ON A THEOREM OF NAGATOMO

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Let C and D be the unit circle and the unit disc respectively. Suppose that $f(z)$ is a meromorphic function in D, then a point ζ on C is said to belong to $A_f(a)$, where $|a|\leqslant\infty$, (B_f^*, resp.) if ζ is the end of a boundary path in D on which $f(z)$ has the asymptotic value a (on which $f(z)$ is bounded, resp.). We set $A_f^*=\bigcup_{a\neq\infty} A_f(a)$, $A_f=A_f^*\cup A_f(\infty)$, and $B_f=B_f^*\cup A_f(\infty)$. A holomorphic (meromorphic, resp.) function $f(z)$ in D is said to belong to the class $\mathscr{A}$ or $\mathscr{B}$ ($\mathscr{A}_m$ or $\mathscr{B}_m$, resp.) of *MacLane* if the set A_f or B_f is dense on C accordingly. Among many other important theorems concerning functions in the class $\mathscr{A}$, *MacLane* [3] had showed that $\mathscr{A}=\mathscr{B}$ and proposed the question:

(A) Are the sum and the product of two functions in $\mathscr{A}$ again functions in $\mathscr{A}$?

Ryan and Barth [5] answered this negatively and raised the following question:

(B) Is the product of a bounded holomorphic function and a function in $\mathscr{A}$ again a function in $\mathscr{A}$? (Their sum is obviously a function in $\mathscr{A}$.)

However, *Barth and Schneider* [1] had recently constructed examples to show that (B) is not true. Hence in order that (B) to be valid, we have to impose some conditions on either or both of the functions. *Nagatomo* [4] had proved an interesting result in this direction.

Theorem N. *Let $b(z)$ be bounded and holomorphic in D such that the set of non-zero Fatou points is residual on C. Then if $f(x)\in\mathscr{A}$, we have $f(z)b(z)\in\mathscr{A}$.*

The proof of the above theorem in [4] is quite long and involving. It is the purpose of this paper to present an elementary and short proof. Moreover, we shall obtain some interesting corollaries which could not be seen directly from the proof given in [4].

Before showing the proof, we shall define some of the terms which will be used later.

(1) If $f(z)$ is a meromorphic function in D, we let $F_f(K)$ ($F_f^*(K)$, resp.), where $0\leqslant K\leqslant\infty$, be the set of Fatou points of $f(z)$ on C at which the Fatou

values are greater than (less than, resp.) K in absolute value. We also set $F_f(\infty)=F_f^*(0)=\phi$.

(2) A set S of points on C is said to be of second category evenly on C if it is of second category on each subarc of C. Note that every residual set on C is a set of second category evenly on C, while the converse is false.

To prove Theorem N, we shall prove the following equivalent form:

If $b(z)$ is a bounded holomorphic function in D, and if $f(z)\in\mathscr{A}$, but $f(z)b(z)\bar{\in}\mathscr{A}$, then $F_b(0)$ is of first category in some subarc of C.

Proof: Since $f(z)b(z)\bar{\in}\mathscr{A}=\mathscr{B}$, then there exists an subarc γ of C such that

$$B_{fb}\cap\gamma=\phi\,. \qquad (*)$$

It is clear that $F_b(0)\cap\gamma=\bigcup_{n=1}^{\infty}F_b(1/n)\cap\gamma$. We shall show that $F_b(0)\cap\gamma$ is of first category. Suppose, on the contrary, that it is of second category, then there exists $n_0>0$, such that $F_b(1/n_0)\cap\gamma$ is of second category. i.e. At each point ζ of $F_b(1/n_0)\cap\gamma$, the radial cluster set of $b(z)$ does not contain the value 0, by [2, Lemma 1], $|1/b(z)|\leqslant M'$, for some $M'>0$, in a neighborhood U of a subarc β of γ in D. Hence

$$0<1/M'\leqslant|b(z)|\leqslant M<\infty\,, \qquad (**)$$

in U, where M is the bound of $b(z)$ in D. Since $f(z)\in\mathscr{A}$, $B_f\cap\beta\neq\phi$, by (**), we have $B_{fb}\cap\beta\neq\phi$, a fortiori, $B_{fb}\cap\gamma\neq\phi$, which contradicts (*). And our proof is complete.

Applying [2, Lemma 2] carefully, we obtain

Corollary 1. *Let $f(z)$ and $g(z)$ be both in $\mathscr{A}$ ($\mathscr{A}_m$, $\mathscr{B}_m$, resp.) such that $F_f(0)\cap F_f^*(\infty)$ is of second category evenly on C, then $f(z)g(z)\in\mathscr{A}$ ($\mathscr{A}_m$, $\mathscr{B}_m$, resp.).*

Corollary 2. *Let $f(z)$ and $g(z)$ be both in $\mathscr{A}$ ($\mathscr{A}_m$, $\mathscr{B}_m$, resp.) and let S be a set of second category evenly on C. Suppose that at each point ζ of S, the radial cluster set of $f(z)$ does not contain the values 0 and ∞. Then $f(z)g(z)\in\mathscr{A}$ ($\mathscr{A}_m$, $\mathscr{B}_m$, resp.).*

Remarks 1. The above corollaries are certainly stronger statements comparing to Theorem N.

2. We could replace radial cluster sets in Corollary 2 by *suitable* chordal cluster sets or *suitable* boundary-path-cluster-sets if we apply [2, Lemma 1a] instead of [2, Lemma 1].

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