is an open subset $V$ of $X$ containing $p$ such that $f(V) \subset U$. The function $f$ is said to be *c-continuous* (*on* $X$) provided $f$ is c-continuous at each point of $X$.

**Theorem 1.** *Let $X$ and $Y$ be topological spaces and let $f: X \to Y$ be a function. Then the following statements are equivalent*:

(1) *$f$ is c-continuous, and*

(2) *if $U$ is an open subset of $Y$ with compact complement, then $f^{-1}(U)$ is an open subset of $X$.*

*These statements are implied by*

(3) *if $C$ is a compact subset of $Y$, then $f^{-1}(C)$ is a closed subset of $X$.*

*and: moreover, if $Y$ is Hausdorff all the statements are equivalent.*

**Proof:** (1)→(2) Suppose (1). Let $U$ be an open subset of $Y$ with compact complement. Let $p \in f^{-1}(U)$. Then $f(p) \in U$ and there is an open set $V_p$ containing $p$ such that $f(V_p) \subset U$. Thus $V_p \subset f^{-1}(U)$ and hence $f^{-1}(U) = \cup \{V_p | p \in f^{-1}(U)\}$ is open.

(2)→(1) Suppose (2). Let $p \in X$ and $U$ be an open subset of $Y$ containing $f(p)$ such that $Y-U$ is compact. Then $f^{-1}(U)$ is open, $p \in f^{-}(U)$, and $f(f^{-1}(U)) \subset U$.

(3)→(2) Suppose (3). Let $U$ be an open subset of $Y$ with compact complement. Then $f^{-1}(Y-U)$ is closed. Thus $f^{-1}(U)=X-f^{-1}(Y-U)$ is open. Now suppose $Y$ is Hausdorff.

(2)→(3) Suppose (2). Let $C$ be a compact subset of $Y$. Since $Y$ is Hausdorff, $C$ is closed and $Y-C$ is open. Then $f^{-1}(Y-C)$ is open. Thus $f^{-1}(C)=X-f^{-1}(Y-C)$ is closed.

The following example shows that Hausdorff is necessary when showing (2)→(3) in the previous theorem.

**Example 1.** Let $X=\{1, 2, 3\}$, $Y=$Reals, $S=\{\phi, \{3\}, \{3, 2\}, \{3, 2, 1\}\}$, and $T$ be the topology for $Y$ generated by $\{(-\infty, -r) \cup (r, \infty) | r \in Y\}$. Define $f: X \to Y$ by $f(x)=x$ for all $x \in X$. Then $f$ is c-continuous but does not satisfy statement (3).

**Proof:** Clearly $f$ is continuous and by the definition it is easily seen that every continuous function is c-continuous. However, $[2, 3]$ is a compact subset of $Y$ and $f^{-1}([2, 3])=\{2, 3\}$ is not closed.

The following example shows that not every c-continuous function is continuous.

**Example 2.** *Let $R$ be the reals with the usual topology and difine $f: R \to R$ by* $f(x)=\begin{cases} 1/x & \text{if } x \neq 0 \\ 1/2 & \text{if } x=0 \end{cases}$. *Then $f$ is c-continuous but not continuous.*

**Proof:** Clearly $f$ is not continuous at 0 and is continuous everywhere else, so it remains only to show that $f$ is c-continuous at 0. Let $U$ be an open subset of $R$ containing $f(0)$ such that $R-U$ is compact. Thus $R-U$ is bounded and there is a number $a>0$ such that $(-\infty, -a) \cup (a, \infty) \subset U$. Then $(-1/a, 1/a)$ is an open set containing 0 and $f((-1/a, 1/a))=(-\infty, -a) \cup (a, \infty) \cup \{f(0)\} \subset U$. Hence $f$ is c-continuous at 0.

**Theorem. 2.** *If $f: X \to Y$ is c-continuous and $A \subset X$, then $f|_A$: $A \to Y$ is a c-continuous function.*

**Proof:** Let $U$ be an open subset of $Y$ with compact complement. Then $f^{-1}(U)$ is open and hence $(f|_A)^{-1}(U)=f^{-1}(U) \cap A$ is an open subset of $A$.

**Theorem. 3.** *If $f: X \to Y$ is continuous and $g: Y \to Z$ is c-continuous, then $gf$: $Y \to Z$ is c-continuous.*

**Proof:** Let $U$ be an open subset of $Z$ with compact complement. Then $g^{-1}(U)$ is open and since $f$ is continuous, $(gf)^{-1}(U)=f^{-1}(g^{-1}(U))$ is open.

The following example shows that if $f$ is c-continuous and $g$ is continuous, then $gf$ need not be c-continuous.

**Example 3.** *Let $R$ be the reals with the usual topology. Define $f$: $R \to R$ by $f(x)=\begin{cases} 1/x \text{ if } x\neq 0 \\ 1/2 \text{ if } x=0 \end{cases}$. Define $g: R\to R$ by $g(x)$ is the distance from $x$ to the nearest integer. Then $f$ is c-continuous, $g$ is continuous and yet $gf$ is not c-continuous.*

**Proof:** Since $f$ has already been shown to be c-continuous and $g$ is clearly continuous we need only show $gf$ is not c-continuous at 0. Let $U=(-\infty, 0)\cup(1/4, \infty)$. Then $U$ is an open set with compact complement containing $(gf)(0)=1/2$. We will show 0 is not an interior point of $(gf)^{-1}(U)$. Now $g^{-1}(U)=\cup\{(i+(1/4), i+(3/4))|i \text{ an integer}\}$ and thus, if $n$ is an integer, $n\bar{\epsilon} g^{-1}(U)$. Hence, if $n$ is an integer $1/n\bar{\epsilon} f^{-1}(g^{-1}(U))=(gf)^{-1}(U)$. Therefore, 0 is not an interior point of $(gf)^{-1}(U)$.

**Theorem 4.** *If $X$ and $Y$ are topological spaces and $X=A\cup B$ where $A$ and $B$ are open (closed) subsets of $X$ and $f: X\to Y$ is a function such that $f|_A$ and $f|_B$ are c-continuous, then $f$ is c-continuous.*

**Proof:** First assume $A$ and $B$ are open. Let $U$ be an open subset of $Y$ such that $Y-U$ is compact. Then $f^{-1}(U)=(f|_A)^{-1}(U)\cup(f|_B)^{-1}(U)$ each of which is an open subset of $X$ and hence $f^{-1}(U)$ is open.

Now assume $A$ and $B$ are closed. Let $x\epsilon X$ and let $W$ be any open subset of $Y$ with compact complement containing $f(x)$. Now either (1) $x\epsilon A\cap B$, (2) $x\epsilon A$ and $x\bar{\epsilon} B$, or (3) $x\epsilon B$ and $x\bar{\epsilon} A$.

*Case* 1: Suppose $x\epsilon A\cap B$. Since $f|_A$ is c-continuous at $x$, there exists a subset $U$ open in $A$ such that $x\epsilon U$ and $f|_A(U)\subset W$. Since $U$ is open in $A$, there exists an open subset $U'$ of $X$ such that $U=U'\cap A$. Since $f|_B$ is c-continuous at $x$, there exists a subset $V$ open in $B$ such that $x\epsilon V$ and $f|_B(V)\subset W$. Since $V$ is open in $B$, there exists an open subset $V'$ of $X$ such that $V=V'\cap B$. Let $Q=U'\cap V'$. Then $Q$ is open in $X$, $x\epsilon Q$, and $f(Q)\subset W$.

*Case* 2: Suppose $x\epsilon A$ and $x\bar{\epsilon} B$. Since $f|_A$ is c-continuous, there is a set $U$ open in $A$ such that $x\epsilon U$ and $f|_A(U)\subset W$. Since $U$ is open in $A$, there is an open subset $U'$ of $X$ such that $U=U'\cap A$. Let $V=U'-B$. Then $V$ is open in $X$, $x\epsilon V$ and $f(V)\subset W$.

*Case* 3: Suppose $x\epsilon B$ and $x\bar{\epsilon} A$. This case follows exactly like case 2.

## 3. Main results

**Theorem 5.** *Let $X$ be a space, $Y$ a Hausdorff space, and $f$ a c-continuous* *function from $X$ into $Y$. If $f(X)$ is a subset of some compact subset of $Y$, then $f$ is continuous.*

**Proof:** Let $D$ be a compact subset of $Y$ containing $f(X)$ and let $U$ be any open subset of $Y$. Since $D$ is compact, $D$ is closed and hence $Y-D$ is open. Thus $U\cup(Y-D)$ is open. But the complement of $U\cup(Y-D)$ is a closed subset of $D$ and thus a compact subset of $D$. It can easily be shown that the complement of $U\cup(Y-D)$ is a compact subset of $Y$. Now $f^{-1}(U)=f^{-1}(U\cup(Y-D))$ and since $f$ is c-continuous $f^{-1}(U)$ is an open subset of $X$. Hence $f$ is continuous.

**Theorem 6.** *Let $X$ be a Baire space and $Y$ be a Hausdorff space which is the countable union of compact sets. Then every c-continuous function from $X$ into $Y$ is continuous on a dense subset of $X$.*

**Proof:** Let $Y=\bigcup_{j=1}^{\infty} C_j$ where each $C_j$ is compact. Let $f:X\to Y$ be c-continuous and let $U$ be a nonempty open subset of $X$. By [1, Prop. 3, p. 193], $U$ with the subspace topology is a Baire space. Suppose $f$ is not continuous at any point of $U$. For each positive integer $n$, let $F_n=\{x|x\in U \text{ and } f(x)\in Y-\bigcup_{j=1}^{n} C_j\}$. Let $n$ be a positive integer. Then $\bigcup_{j=1}^{n} C_j$ is a compact subset of a Hausdorff space and hence closed. Thus $Y-\bigcup_{j=1}^{n} C_j$ is open and since $f$ is c-continuous, $f^{-1}(Y-\bigcup_{j=1}^{n} C_j)$ is open. Hence $F_n=f^{-1}(Y-\bigcup_{j=1}^{n} C_j)\cap U$ is an open subset of $U$. Suppose $F_n$ is not a dense subset of $U$. Then there exists some nonempty open subset $V$ of $U$ containing no points of $F_n$. Let $x\in f(V)$. Then there is a $y\in V$ such that $x=f(y)$. Since $y\in V$, $y\bar{\in} F_n$. But since $V\subset U$, $y\in U$ and hence $f(y)\bar{\in} Y-\bigcup_{j=1}^{n} C_j$. Therefore, $x=f(y)\in \bigcup_{j=1}^{n} C_j$. Thus $f(V)\subset \bigcup_{j=1}^{n} C_j$. By Theorem 2, $f|_V : V\to Y$ is c-continuous. But by Theorem 5, $f|_V$ is continuous. Therefore, since $V\neq\phi$, $f$ is continuous at a point of $U$ which is a contradiction. Thus $F_n$ is dense in $U$. Hence each $F_n$ is open and dense in $U$. Since $U$ is a Baire space, $\bigcap_{j=1}^{\infty} F_j$ is dense in $U$. But clearly $\bigcap_{j=1}^{\infty} F_j=\phi$. Thus $f$ is continuous at a point of $U$ and the theorem is proved.

**Theorem 7.** *Let $X$ be a saturated space and let $Y$ be a locally compact regular space. If $f: X\to Y$ is c-continuous, then $f$ is continuous.*

**Proof:** Let $p\in X$ and let $M$ be an open subset of $Y$ containing $f(p)$. Since $Y$ is regular, there is an open set $U$ such that $f(p)\in U\subset \bar{U}\subset M$. Let $y\in Y-\bar{U}$.

Since $Y$ is regular, $\bar{U}$ is closed and thus there is an open set $V_y$ containing $y$ such that $V_y \cap \bar{U} = \phi$. Since $Y$ is locally compact, there exists a compact set $C_y$ containing $y$ such that $C_y \cap U = \phi$. Then $Y - C_y$ is an open set containing $f(p)$ such that $Y - C_y$ has compact complement. Since $f$ is c-continuous there is an open set $N_y$ of $p$ such that $f(N_y) \subset Y - C_y$. Let $N = \cap \{N_y \mid y \in Y - \bar{U}\}$. Now $N$ contains $p$ and since $X$ is a saturated space, $N$ is open. Clearly $f(N) \subset \bar{U} \subset M$. Hence $F$ is continuous.

**Theorem 8.** *Let $X$ be a locally connected space and let $(Y, d)$ be a metric space in which every closed and bounded set is compact. If $f: X \to Y$ is a c-continuous, connected function, then $f$ is continuous.*

**Proof:** Let $x \in X$ and let $N_\epsilon(f(x))$ be a basic open set in $Y$ containing $f(x)$. Let $V = Y - \overline{N_{2\epsilon}(f(x))}$. Then $N_\epsilon(f(x)) \cup V$ is an open subset of $Y$ whose complement is closed and bounded and hence compact. Since $f$ is c-continuous, there is an open set $W$ containing $x$ such that $f(W) \subset N_\epsilon(f(x)) \cup V$. Since $X$ is locally connected, there exists an open connected set $W'$ such that $x \in W' \subset W$. Since $W'$ is connected and $f$ is a connected function, $f(W')$ is connected. Now $N_\epsilon(f(x))$ and $V$ are mutually separated and $f(x) \in f(W') \cap N_\epsilon(f(x))$. Hence $f(W') \subset N_\epsilon(f(x))$ and $f$ is continuous.

The following example shows that c-continuous connected functions need not be continuous and that locally connected is necessary in the previous theorem.

**Example 4.** *Let $Q$ be the rationals and let $Q^1 = [0, 1] \cap Q$ where $Q$ and $Q^1$ both have the induced topology from the reals. Define $f: Q^1 \to Q$ by $f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$. Then $f$ is a c-continuous, connected function which is not continuous, and moreover, since $Q$ is a metric space in which every closed and bounded set is compact, locally connected is necessary in the previous theorem.*

**Proof:** The proof is clear.

**Theorem 9.** *Let $X$ be a space and $Y$ be a Hausdorff space. If $f: X \to Y$ is a one-to-one, continuous, onto function, then $f^{-1}: Y \to X$ is c-continuous.*

**Proof:** Let $C$ be a compact subset of $X$. Since $f$ is continuous, $f(C)$ is compact and since $Y$ is Hausdorff, $f(C)$ is closed. By Theorem 1, $f^{-1}$ is c-continuous.

**Corollary 9.1.** *Let $X$ be a compact space and $Y$ a Hausdorff space. If $f: X \to Y$ is a one-to-one onto continuous function, then $f$ is a homeomorphism.*

**Proof:** By Theorem 9, $f^{-1}$ is c-continuous. By Theorem 5, $f^{-1}$ is continuous.

**Theorem 10.** *Let $X$ be a topological space and let $(Y, d)$ be a metric space* *such that every closed and bounded subset is compact. If* $\{f_n\}_{n=1}^{\infty}$ *is a sequence of c-continuous functions from* $X$ *into* $Y$ *which converges uniformly to* $f$, *then* $f$ *is c-continuous.*

**Proof:** Let $p \epsilon X$ and let $U$ be an open subset of $Y$ with compact complement containing $f(p)$. There is a positive number $\epsilon$ such that $N_\epsilon(f(p)) \subset U$. There exists a positive number $a$ such that $Y - \overline{N_a(f(p))} \subset U$ for if no such number $a$ exists, the complement of $U$ would be unbounded and hence not compact. Let $V = N_{\epsilon/2}(f(p)) \cup (Y - \overline{N_{a+\epsilon}(f(p))})$. Now the complement of $V$ is closed and bounded and hence compact. Since $\{f_n\}_{n=1}^{\infty}$ converges uniformly to $f$, there is a positive integer $n$ such that if $x \epsilon X$, then $d(f_n(x), f(x)) < \epsilon/2$. Thus $d(f_n(p), f(p)) < \epsilon/2$ and $f_n(p) \epsilon V$. Since $f_n$ is c-continuous, there exists an open subset $W$ of $p$ such that $f_n(W) \subset V$. Let $y \epsilon W$. Then either $f_n(y) \epsilon N_{\epsilon/2}(f(p))$ or $f_n(y) \epsilon Y - \overline{N_{a+\epsilon}(f(p))}$.

If $f_n(y) \epsilon N_{\epsilon/2}(f(p))$ then $d(f(y), f(p)) \leq d(f(y), f_n(y)) + d(f_n(y), f(p)) < \epsilon/2 + \epsilon/2 = \epsilon$ and hence $f(y) \epsilon U$.

If $f_n(y) \epsilon Y - \overline{N_{a+\epsilon}(f(p))}$, then since $d(f_n(y), f(y)) < \epsilon/2$, $f(y) \epsilon Y - \overline{N_a(f(p))}$ and once again $f(y) \epsilon U$. Hence $f(W) \subset U$ and $f$ is c-continuous.

## Bibliography

[1] Nicolas, Bourbaki, *Elements of Mathematics, General Topology, Part* 2, Addison-Wesley, Reading, Mass., 1966.

[2] Francois Lorrain, *Notes on Topological Spaces with Minimum Neighborhoods*, American Mathematical Monthly, 76 (1969), 616–627.

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