A METHOD FOR DERIVING DIFFERENTIAL EQUATIONS OF SPECIAL FUNCTIONS*

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Every mathematician working in the field of the so-called "special functions of analysis" knows what a vast literature exists on the subject and so all the more surprising is the fact that one does not find therein a general method for deriving the differential equation of the function from its generating function [20]. We know that a large class of special functions satisfy a second order differential equation which is rather easy to get from a number of easily derivable second order recurrence relations satisfied by them. In recent years, however, generalized potential problems associated with extended Laplace's equation and satisfying recurrence relations of an order higher than the second have been studied by various workers in the field. In two such papers by Gould ([9], [10]) the differential equation is conspicuous by its absence. It was in an effort to find these missing differential equations that the following problem was solved for a very large class of special functions ([2], [6], [17]): *Given the generating function of a polynomial or special function how should one proceed to be able to get its differential equation.*

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1. Let us first derive the differential equation satisfied by the generalized Humbert polynomials $P_n(m, x, y, p, C) \equiv P_n$ studied by Gould [9] in great detail. He defines them by

$$(C-mxt+yt^m)^p = \sum_{n=0}^{\infty} t^n P_n(m, x, y, p, C) , \tag{1.1}$$

where $m \geqslant 1$ is an integer and the other parameters are in general unrestricted. Differentiating (1.1) partially with respect to x and then with respect to t we easily get relations (2.5) and (2.7) of Gould [9], viz.

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$$(1.2)\qquad (xD-n)P_n=yP'_{n-m+1}\left(D\equiv\frac{\partial}{\partial x},\quad P'_n\equiv DP_n\right)$$

and

$$(1.3)\qquad [(m-1)xD+(n-mp)]P_n=CP'_{n+1}\,,\quad (n\geqslant 0;\ m-1\leq n)\,.$$

Now iteratively applying the linear differential operator in (1.3) $(m-1)$ times on (1.2) and using at each step the simple

Lemma: $(axD+b)D^n\equiv D^n(axD+b-na)$

we easily get the required differential equation in the following form:

$$(1.4)\qquad \{(m-1)xD+n-mp+m(m-2)\}\{(m-1)xD+n-mp+m(m-3)\}\cdots\{(m-1)xD+n-mp\}(xD-n)P_n=yC^{m-1}P_n^{(m)}\,.$$

Particular cases: (i) *Legendre polynomials*: Putting $m=2$, $y=1$, $C=1$, $p=-\frac{1}{2}$ in (1.4) we get the well-known differential equation for Legendre polynomials [18], viz.

$$(1.5)\qquad (xD+n+1)(xD-n)P_n=P''_n\,,$$

i.e., $(1-x^2)P''_n-2xP'_n+n(n+1)P_n=0$.

(ii) *Humbert polynomials*: Put $m=3$, $y=1$,, $C=1$, $p=-\nu$ in (1.4) to get the differential equation satisfied by Humbert polynomials ([12], [13]):

$$(1.6)\qquad (1-4x^3)y'''-6(3+2\nu)x^2y''-[(n+3\nu+5)(2-3n+3\nu)+10n]\,xy'+n(n+3\nu)(n+3\nu+3)y=0\,,$$

where $y=P_n(3,x,1,-\nu,1)$.

(iii) As particular cases of (1.4) we can get the differential equations for the well-known polynomials of *Louville* [16], *Tchebycheff* [19], *Gegenbauer* ([7], [8]), *Pincherle* [11] *and Kinney* [14].

2. Gould [10] has given the following two generalizations of Hermite polynomials.

$$(2.1)\qquad x^{-a}(x-t)^a\, e^{p[x^r-(x-t)^r]}=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n^r(x,a,p)\,,$$

$$(2.2)\qquad e^{tx+ht^r}=\sum_{n=0}^{\infty}\frac{t^n}{n!}g_n^r(x,h)\,.$$

Differentiating (2.1) first with respect to x and then with respect to t we easily get the following two recurrence relations satisfied by $H_n\equiv H_n^r(x,a,p)$:

(2.3) $$\left(D-prx^{r-1}+\frac{a}{x}\right)H_n=-H_{n+1}\,,$$ (Compare (3.4) of Gould [10])

(2.4) $$(xD+n)H_n=-pr\sum_{s=1}^{r}(-1)^s\binom{r}{s}x^{r-s}\frac{n!}{(n-s)!}H_{n-s}\,.$$

Utilizing these two we immediately get an $(r+1)^{th}$ order differential equation satisfied by $H_n^r(x, a, p)$, viz.

(2.5) $$\begin{aligned}&(xD+n+r)\left(D-prx^{r-1}+\frac{a}{x}\right)^r H_n\\ &\quad=-pr\sum_{s=1}^{r}\binom{r}{s}x^{r-s}\frac{(n+r)!}{(n+r-s)!}\left(D-prx^{r-1}+\frac{a}{x}\right)^{r-s}H_n\,.\end{aligned}$$

For $a=0$ this reduces to an r^{th} order differential equation of the form [1]

(2.6) $$\begin{aligned}&D(D-prx^{r-1})^{r-1}H_n\\ &\quad=-pr\sum_{s=1}^{r-1}\binom{r-1}{s}x^{r-1-s}\frac{(n+r-1)!}{(n+r-1-s)!}(D-prx^{r-1})^{r-1-s}H_n\,,\end{aligned}$$

which in turn gives for $p=1$ and $r=2$ the following well-known differential equation satisfied by Hermite polynomials [18]

(2.7) $$D(D-2x)H_n=-2(n+1)H_n \quad \text{i.e.}\quad H_n''-2xH_n'+2nH_n=0\,.$$

Using the same technique on (2.2) we easily get the differential equation satisfied by $g_n\equiv g_n^r(x, h)$ in the form

(2.8) $$(xD-n)g_n=-rhD^r g_n\,,$$

which in turn will give the standard differential equation of Hermite polynomials as a particular case, i.e. for $r=2$, $h=-1$.

3. We can apply the same method to get differential equations satisfied by Laguerre polynomials, Bessel functions and hypergeometric functions by utilizing the following well-known pairs of recurrence relations of the respective functions ([5], [18]):

(3.1) $$(D-1)L_n=\frac{1}{n+1}L_{n+1}' \text{ and } (n-xD)L_n=n^2L_{n-1};$$

(3.2) $$\left(D+\frac{n}{x}\right)J_n=J_{n-1} \text{ and } \left(D-\frac{n}{x}\right)J_n=-J_{n+1};$$

(3.3) $$(a+xD)F_a=aF_{a+1} \text{ and } \{(c-a-bx)+x(1-x)D\}F_a=(c-a)F_{a-1}\,.$$

The respective differential equations have the following form but can easily be reduced to the standard differential equations:

$$(n-xD)(D-1)L_n=(n+1)L'_n; \tag{3.4}$$

$$\left(D-\frac{n-1}{x}\right)\left(D+\frac{n}{x}\right)J_n=-J_n; \tag{3.5}$$

$$\{(c-a-1-bx)+x(1-x)D\}(a+xD)F_a=a(c-a-1)F_a. \tag{3.6}$$

4. We will now give this method of finding the differential equation of a special function a somewhat more explicit formulation.

Let $F(x, t)$ be a generating function for the set $\{f_n(x)\}$ of special functions, i.e. let

$$F(x, t)=\sum f_n(x)t^n \tag{4.1}$$

where

(i) the formal power series expansion of $F(x, t)$ in powers of t may or may not converge;

(ii) $n=0, 1, 2, \ldots$ or $n=0, \pm1, \pm2, \ldots$;

(iii) both $F(x, t)$ and $f_n(x)$ may depend on a number of parameters.

Then

$$\frac{\partial F}{\partial x}=\sum\frac{\partial f_n}{\partial x}t^n \tag{4.2}$$

$$\frac{\partial F}{\partial t}=\sum nf_nt^{n-1}. \tag{4.3}$$

From these three equations (4.1), (4.2) and (4.3) we can easily get the two linear differential-difference relations satisfied by the set of functions $\{f_n(x)\}$ which will be used to obtain the required differential equation.

If the special function is defined not by means of its generating function but in some other way we can still easily obtain the two differential recurrence relations we need. From the cases discussed in this paper we notice that only two linear differential-difference relations and no more are required to obtain the desired differential equation. The technique in all cases is to apply the 2 linear differential operators iteratively on $f_n(x)$ so that the index n is restored back. In all cases solved above (except one) the index of the function $f_n(x)$ was lowered in one relation and raised in the other and our technique was ideally suited to meet the situation (notice that a lemma was needed for the case of the generalized Humbert polynomials); in the exceptional case of (2.2) the matter was simpler still.

To illustrate further let us take up a generalization of Bessel functions* given by the following generating function:

$$e^{cx(t^l-y/t^m)}=\sum_{n=-\infty}^{+\infty} J_n(x, l, m, c, y)t^n ,$$

where m, l are positive integers and the other parameters are in general unrestricted. Differentiating the above generating function with respect to x and t respectively and eliminating we get the following two differential recurrence relations:

$$\left(mD+\frac{n}{x}\right)J_n=c(l+m)J_{n-l} ,$$

$$\left(lD-\frac{n}{x}\right)J_n=-cy(l+m)J_{n+m} .$$

If $l=sp$ and $m=sr$ where p and r are positive integers prime to each other we see that the index n of J_n will be restored back if we apply on J_n the first operator r times and the second operator p times. The required $(r+p)^{th}$ order differential equation is of the form

$$\begin{aligned}&\left(lD-\frac{n-rl+m(p-1)}{x}\right)\left(lD-\frac{n-rl+m(p-2)}{x}\right)\cdots\left(lD-\frac{n-rl}{x}\right)\\&\quad\cdot\left(mD+\frac{n-(r-1)l}{x}\right)\cdots\left(mD+\frac{n-l}{x}\right)\left(mD+\frac{n}{x}\right)J_n\\&=(-y)^pc^{r+p}(l+m)^{r+p}J_n .\end{aligned}$$

If, however, the special function depends on two sets of indices b and c then we require 3 differential-difference relations but the technique of deriving the differential equation satisfied by $f_{b,c}(x)\equiv f_{b,c}$ is the same, that is, we restore back the 2 indices b and c by applying the 3 operators iteratively. For example** the 3 recurrence relations

$$\left(\frac{m}{c-1}xD+1\right)f_{b,c}=f_{b,c-1} ,$$

$$\left(\frac{x}{b}D+1\right)f_{b,c}=f_{b+1,c} ,$$

* This generalization of Bessel function will be studied later in a separate paper.

** The author came across a more general example of such a function in one of his papers, "Some generalizations of Laguerre Polynomials-I" in press.

$$Df_{b,c}=\frac{b}{(c)_m}f_{b+1,c+m}\ ,$$

give the following differential equation of the m^{th}-order

$$D\left(\frac{m}{c-m}xD+1\right)\left(\frac{m}{c-m+1}xD+1\right)\cdots\left(\frac{m}{c-1}xD+1\right)y=\frac{b}{(c-m)_m}\left(\frac{x}{b}D+1\right)y$$

where $(c)_m=\dfrac{\Gamma(c+m)}{\Gamma(c)}$ and $y\equiv f_{b,c}(x, m)=\sum_{s=0}^{\infty}\dfrac{(b)_s}{(c)_{ms}}\dfrac{x^s}{s!}$.

This seems to be a very simple, elegant and a general method of finding differential equations of a large class of special functions. The technique however is best illustrated by specific examples; as there are so many variations in the form of the differential-difference relations it does not seem to be possible to give a general method of how to restore back the indices. It may be pointed out here that this technique of finding differential equations from differential-difference relations is not entirely new and has been applied earlier in stray complicated cases with great success. The author first came across such a method in the work by Delerue in 1951 [4] and then he himself used it in 1954 [3] but he has not seen it being used systematically as a general method to derive differential equations of wide classes of functions satisfying a higher order differential equation than the second.

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