ON FIXED AND PERIODIC POINTS IN METRIC SPACES

By

HIROSHI FUKUSHIMA

(Received March 30, 1970)

1. Introduction and notations. Let (X, d) be a metric space. A mapping f of X into itself is said to be *non-expansive* if

$$d(f(x), f(y)) \leqq d(x, y) \quad \textit{for every } x, y \in X.$$

A point $x \in X$ is called a *periodic point* for f if there exists a positive integer p such that $f^p(x) = x$. If $p = 1$, then x is called a *fixed point* for f.

According to [1], we describe the following definition and notations. For each $x \in X$, $L(x)$ denotes the set of points of X which are limits of all convergent subsequences of the sequence $\{f^n(x)\}$, and $O(f^p(x))$ denotes the sequence of iterates of $f^p(x)$, that is, $O(f^p(x)) = \{f^p(x), f^{p+1}(x), \cdots\}, p = 0, 1, 2, \cdots$, where it is understood that $f^0(x) = x$. For a subset Y of X, $d(Y)$ denotes the diameter of Y. If $d(O(x)) < \infty$, then the sequence $\{d(O(f^n(x))); n = 0, 1, 2, \cdots\}$ has limit $r(x) \geqq 0$ which is called the ***limiting orbital diameter*** of f at x. f is said to have diminishing orbital diameters if for each $x \in X$, $d(O(x)) < \infty$ and the limiting orbital diameter of f at x is less than $d(O(x))$ when $d(O(x)) > 0$.

In [1] *Belluce and Kirk* proved that if f is a non-expansive mapping which has diminishing orbital diameters, and if for some $x \in X$, a subsequence of the sequence $\{f^n(x)\}$ has limit z, then $f(z) = z$ and $\lim_{n\to\infty} f^n(x) = z$.

Further in [2] *Kirk* obtained the very same results by exchanging a non-expansive mapping f for a mapping g which satisfies the condition that there exists a constant C such that for each positive integer p and for each $x, y \in X$, $d(g^p(x), g^p(y)) \leqq Cd(x, y)$. If $C = 1$, then g is non-expansive.

Our purposes are to generalize the above Kirk's theorem, and to give some examples.

Let $I(x)$ denote the number of all isolated points of $O(x)$, then $I(x) = 0$ implies that $O(x)$ is dense in itself.

2. Periodic points and isolated points.

Theorem 1. *Let f be a mapping of a metric space X into itself which satisfies* *the following conditions* ;

(i) $I(x) \geqq 1$ *for each point* $x \in X$

(ii) *for every positive number* ε *and for each point* $x \in X$, *there exist a positive number* $\delta(x, \epsilon)$ *and a positive integer* $M(x)$ *such that* $d(x, y) < \delta(x, \epsilon)$ *implies* $d(f^p(x), f^p(y)) < \varepsilon$ *for every positive integer* $p \geqq M(x)$

(iii) $z \in L(x)$ *for some point* $x \in X$.

Then z *is a periodic point for* f *and so* $I(z)$ *is the number of all periodic points in* $O(z)$.

Proof. Let $z = \lim_{i \to \infty} f^{n_i}(x)$ and $\varepsilon\ (>0)$ given. Then there exists a positive integer $N_1(z, \frac{\varepsilon}{2})$ such that $N_1(z, \frac{\varepsilon}{2}) < i$ implies $d(f^{n_i}(x), z) < \frac{\varepsilon}{2}$, and also exist two positive integers $N_2(z, \frac{\varepsilon}{2})$ and $M(z)$ such that $N_2(z, \frac{\varepsilon}{2}) < i$ and $M(z) \leqq p$ imply $d(f^p(f^{n_i}(x)), f^p(z)) < \frac{\varepsilon}{2}$. Here anew we put $N(z, \varepsilon) = \max\{N_1(z, \frac{\varepsilon}{2}), N_2(z, \frac{\varepsilon}{2})\}$, $i_1 = N(z, \varepsilon) + 1$ and $j_1 = n_{i_1} + M(z)$. Then $d(f^{n_{j_1} - n_{i_1}}(z), z) \leqq d(f^{n_{j_1} - n_{i_1}}(z), f^{n_{j_1} - n_{i_1}}(f^{n_{i_1}}(x))) + d(f^{n_{j_1}}(x), z) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Thus it is easily seen that there exists a subsequence $\{f^{m_i}(z);\ m_1 < m_2 < \cdots < m_i < \cdots\}$ of the sequence $\{f^n(z)\}$ such that $z = \lim_{i \to \infty} f^{m_i}(z)$. Hence $f^p(z) = \lim_{i \to \infty} f^{p + n_i}(z)$ for every positive integer $p \geqq M(z)$. If a set $\{f^{p+m_i}(z);\ i = 1, 2, 3, \cdots\}$ is infinite for each positive integer $p \geqq M(z)$, then $I(f^{M(z)}(z)) = 0$. It is in contradiction to the hypothesis. Therefore for some $q \geqq M(z)$, the set $\{f^{q+m_i}(z);\ i = 1, 2, 3, \cdots\}$ is finite. Since $f^q(z) = \lim_{i \to \infty} f^{q+m_i}(z)$, there exists a positive integer t such that $f^{q+m_i}(z) = f^q(z)$ for every $i \geqq t$. Hence $f^q(z)$ is a periodic point. Then $O(z)$, consequently the set $\{f^{m_i}(z);\ i = 1, 2, 3, \cdots\}$, is finite. Thus z is periodic and $I(z)$ is the minimal of q such that $f^q(z) = z$.

Corollary 1. *Let* f *be a mapping of a metric space* X *into itself.*

(i) f *has diminishing orbital diameters*

(ii) *for every positive number* ε *and for each point* $x \in X$, *there exist a positive number* $\delta(x, \varepsilon)$ *and a positive integer* $M(x)$ *such that* $d(x, y) < \delta(x, \varepsilon)$ *implies* $d(f^p(x), f^p(y)) < \varepsilon$ *for every* $p \geqq M(x)$

(iii) $z \in L(x)$ *for some point* $x \in X$.

Then $f(z) = z$ *and* $\lim_{n \to \infty} f^n(x) = z$.

Proof. From the proof of Theorem 1, (ii) and (iii) imply that $f^p(z)$ is a limit point of $O(f^{M(z)}(z))$ for every $p \geqq M(z)$. Then $r(f^{M(z)}(z))=d(O(f^{M(z)}(z)))$ and so $d(O(f^{M(z)}(z)))=0$ by (i). Hence $f^{M(z)+1}(z)=f^{M(z)}(z)$ and further $f(z)=z$. Let $z=\lim_{i\to\infty} f^{n_i}(x)$ and $\varepsilon(>0)$ given. Then there exists a positive integer $N(z, \varepsilon)$ such that $N(z, \varepsilon)<i$ implies $d(f^p(f^{n_i}(x)), f^p(z))<\varepsilon$ for every $p \geqq M(z)$. Hence $\lim_{n\to\infty} f^n(x)=z$.

Remark. $I(x)=0$ implies that $r(x)=d(O(x))$.

The conditions of *Kirk's* theorem imply those of Corollary 1. The following example shows that Corollary 1 is more general than *Kirk's* theorem.

Example 1. Let $X=[0,1]$ with the usual distance and $s=s(t)=2^{t-1}+1$ where $t=1,2,3,\cdots$. Let f be a mapping of X into itself defined by;

(1) $f(x)=sx-1$ if x is a rational number in $\left[\frac{2}{2s-1}, \frac{4}{4s-3}\right]$ for $2^{2u-1} \leqq t < 2^{2u}$, or if x is an irrational number in $\left[\frac{2}{2s-1}, \frac{4}{4s-3}\right]$ for $2^{2u} \leqq t < 2^{2u+1}$, where u is a positive integer

(2) $f(x)=0$ for others.

Then for every point $x \in \left[\frac{2}{2s-1}, \frac{4}{4s-3}\right]$, $f^{2^v-t+1}(x)=0$ where $2^{v-1} \leqq t < 2^v$ and v is a positive integer.

Thus the conditions of Corollary 1 are all satisfied, but there exists no constant C such that $d(f^p(x), f^p(y)) \leqq Cd(x,y)$ for each $x, y \in X$ and for every positive integer p.

The next theorem is easily obtained from the proof of Theorem II [2].

Theorem 2. *Let (X, d) be compact and f a continuous mapping of X into itself. Then f has at least one periodic point in X if $I(x) \geqq 1$ for each $x \in X$.*

Proof. Let x be an arbitrary point of X. From the proof of Theorem II [2], there exists a minimal subset K of $L(x)$ with respect to being non-empty, compact and mapped into itself by f. Let $d(K)>0$ and $z \in K$. Then the closure $\overline{O(z)}$ of $O(z)$ coincides with K by the minimality of K. Suppose that $O(z)$ is infinite. Then by the hypothesis, there exists a positive integer p for which $f^p(z)$ is an isolated point of $O(z)$. Thus $\overline{O(f^{p+1}(z))}$ is a proper subset of K which is mapped into itself by f. This contradicts the minimality of K. Hence $O(z)$ is finite. Again the minimality of K implies that $f(K)=K$ and $z=f^{I(z)}(z)$. When $d(K)=0$, it is clear that $K=\{z\}$ and $f(z)=z$.

The following example shows that if the condition (ii) is exchanged for the continuity of f in Corollary 1, then it no longer assures the existence of a fixed point for f, and shows too that it does not guarantee the existence of a periodic point to change the condition (ii) of Theorem 1 for the global continuity of f on X, or to substitute the locally compactness and the totally boundedness for the compactness of X in Theorem 2.

Example 2. Let X be a subset of the plane, with the Euclidean distance, defined as follows;

$$X=X_1\cup X_2\cup X_3,\ X_1=\left\{\left(\frac{1}{s}+1,\frac{1}{s}\right);\ s=2,3,4,\cdots\right\},$$

$$X_2=\left\{\left(\frac{1}{s},\frac{1}{t}\right);\ s,t=2,3,4,\cdots;\ s\leqq t\right\},\ X_3=\left\{\left(\frac{1}{s},0\right);\ s=1,2,3,\cdots\right\}.$$

Define a mapping f of X into itself by;

(1) $\quad f\left(\left(\frac{1}{s}+1,\frac{1}{s}\right)\right)=\left(\frac{1}{2},\frac{1}{s}\right)$

(2) $\quad f\left(\left(\frac{1}{s},\frac{1}{t}\right)\right)=\left(\frac{1}{s+1},\frac{1}{t}\right)$ if $s<t$

(3) $\quad f\left(\left(\frac{1}{s},\frac{1}{t}\right)\right)=\left(\frac{1}{s+1}+1,\frac{1}{t+1}\right)$ if $s=t$

(4) $\quad f\left(\left(\frac{1}{s},0\right)\right)=\left(\frac{1}{s+1},0\right).$

Then we see easily the following in this example;

(i) X is locally compact and totally bounded, but not compact

(ii) f is continuous on X

(iii) f has diminishing orbital diameters

(iv) the condition (ii) of Theorem 1 and Corollary 1 is not satisfied

(v) $L(x)=X_3=O((1,0))$ for each point $x\epsilon X_1\cup X_2$, and $I(x)\geqq 1$ for each $x\epsilon X$.

But clearly f has no periodic point.

The following example shows that if f is discontinuous, then the other conditions of Theorem II [2] don't assure the existence of a fixed point for f.

Example 3. Let $X=\{\pm(1+\left(\frac{1}{2}\right)^k);\ k=0,1,2,\cdots\}\cup\{\pm 1\}$ with the usual distance and f a mapping of X into itself defined by;

$$f(-x)=x \text{ if } x>0,\ f\left(1+\left(\frac{1}{2}\right)^k\right)=-\left(1+\left(\frac{1}{2}\right)^{k+1}\right) \text{ and } f(1)=-2.$$

Then X is compact and f has diminishing orbital diameters, but there exists no fixed point.

3. Mappings each of whose iterative sequences converges to some mapping.

Theorem 3. *Let f be a mapping of a metric space X into itself and the sequeuce $\{f^n\}$ of iterates of f pointwise converge to a mapping g which is commutative with f^p for some positive integer p. Then f has a fixed point.*

Proof. For an arbitrary point $x \epsilon X$, $\lim_{n\to\infty} f^n(x)=g(x)$ and $\lim_{n\to\infty} f^{n+p}(x)=g(f^p(x))$ $=f^p(g(x))$. Thus $f^p(g(x))=g(x)$. Since $\lim_{n\to\infty} f^n(g(x))=g^2(x)$ and $f^{mp}(g(x))=g(x)$ for every positive integer m, $d(g(x), f(g(x)))=\lim_{n\to\infty} d(f^{np}(g(x)), f^{np+1}(g(x)))=0$.

It is known that if $\lim_{n\to\infty} f^n(x)=\eta \epsilon X$ for some point $x \epsilon X$ and if f is continuous at η, then $f(\eta)=\eta$. The following example shows that Theorem 3 is independent of the above statement.

Example 4. Let $X=[0,1]$ with the usual distance and f a mapping of X into itself defined by;

(1) $f(0)=f(1)=1,\ f\left(\frac{1}{3}\right)=\frac{1}{3}$

(2) $f(x)=0$ if $x \epsilon \left(0, \frac{1}{3}\right)$

(3) $f(x)=\frac{1}{2}x$ if $x \epsilon \left(\frac{1}{3}, 1\right)$.

Then it is easily seen that;

(i) for each $x \epsilon \left[0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, 1\right]$, $\lim_{n\to\infty} f^n(x)=1$

(ii) f is discontinuous at $\frac{1}{3}$ and 1 which are fixed points

(iii) the limit mapping g is commutative with f^2, but not with f.

Finally we are very much grateful to Professor Masae Orihara.

REFERENCES

[1] L. P. Belluce and W. A. Kirk. ***Fixed-point theorems for certain classes of nonexpansive mappings,*** Proc. Amer. Math. Soc., 20 (1969) pp 141–146.

[2] W. A. Kirk. ***On mappings with diminishing orbital diameters,*** J. London Math. Soc., 44 (1969) pp 107–111.

Department of Mathematics
Kanagawa University
Rokkakubashi, Kanagawa-ku
Yokohama 221 Japan