# AN APPLICATION OF A THEOREM OF STEIN AND WEISS

By

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The purpose of this note is to present in detail an important application (concerning the pointwise convergence of Fourier series) of the interpolation theorem found in [3] that was communicated to the author by Professor *E. M. Stein* who also provided an outline of the proof given here. The definitions and notation used in this note are the same as those used in [3].

Let $S_n(x, f)$ denote the $n^{\text{th}}$ partial sum of the Fourier series for $f \epsilon L^1(-\pi, \pi)$.

Let $S_n^*(x, f) = \text{P. V.} \displaystyle\int_{-4\pi}^{4\pi} \frac{\varepsilon^{-int} f^0(t)}{x-t}\, dt$ where $x \epsilon (-\pi, \pi)$ and $f^0$ denotes the $2\pi$ periodic extension of $f$ to $(-4\pi, 4\pi)$.

Let $Mf(x) = \sup\limits_{n \geq 0} |S_n(x, f)| \quad x \epsilon (-\pi, \pi)$.

Let $M^*f(x) = \sup\limits_{|n| \geq 0} |S_n^*(x, f)| \quad x \epsilon (-\pi, \pi)$.

**Theorem 1.** (*Carleson-Hunt*). *$M^*$ is of restricted weak type $(p, p)$ for $1 < p < \infty$.*

**Proof.** This is established in [1].

**Theorem 2.** *For $1 < p < \infty \|Mf\|_p \leq A_p(\|f\|_p (2\pi)^{1/p} + \|M^*f\|_p)$ for $f$ in $L^p(-\pi, \pi)$ where $A_p > 0$ is a constant independent of $f$.*

**Proof.** This is an immediate consequence of the easily established fact: $Mf(x) \leq A_p(\|f\|_p + M^*f(x))$ for almost every $x \epsilon (-\pi, \pi)$, $f$ in $L^p(-\pi, \pi)$ where $A_p > 0$ is a constant independent of $f$ for $1 < p < \infty$.

**Theorem 3.** *For $1 < p < \infty$ for each measurable set $E \subset (-\pi, \pi)$ we have $\|M\chi_E\|_p \leq B_p \|\chi_E\|_p$ where $B_p > 0$ is a constant independent of $E$.*

**Proof.** This is an immediate consequence of Theorem 1, Theorem 2, and Lemma 2 on page 266 in [3] with $p_0 = q_0 = ((p+1)/2)$, $p_1 = q_1 = (p+1)$ and $t = (1 - 1/p)$; since the hypothesis of linearity is not used in the proof of Lemma 2.

Fix integer $N > 0$.

Let $M_N f(x) = \max\limits_{0 \leq n \leq N} |S_n(x; f)|$

Let $\alpha$ denote any simple function with domain $(-\pi, \pi)$ and range in $[0, 1, \ldots, N]$. We say $\alpha$ is an $N^{\text{th}}$ order simple function.

Let $T_\alpha f(x) = S_{\alpha(x)}(x; f) \quad x \epsilon (-\pi, \pi)$.

Clearly, $T_\alpha$ is linear for every $N^{th}$ order simple function.

**Theorem 4.** *For* $1<p<\infty \|T_\alpha \chi_E\|_p \leq B_p\|\chi_E\|_p$ *for each measurable set* $E\subset(-\pi, \pi)$ *and for each* $N^{th}$ *order simple function* $\alpha$ *where* $B_p$ *is that of Theorem* 3.

**Proof.** This is immediate by Theorem 3 and the fact that $\|T_\alpha f\|_p \leq \|Mf\|_p$ for each $1<p<\infty$, for each $f\in L_p(-\pi, \pi)$ and for each $N^{th}$ order simple function $\alpha$.

**Theorem 5.** *For* $1<p<\infty$ $\lambda_{T_\alpha \chi_E}(y) \leq B_p^p y^{-p}\|\chi_E\|_p^p$ *for each measurable set* $E\subset(-\pi, \pi)$ *and for each* $N^{th}$ *order simple function* $\alpha$ *where* $B_p$ *is that of Theorem* 4.

**Proof.** This is an immediate consequence of Theorem 4 and (c) on page 284 in [3].

**Theorem 6.** *For* $1<p<\infty \|T_\alpha f\|_p \leq C_p\|f\|_p$ *for every simple function* $f$ *in* $L_p(-\pi, \pi)$ *and for each* $N^{th}$ *order simple function* $\alpha$ *where* $C_p>0$ *depends only on* $p$.

**Proof.** This is an immediate consequence of Theorem 5, Theorem II in [3] and the remark in the footnote on the bottom of pape 264 in [3] with $p_0=q_0=((p+1)/2)$, $p_1=q_1=(p+1)$ and $t=(1-1/p)$.

**Theorem 7.** *Let* $f\epsilon L^p(-\pi, \pi)$ $1<p<\infty$. *Let* $\alpha$ *be any* $N^{th}$ *order simple function. There exists a sequence of simple functions* $\{f_n\}\subset L^p(-\pi, \pi)$ *such that* $\|f_n\|_p\rightarrow\|f\|_p$ *and* $T_\alpha(f-f_n)(x)\rightarrow 0$ *for* $x\epsilon(-\pi, \pi)$.

**Proof.** This is an immediate consequence of the Lebesgue dominated convergence theorem and the fact that there exists a sequence of simple functions $\{f_n\}\subset L^p(-\pi, \pi)$ such that $\|f_n\|_p\rightarrow\|f\|_p$, $f_n(x)\rightarrow f(x)$ for $x\epsilon(-\pi, \pi)$ and $|f_n(x)|\leq|f(x)|$ for $x\epsilon(-\pi, \pi)$ and $n\geq 0$.

**Theorem 8.** *For* $1<p<\infty \|T_\alpha f\|_p \leq C_p\|f\|_p$ *for every* $f$ *in* $L^p(-\pi, \pi)$ *and for each* $N^{th}$ *order simple function* $\alpha$ *where* $C_p$ *is that of Theorem* 6.

**Proof.** Fix $f\epsilon L^p(-\pi, \pi)$. Let $\{f_n\}$ be the sequence of Theorem 7. Then $|T_\alpha f_n(x)|\rightarrow|T_\alpha f(x)|$ for $x\epsilon(-\pi, \pi)$; so that by Fatou's theorem and Theorem 6 we have

$$\|T_\alpha f\|_p \leq \underline{\text{limit}}_{n\rightarrow\infty} \|T_\alpha f_n\|_p \leq C_p \underline{\text{limit}}_{n\rightarrow\infty} \|f_n\|_p = C_p\|f\|_p .$$

**Theorem 9.** *For* $1<p<\infty \|Mf\|_p \leq C_p\|f\|_p$ *for every* $f\epsilon L^p(-\pi, \pi)$ *where* $C_p$ *is that of Theorem* 8.

**Proof.** Fix $f_0$ in $L^p(-\pi, \pi)$. It is easily shown that there exists an $N^{th}$ order simple function $\alpha_0$ such that $|T_{\alpha_0} f_0(x)|=M_N f_0(x)$ for all $x\epsilon(-\pi, \pi)$. Hence by Theorem 8 $\|M_N f_0\|_p \leq C_p\|f_0\|_p$. But $M_N f_0(x)$ increases monotonically to $Mf_0(x)$

for each $x$ in $(-\pi, \pi)$. Hence $\|Mf_0\|_p \leq C_p \|f_0\|_p$.

**Theorem 10.** (*Carleson-Hunt*). *If $f$ in $L^p(-\pi, \pi)$ $1<p<\infty$, then $S_n(x; f)$ converges to $f(x)$ for almost every $x$ in $(-\pi, \pi)$.*

**Proof.** This is a straightfoward consequence of Theorem 9. For details cf. [1] or [2].

**Remark.** The technique used in the proof of Theorem 9 is similar to that used in Chapter XIII in [4], and I would like to thank Professor *E. M. Stein* for calling my attention to it. I would also like to acknowledge my indebtedness to Professor *R. A. Hunt* for his help in reading [1] and writing [2].

## REFERENCES

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