TRIANGULATION OF A TOPOLOGICAL MANIFOLD WITH TRANSVERSE K-PLANE FIELD

By

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1. Introduction.

Let M be a topological n-manifold imbedded in euclidean $(n+k)$-space R^{n+k}. Let Q be a k-plane through the origin in R^{n+k} (i. e., $Q \varepsilon G_{k,n}$).

Let x be a point of M and U a neighborhood of x in M; suppose that no line joining two points of $\bar{U}$ is parallel to Q. Then Q is said to be transverse to M at x. And U is called an admissible neighborhood of x for Q. A transverse k-plane field on M is a continuous map φ of M into $G_{k,n}$ such that $\varphi(x)$ is transverse to M at x, for all $x \varepsilon M$.

If a topological n-manifold M^n in an euclidean $(n+k)$-space R^{n+k} admits a transverse k-plane field then M^n has a differentiable structure [2]. And a differentiable manifold M^n has a C^r-triangulation [1], [3].

Hence the topological n-manifold M^n with the transverse k-plane field in R^{n+k} is triangulable. In this paper we immediately triangulate the compact topological n-manifold M^n with the transverse k-plane field in R^{n+k} but not using the differentiable structure induced by the k-plane field. The method is analogous to *Whitney's* method [3] which is used the triangulation of a differentiable manifold.

Let K be a (locally finite) simplicial complex and $\pi^*: |K| \longrightarrow M$ be a homeomorphism. Then we call K a triangulation of M. If for any vertex v of K $St(v, K)$ is piecewise linearly homeomorphic to the n-simplex, then we call K a combinatorial triangulation of M.

Let K and L be two complexes. If K and L have simplicial subdivisions K' and L' resp. such that K' is isomorphic to L', then we say that K is combinatorially equivalent to L. If any triangulation of M are combinatorially equivalent, then we say that M has a unique triangulation. By a closed n-manifold we mean a compact topological n-manifold without boundary.

Theorem. *If the closed n-manifold M^n in R^{n+k} satisfies the following two conditions, then M^n has a combinatorial triangulation.*

Furthermore if $(n+k) \geqq 4$, M^n has a unique triangulation.

Conditions 1) M^n *admits a transverse k-plane field* $\varphi : M^n \longrightarrow G_{k,n}$.

2) *The angle between* $\varphi(x)$ *and the line yz is greater than* $\cot^{-1} \frac{1}{4}$ *i.e.,* $\cot \alpha(\overleftrightarrow{yz}, \varphi(x)) > \frac{1}{4}$ *where* y, z *are any points contained in some admissible neighborhood of* x *for* $\varphi(x)$.

2. Definitions and Notations.

R^m will mean an m-dimensional euclidean space.

We use $G_{k,n}$ to denote the *Grassmannian manifold* consisting of k-planes through the origin in R^{n+k}. Let x and y are vectors in R^{n+k}, then $\alpha(x, y)$ will denote the *angle* between them, on the understanding that $0 \leqq \alpha(x, y) \leqq \pi$.

If $Q \varepsilon G_{k,n'}$, then $\alpha(x, Q)$ will denote the angle between x and its orthogonal projection in Q, with $\alpha(x, Q) = \frac{\pi}{2}$ if x is orthogonal to Q.

Thus $0 \leqq \alpha(x, Q) \leqq \frac{\pi}{2}$. If $P \varepsilon G_{i,j}$, $Q \varepsilon G_{l,m}$ where $i+j = l+m$, $0 < i \leqq l, m > 0$, then $\alpha(P, Q)$ will denote

$$\alpha(P, Q) = max\,\{\alpha(x, Q)\,;\, 0 \neq x \varepsilon P\}.$$

If $x \varepsilon R^{n+k}$, $P \varepsilon G_{k,n}$, then $x+P$ will denote the flat k-space consisting of the vectors $x+y \varepsilon R^{n+k}$ for every $y \varepsilon P$. Let M^n be an n-dimensional topological manifold in R^{n+k}. A k-plane $P_p \varepsilon G_{k,n}$ will be described as *transverse to* M^n *at a point* $p \varepsilon M^n$ if and only if there is a neighborhood $W \subset M$ of p and a number δ such that $0 < \delta < \pi/2$ and $\alpha(\overleftrightarrow{xy}, P_p) > \delta$ if $x, y \varepsilon W, x \neq y$. Let W, P_p satisfy the above condition and let $Q_p \varepsilon G_{n.k}$ be the n-plane through p and orthogonal to P_p. Then the orthogonal projection π_p : $R^{n+k} \longrightarrow Q_p$ maps a neighborhood W of p in M in 1-1 fashion and hence homeomorphically, on an open subset $U \subset Q_p$. We call the neighborhood W an *admissible neighborhood* of p for a transverse k-plane P_p.

A *transverse k-plane field to* M^n *in* R^{n+k} will mean a continuous map $\varphi : M^n \longrightarrow G_{k,n}$ such that $\varphi(x)$ is transverse to M^n at x for every $x \varepsilon M$. In this paper M^n always means a closed topological n-manifold with a transverse k-plane field in R^{n+k}.

From Proposition 0 every transverse k-plane field $\varphi : M \longrightarrow G_{k,n}$ can be ε-approximated by a transverse k-plane field φ which is a Lipschitz map and transversally homotopic to φ [see 2. p 159].

Hence we always consider that every transverse k-plane field φ is a Lipschitz map.

And as a matter of convenience we shall often denote M^n, R^{n+k} by M, R respectively.

Let P and P' be planes of dimensions n and k respectively in R^{n+k} with just one point in common. Then *independence of P and P'* we define to be

$$\text{ind}(P, P')=\inf \{ |v-\pi_P v| ; \quad v\varepsilon P', \quad |v|=1\}$$

where π_P be the orthogonal projection into P.

By a complex K we shall mean a locally finite simplicial complex.

And $|K|$ will denote the underlying space of K.

For any oriented complex K we call the continuous mapping f of $|K|$ into oriented R^n simplexwise positive if for each n-simplex σ_i^n of K, f is linear and one-one in σ_i^n, and orientation preserving there.

The *star* $St(\sigma)$ of σ in K is the point set consisting of all int (σ') such that σ is a face of σ'. The closed star $\overline{St}(\sigma)$ is the closure in $|K|$ of $St(\sigma)$. The *star boundary* $\partial St(\sigma)$ is $\overline{St}(\sigma)-St(\sigma)$.

For any complex K, let K^k or $(K)^k$ denote the subcomplex containing all simplexes of K of dimension $\leqq k$. With a mapping $f: K \longrightarrow R^n$, any point q of $f(K)-f(K^{n-1})$ is in the image of a certain number h of n-simplexes of K; we say q is covered h-times.

Proposition 0. (*J. H. C. Whitehead*) *Every transverse k-plane field $\varphi; M \longrightarrow G_{k,n}$ on M can be ε-approximated by a transverse k-plane field φ which is a Lipschitz map and transversally homotopic to φ* [2. Th 1.10].

3. Local properties of $\mathbf{M}^n$ in $\mathbf{R}^{n+k}$.

Let $\pi_p: R^{n+k} \longrightarrow Q_p$ be an orthogonal projection. Then $\pi_p | W: W \longrightarrow U$ is one to one where W is an admissible neighborhood for P_p.

We define to be $h=(\pi_p | W)^{-1}$ and

$$(1) \quad \begin{cases} U_\eta(p)=\{x\varepsilon R^{n+k}; |p-x| < \eta\}. \\ Q_{p,\eta}=Q_p \cap U_\eta(p). \\ M_{p,\eta}=h_p(Q_{p,\eta}). \end{cases}$$

Lemma 1. *Let M^n in R^{n+k} be compact. Then there is a $\eta_0 > 0$ such that M_{p,η_0} is defined for all $p\varepsilon M^n$.*

Moreover

$$(2) \quad d(p, M-M_{p,\eta_0}) \geqq \eta, \qquad \eta \leqq \eta_0.$$

Proof. Since at any point p of M^n there is a transversal k-plane P_p, there exist an admissible neighborhood W and orthogonal projection $\pi_p: R^{n+k} \longrightarrow Q_p$ such that $\pi_p | W$ maps homeomorphically on an open subset $U \subset Q_p$. Then there exists a positive number η such that $Q_{p,\eta'}$ is contained in U, Then $M_{p,\eta'}$ is defined by $h_p=(\pi_p | W)^{-1}$.

Since M^n is compact, for some $\eta_0' < \eta'$, M_{p,η_0} is defined for all $p\varepsilon M$.

From the definition of π_p (2) is clear.

Lemma 2. *Let M^n in R^{n+k} be compact. Then for any $\omega > 0$ there is a positive number $\eta_1 \leqq \eta_0$ with the following property.*

For any point $p\varepsilon M$ and vector $v=q'-q$ in Q_q where q is a point in M_{p,η_1} and q' is a point in Q_q,

$$(3)\qquad |v-\pi_p v| \leqq \omega |\pi_p v| \leqq \omega |v|.$$

Proof. Let sup $\alpha(Q_x, Q_p)=\kappa_1$ for $x\varepsilon M_{p,\eta}$. Since M^n admits a transverse k-plane fields, we may choose η_1 so that tan $\kappa_1 \leqq \omega$ for any point x in M_{p,η_1}. Let $\alpha(Q_q, Q_p)=\theta$, then

$$|v-\pi_p v| = |\pi_p v| \tan\alpha(v, \pi_p v) \leqq |\pi_p v| \tan\theta \leqq |\pi_p v| \tan\kappa_1 \leqq \omega |\pi_p v| \leqq \omega |v|.$$

Lemma 3. *Let M^n and η_1 be as in Lemma 2 and let $v=x-y$ for any point x, y in M_{p,η_1}.*

Let κ_2 be a positive number satisfying inf $\alpha(\delta z, P_p)=\kappa_2$ for any $z, z+\delta z\varepsilon M_{p,\eta_1}$. (The existence of κ_2 follows from $\eta_1 \leqq \eta_0$.)

Then v satisfies (3) for $\omega \geqq$ cot κ_2. Moreover

$$(4)\qquad |p'-\pi_p(p')| > \omega\eta, \quad p'\varepsilon M_{p,\eta}, \quad \eta \leqq \eta_1.$$

$$(5)\qquad M_{p,\eta} \subset U_{\omega\eta}(Q_{p,\eta}), \quad Q_{p,\eta} \subset U_{\omega\eta}(M_{p,\eta}), \quad \eta \leqq \eta_1.$$

Proof. $|\pi_p v-v| = |\pi_p v|$ cot $\alpha(v, P_p) \leqq |\pi_p v|$ cot $\kappa_2 \leqq \omega |\pi_p v| \leqq \omega |v|$ proving (3). $|p'-\pi_p(p')| = |v-\pi_p v| < \omega |\pi_p v| < \omega\eta$ where $v=p'-p$ proving (4).

Relation (5) follows from (4).

Let P and P' be planes of dimensions n and k respectively in R^{n+k} with just one point in common. Then to each point $p\varepsilon R^{n+k}$ corresponds a unique point $q=\pi'(p)\varepsilon P$ such that $q-p$ is a vector in P' where $\pi'; R^{n+k} \longrightarrow P$ is the projection along P'.

Lemma 4. *Given M^n in R^{n+k}, let ω and η_1 be as in* Lemma 3. *Take $p\varepsilon M$, and let P be an k-plane such that*

$$(6)\qquad \text{ind}(Q_p, P') \geqq \omega' > \omega.$$

Then π' considered in M_{p,η_1}, is an imbedding in Q_p. We have

$$(7)\qquad |\pi'(q)-q| < \omega\eta/\omega' \text{ if } q\varepsilon M_{p,\eta_1}, \quad \eta \leqq \eta_1.$$

(8) $$Q_{p,c} \subset \pi'(M_{p,\eta}), \quad c=(1-\omega/\omega')\eta, \quad \eta \leqq \eta_1.$$

Proof. Suppose $\pi'(x)=\pi'(y)$ for some points x, y in M_{p,η_1} and let $v=x-y$ be a secant vector, then v is in P'. On the other hand, (3) holds and since $\omega < \omega'$, v is not in P' from (6). This is contradiction.

Therefore π' is one-one in M_{p,η_1}, and hence is an imbedding.

Let $v=q-\pi'(q)$ and $\alpha(v, Q_p)=\theta$, then sin $\theta \geqq \omega'$. And by Lemma 3 $|\pi_p(q)-q| < \omega\eta$, hence $|\pi'(q)-q| = |\pi_p(q)-q|$ coses $\theta < \frac{\omega\eta}{\omega'}$, (7) holds.

Furthermore $|\pi'(q)-q|$ cos $\theta < \frac{\omega\eta}{\omega'}\sqrt{1-\omega'^2} < \frac{\omega\eta}{\omega'}$.

Relation (8) follows from this.

Lemma 5. *There is a positive function $\delta(p)$ defined on M with the following properties. For each $p \varepsilon M$ if $P_p^* = P_p \cap U_{\delta(p)}(p)$, then P_p^* fill out a neighborhood U^* of M in an one-one way.*

Set $\pi^*(q)=p$ if $q \varepsilon p_p^* \cap U^*$. Then $\pi_p | P_p = \pi^*$ and

(9) $|\pi^*(q)-q| = |\pi_p(q)-q| = 2d(M, q)$ *for* $q \varepsilon U^*$ *and for* $\frac{1}{4} >$ cot κ_2 *where* κ_2 *is similar to Lemma* 3.

Proof. Since the transverse k-plane field $\varphi : M \longrightarrow G_{k,n}$ is a Lipschitz map by Lemma 0, first part of the statement follows from [2. Th 1.5].

Let $\omega > 0$, η_1 be as in Lemma 3. Let $\delta(p) \leqq \frac{\eta_1}{2}$ and take $\omega \leqq \frac{1}{4}$ (such ω exist from the assumption $\frac{1}{4} >$ cot κ_2). Set $u=q-p$, $|u|=a$, then $d(q, Q_p) = |\pi_p(q)-q| = |p-q| = |u| = a$. Since $2a \leqq \eta_1$, by Lemma 3, $M_{p,2a} \subset U_{2a\omega}(Q_p)$.

Let $d(q, M_{p,2a})=d(q, s)$, $d(s, Q_p)=d(s, t)$, then $d(q, s) \geqq d(q, t)-d(s, t) \geqq d(q, p) -d(s, t) \geqq \frac{a}{2}$. $d(q, M-M_{p,2a}) \geqq d(p, M-M_{p,2a})-a \geqq a$ and hence hold (9).

Lemma 6. *Take ω, δ and $\eta_1 \leqq \eta_0$ as in Lemma 3, Take any $p, p' \varepsilon M$ such that $|p-p'| < \eta_1$. Then P_p intersects Q_p in a unique point and*

(10) $|\pi_p v| \leqq \omega |v|$ *if v is in $P_{p'}$ and $|v| >$ cot δ*

Proof. Since $|p-p'| < \eta_1$, $p' \varepsilon M_{p,\eta_1}$.

Take any unit vector u in Q_p. Since v is orthogonal to Q_p, $\pi_{p'} u \cdot v = 0$. Hence (3) gives $|\pi_p v| \leqq |u \cdot v| = |(u-\pi_{p'} u) v| \leqq \omega |v|$.

If the statement about intersection were false, then there would be a unit vector u in both $P_{p'}$ and Q_p.

But then (10) would give $|u| = |\pi_p u| < |u|$, a contradiction.

4. Fullness.

Given the r-simplex $\sigma\,(r>0)$ in Euclidean space R^m (σ could be any set to which are attached a dimension r, a volume $|\sigma|$, and a diameter diam (σ)), and a defining set of vectors $v_1, \cdots, v_r$ for σ, then by [3. p 125] its *fullness* $F(\sigma)$ *and volume* $|\sigma|$ *are*

(1) $$F(\sigma)=|\sigma|/\delta_\sigma^r, \quad \delta_\sigma=\text{diam}\,(\sigma).$$

(2) $$|\sigma|=|v_1\vee\cdots\vee v_r|/r!\leqq|v_1|\cdots|v_r|/r!=\delta_\sigma^r/r!;\ \text{hence}$$

(3) $$F(\sigma)=1/r!,\quad \dim\sigma=r.$$

(4) $$r!\,F(\sigma^r)\leqq k!\,F(\sigma^k),\quad \sigma^k \text{ a face of } \sigma^r.$$

and the following propositions is due to [3. pp 125–127].

Proposition 1. *For any r-simplex $\sigma=p_0\cdots p_r$ and point $p=\mu_0 p_0+\cdots+\mu_r p_r$ in σ*

(5) $$d(p,\partial\sigma)\geqq r!\,F(\sigma)\,\delta_\sigma\,\inf\,\{\mu_0,\cdots,\mu_r\}.$$

Proposition 2. *Given $r, F_0>0$, and $\varepsilon>0$, there is a $\rho_0>0$ with the following property. Take any simplex $\sigma=p_0\cdots p_r$ with $F(\sigma)\geqq F_0$, and take any points $q_0,\cdots,q_r$, with $|q_i-p_i|\leqq\rho_0\,\delta_\sigma$ (all i).*

Then $\sigma'=q_0\cdots q_r$ is a simplex and $F(\sigma')\geqq F_0-\varepsilon$.

Proposition 3. *Given vectors $u_1,\cdots,u_r$* and *numbers $a_1,\cdots,a_r$,*

(6) $$|\Sigma a_i u_i|\geqq\sup\,\{|a_1|,\cdots,|a_r|\}\,|u_1\vee\cdots\vee u_r|\ \textit{if each}\ |u_i|=1.$$

Proposition 4. *Let $u_1,\cdots,u_r$ be independent unit vectors parallel to edges of the r-simplex σ. Then*

(7) $$|\Sigma a_i u_i|\geqq r!\,\sup\,\{|a_1|,\cdots,|a_r|\}\,F(\sigma).$$

(8) $$|a_i|\leqq|\Sigma a_j u_j|/r!\,F(\sigma),\quad i=1,\cdots,r.$$

Proposition 5. *Let π denote an orthogonal projection into a plane P. Let $\sigma=p_0\cdots p_r$ be a simplex and suppose*

(9) $$\sigma\subset U_\zeta(P),\,|p_i-p_0|\geqq\delta>0\quad(i=1,\cdots,r)$$

Then for any unit vector u in σ,

(10) $$|u-\pi u|\leqq 2\zeta/(r-1)!\,F(\sigma)\,\delta.$$

Proposition 6. *Let σ be an s-cell and let P be an n-plane in R^m, such that*

(11) $$s+n\geqq m,\quad d(P,\sigma)<d(P,\partial\sigma).$$

Then if $s+n=m$, P intersects σ in a single point, and

(12) $\operatorname{ind}(P, P(\sigma)) > d(P, \partial\sigma) / \operatorname{diam}(\sigma)$.

Proposition 7. *Let P be a plane in R^m, let Q be a plane in P, let E be a closed set in P, let p be a point of R^m not in E, and let Q^* be the join p^*E. Then*

$$(d(Q^*, Q) = d(E, Q)\, d(p, P) / \operatorname{diam}(Q^*).$$

5. The complex L.

If we take a cubical subdivision of R^m and the barycentric subdivision L of this, all simplexes of L have the same fullness. Let N be the largest number of simplexes in any star of a vertex of L.

Choose $\rho_0 < 1/4m^{1/2}$ by proposition 2 so that for any n-simplex $\sigma = p_0 \cdots p_n$, if $F(\sigma) \geqq 2F_0$, and $|q_i - p_i| \leqq \rho_0 \delta_\sigma$, then $\tau = q_0 \cdots q_n$ is a simplex, with $F(\tau) \geqq F_0$.

There is a number $\rho_1 > 0$ with the following property. Let Q be any ball in R^m, of any radius a, and let Q' be the part of Q between any two parallel $(m-1)$-planes whose distance apart is $\leqq 2\rho_1 a$. Then we have the inequality on volumes

(1) $|Q'| < |Q| / N$.

Set

(2) $\rho = \rho_0 \rho_1 / 4, \quad \alpha_r = \rho^r \rho_0 \rho_1 / 2, \quad \alpha = \alpha_{s-1} / 4. \quad s = m - n.$

(3) $\beta = F_0 \alpha / m^{1/2} N \quad F_1 = \beta^n / 2^n, \quad \gamma = (n-1)!\, F_1 \beta / 2.$

Choose $\rho_0' \leqq 1/4$ by proposition 2, using n, F_1 and $F_1 / 2$ in place of r, F_0 and ε.

Set

(4) $\omega = \inf\{\alpha\gamma / 128,\ \rho_0' \alpha\beta / 8\}$.

Say the projection π^* of Lemma 5 is defined in the neighborhood $U^* = U_{\delta_0}(M)$. We take $\omega \leqq \frac{1}{4}$ in Lemma 5.

Choose η_0 by Lemma 1, Choose $\eta_1 \leqq \eta_0$ by Lemma 2 and set

(5) $\eta = \inf\{\eta_1, \alpha\delta_0 / 4\omega\}, \quad \delta = \eta / 8, \quad h = 2\delta / m^{1/2},$

(6) $a = 2\alpha\delta, \quad b = \beta\delta, \quad c = \gamma\delta.$

Let L be a cubical subdivision of R^m, the cubes being of side length h, and let L be the barycentric subdivision of L_0. Then each 1-simplex of L is of length $\geqq h/2$, and the m-simplexes have diamenter δ.

6. The complex L*.

Let the vertices of L be $p_1, p_2, \cdots$, we shall new points $p_1^*, p_2^*, \cdots$, with

(1) $|p_i^*-p_i|<\rho_0\delta$, all i.

By the choice of ρ_0, this will define a new triangulation of R^m, and using $\rho_0\delta<h/8$ and (5.5) gives, for all simlexes τ of L^* of dimension $\geqq 1$

(2) $h/4<\operatorname{diam}(\tau)<2\delta$, $F(\tau)\geqq F_0$.

We shall obtain also

(3) $d(M^n,\tau^r)>a_r\delta$, all τ^r in L^*, $r\leqq s-1$, $s=m-n$

and hence if L^{*s-1} denote the $(s-1)$-skeleton of L^*,

(4) $d(M, L^{*s-1})>2a$.

Proof of (3). Suppose $p_i^*, \cdots, p_{i-1}^*$ have been found, so that the complex L_{i-1}^* with these vertices satisfies (3); We shall find p_i^*, so that L_i^* satisfies (3).

Case I, $d(p_i, M)\geqq 3\delta$, Then we set $p_i^*=p_i$. Because of (2), (3) will hold for L_i^*.

Case II, there is a point $p\varepsilon M$, $|p-p_i|<3\delta$. Let $\tau_1', \cdots, \tau_\nu'$ $(\nu\leqq N-1)$ be the simplexes of L_{i-1}^* of dimension $\leqq s-2$ such that $\tau_j=p_i^*\tau_j'$ will be a simplexes of L_i^*. Let Q_j be the plane spanned by τ_j' and Q_p $(j\geqq 1)$; its dimension is at most $(s-2)+n+1<m$.

Set

(5) $P_j=U_{\rho_0\delta}(p_i)\cap U_{\rho_1\rho_0\delta}(Q_j)$, $j=0, 1, \cdots, \nu$.

By the choice of $\rho_1, |P_j|<|U_{\rho_0\delta}(p_i)|/N$; hence there is a point p_i^* satisfying (1), such that

(6) $d(p_i^*, Q_j)>\rho_1\rho_0\delta$, $j=0, 1, \cdots, \nu$.

We show now that

(7) $d(\tau_j', Q_p)>2a_{r-1}\delta/3$ if $\dim(\tau_j')=r-1$.

Since τ_j' is in $L_{i-1}^*, d(\tau_j', M)>a_{r-1}\delta$.

By (3.5) $Q_{p,\eta}\subset U_{\omega\eta}(M^n)$; Since $\omega<\alpha_{s-1}/24, \omega\eta<\alpha_{r-1}\delta/3$, and (7) holds with $Q_{p,\eta}$ in place of Q_p. Since $|p-p_i|<3\delta$ and $d(p_i, \tau_j')<2\delta, d(\tau_j', Q_q-Q_{p,\eta})>3\delta$, which gives (7).

Applying proposition 7 gives

$$\begin{aligned} d(\tau_j, Q_p)&=d(\tau_j', Q_p)\,d(p_i^*, Q_j)/\operatorname{diam}(\tau_j)>(2\alpha_{r-1}\delta/3)\rho_1\rho_0\delta/2\delta \\ &=4\alpha_{r-1}\rho\delta/3=4\alpha_r\delta/3. \end{aligned}$$

Since $\omega\eta<\alpha_r\delta/3$, using (3.5) and (3.2) and the same argument as above gives (3), for $\tau^r=\tau_j, j\geqq 1$.

Using $j=0$ in (6) and the same argument again gives (3) for $\tau^r=p_i^*$; hence (3) and (4) are proved.

7. The intersection of M with L*. Let M^n be a topological n-manifold imbedded in R^m and $s=m-n$.

(a) For any point $p\varepsilon M^n$ and r-simplex σ^r of L^*,

(1) $$d(Q_p, \sigma^r) > a \text{ if } \sigma^r \subset U_{7\delta}(p),\ r \leqq s-1.$$

For $d(Q_p - Q_{p,\eta}, \sigma^r) > \eta - 7\delta > a$ and $Q_{p,\eta} \subset U_{\omega\eta}(M), \omega\eta < a$; using (6.4) gives (1).

(b) If M^n intersects $\sigma^r, p\varepsilon M^n$, and $\sigma^r \subset U_{7\delta}(p)$, then Q_p intersects σ^r. For if $p'\varepsilon M^n \cap \sigma^r$, then by (3.2), $p'\varepsilon M^n_{p,\eta}$.

By (3.5), $d(p', Q_p) < \omega\eta < a$. Let σ^t be a face of smallest dimension of σ^r with $d(\sigma^t, Q_p) \leqq a$.

By (1), $t \geqq s$, and by Proposition 6, Q_p intersect σ^t.

(c) If $r=s$ in (b), and $P(\sigma^s)$ is the plane of σ^s, then

(2) $$\text{ind } (Q_p, P(\sigma^s)) > \alpha.$$

This follows from Proposition 6, (1) and (6.2)

(d) If $p\varepsilon M, \sigma^r \subset U_{7\delta}(p)$, and Q_p intersect σ^r, then $r \geqq s$, and $M_{p,\eta}$ intersects σ^r.

Let σ^t be a smallest face of σ^r such that $d(Q_p, \sigma^t) \leqq a$. By (1) and Proposition 6, $t=s$ (hence $r \geqq s$), Q_p has a point p' in σ^s, and (2) holds. Let π' be the projection into Q_p along planes parallel to σ^s. By Lemma 4, $\pi'(M_{p,\eta})$ covers $Q_{p,\zeta}$ with $\zeta=(1-\omega/\alpha)\eta > 7\delta$.

Since $|p'-p| < 7\delta$, there is a $p^*\varepsilon M_{p,\eta}$ with $\pi'(p^*)=p'$; hence $p^*\varepsilon P(\sigma^s)$.

By (3.7) $|p'-p^*| < \omega\eta/\alpha \leqq \rho_0'\beta\delta < \beta\delta < a$.

Since $p'\varepsilon\sigma^s$, (6.4) shows that $p^*\varepsilon\sigma^s$.

(e) M^n intersects any σ^s in at most one point. For suppose M^n had the distinct points p, p' in σ^s. Then by (6.2), $p'\varepsilon M_{p,\eta}$ and $M_{p,\eta}$ has a secant vector $v=p'-p$ in σ^s. By Lemma 3, $|v-\pi_p v| \leqq \omega|v|$.

But (2) gives $|v-\pi_p v| > \alpha|v| > \omega|v|$, a contradiction.

(f) If M intersect $\sigma^r=q_0 \cdots q_r$, then for each k, M intersects some s-face of σ^r containing q_k. For take $p\varepsilon M^n \cap \sigma^r$. Let σ^t be a face of smallest dimension of σ^r containing q_k. For take $p\varepsilon M^n \cap \sigma^r$. Let σ^t be a face of smallest dimension of σ^r containing q_k which Q_p intersects. Suppose $t > s$. Then if σ^{t-1} is the face of σ^t opposite q_k, Q_p

intersects some s–face of σ^{t-1}. Because of (c) Q_p contains interior points of σ^t, and hence intersects $\partial\sigma^t-\sigma^{t-1}$, a contradiction.

Hence $t=s$. By (d), M^n also intersects σ^s.

8. The complex K.

In each simplex σ of L^* intersecting M^n we shall choose a point $\varphi(\sigma)$; these are the vertices of K. For each sequence $\sigma_0\subset\sigma_1\subset\cdots\subset\sigma_r$ of distinct simplexes of L^* such that M^n intersects σ_0 (and hence all the σ_i),

(1) $$\sigma^{*r}=\varphi(\sigma_0)\cdots\varphi(\sigma_r)$$

will be a simplex of K.

First, for each σ^s which M^n intersects, there is just one point of intersection by $(7(e))$; let $\varphi(\sigma^s)$ be this point.

Next for any $\sigma^r\,(r>s)$ which M^n intersects let $\sigma_1^s,\cdots,\sigma_k^s$ be the s–faces of σ^r intersecting M^n (see $7(f)$);

set

(2) $$\varphi(\sigma^r)=(1/k)\,\varphi(\sigma_1^s)+\cdots+(1/k)\,\varphi(\sigma_k^s).$$

We show that for any $\sigma^s=q_0\cdots q_s$ of L^* intersecting M^n,

(3) $$\mu_i>2\alpha\ (i=0,\cdots,s)\ \text{if}\ \varphi(\sigma^s)=\sum_i\mu_i\,q_i.$$

For let σ_i be the $(s-1)$–face opposite q_i. Let A_i and A_i' be the height from q_i and $\varphi(\sigma^s)$ respectively to $P(\sigma_i)$. By (6.4) and (6.2)

$$\mu_i\equiv A_i'/A_i>2a/2\delta=2\alpha$$

Next, if M^n intersects $\sigma^r=q_0\cdots q_r$, then

(4) $$\mu_i>2\alpha/N\quad(i=0,\cdots,r)\quad\text{if}\ \varphi(\sigma^r)=\sum_i\mu_i\,q_i.$$

Given K, let σ^s be an s–face of σ^r containg q_i, which intersects $M^n\,(7(f))$.

By (3), the barycentric coordinate μ' of $\varphi(\sigma^s)$ correspoding to q_i is at least 2α. By (2), μ_i is the average of at most N barycentric coordinates, one of which is μ'; hence (4) holds.

The vertices of each simplex σ^* of K have a natural order; let $h(\sigma^*)$ be the height from the last vertex (vertex in the simplex of highest dimension of L^*). We prove

(5) $$h(\sigma^*)\geqq rb.$$

For if σ^{*r} is as in (1), the $(r-1)$–face σ^{*r-1} opposite $\varphi(\sigma_r)$ lies in σ_{r-1}. If dim

$(\sigma_r)=t \geqq r$, (4.5), (4) and (6.2) give

$$h(\sigma^{*r}) \geqq d(\varphi(\sigma_r), \sigma_r) \geqq t!\,F(\sigma_r)\,\delta_{\sigma_r}(2\alpha/N) \geqq rF_0(\delta/2m)(2\alpha/N)=rb.$$

Since $|\sigma^{*r}|=h(\sigma^{*r})|\sigma^{*r-1}|/r$, we see at once, by induction, that $|\sigma^{*r}|=b^r$, and hence

(6) $$F(\sigma^{*r}) \geqq b^r/(2\delta)^r=\beta^r/2^r \geqq F_1.$$

9. Imbedding of simplexes in M.

We show that K is near M, and that any n-simplex near an n-simplex of K is imbedded in M by π^*. (π^* is defined by Lemma 5). We prove first, for any simplex σ^* of K,

(1) $$\text{if } \sigma^* \subset U_{6\delta}(p),\quad p\varepsilon M,\quad \text{then } \sigma^* \subset U_{\omega\eta}(Q_{p,\eta}).$$

Say σ^* is in the simplex σ of L^*; then $\sigma \subset U_{8\delta}(p)$.

Each vertex p_i of σ^* is an average of points $\varphi(\sigma_j^s)$, which are in $M_{p,\eta}$ and hence in $U_{2\eta}(Q_{p,\eta})$ by (3.5); therefore (1) holds for the p_i and hence for σ^*. As a consequence of this and (3,9),

(2) $$K \subset U_{2\omega\eta}(M),\quad |\pi^*(q)-q| < 4\omega\eta \quad (q\varepsilon K).$$

Lemma 7. *Let $\sigma=p_0\cdots p_n$ be an n-simplex of K (vertices in increasing order), and let $p_0', \cdots, p_n'$ be any points such that*

(3) $$|p_i'-p_i| \leqq \omega\eta/\alpha, \quad i=0, \cdots, n.$$

Then $\sigma'=p_0'\cdots p_n'$ is a simplex in $U^*=U_{\delta_0}(M)$, and π^* imbeds σ' in M.

Proof. First, since $\omega\eta/\alpha \leqq \rho_0'\beta\eta/8=\rho_0'b$, $F(\sigma) \geqq F_1$ and diam $(\sigma) \geqq b$, by (8.6), (8.5) the choice of ρ_0' give $F(\sigma') \geqq F_1/2$.

Next, because of (2) and (3), $\sigma' \subset U_\zeta(M)$, with

(4) $$\zeta=2\omega\eta/\alpha+2\omega\eta < 4\omega\eta/\alpha \leqq \delta_0;$$

hence $\sigma' \subset U_{\delta_0}(M)$, and π^* is defined in σ'

Now take any $q\varepsilon\sigma'$. Say $q\varepsilon P_p, p\varepsilon M$; then $\pi^*(q)=\pi_p(q)=p$. By (3.9) and part of (4), $|q-p| \leqq 2(4\omega\eta/\alpha) < \delta$;

hence $\sigma \subset U_{4\delta}(p)$, and (1) gives $\sigma \subset U_{\omega\eta}(Q_p)$.

Therefore $\sigma' \subset U_{3\omega\eta/\alpha}(Q_p)$. Also (8.5), $|p_i-p_0| \geqq b$; hence $|p_i'-p_0'| \geqq b-3\omega\eta/\alpha \geqq b-3\rho_0'b \geqq b/2$.

Hence, if u is any unit vector in σ', Proposition 5 gives $|u|-|\pi_p u| \leqq |u-\pi_p u| \leqq 2(3\omega\eta/\alpha)/(n-1)!(F_1/2)(b/2)=96\omega/\alpha\gamma \leqq 3/4$.

Hence $|\pi_p u| \geqq 1/4$ and u is not in P_p^* by Lemma 6.

Therefore π^* maps each non zero vector in σ' at q into a non-zero vector, and π^* in non-degenerate at q. Also, if q' is another point of σ', then using $v=q'-q$ shows that q' is not in P_p^*, and hence $\pi_p(q') \neq \pi_p(q)$.

This complete the proof.

10. The complex $\mathbf{K}_p$.

For each $p \varepsilon M$, let L_p^* be the subcomplex of L^* containing all simplexes which touch $\bar{U}_{4\delta}(p)$, together with their faces; then

(1) $\qquad L_p^* \subset U_{6\delta}(p)$.

Let K_p'' be the complex in Q_p formed by the intersections of Q_p with the simplexes of L_p^*, and let K_p' be the barycentric subdivision of K_p''. By (b) and (d) of § 7, Q_p intersects a simplex of L_p^* if and only if M does. Hence, if K_p is the subcomplex of K containing those simplexes with vertices $\varphi(\sigma)$, σ in L_p^*, there is a one-one correspondence g_p of the vertices of K_p onto K_p', and this defines a simplicial mapping g_p which is an isomorphism of K_p onto K_p'.

We prove

(2) $\qquad |g_p(q)-q| < \omega\eta/\alpha, \quad q \varepsilon K_p$.

First suppose $q=\varphi(\sigma^s)$ for some σ^s in L_p^*. Then $v=q-g_p(q)$ is in σ_s, and using (3.4) and (7 (c)) gives

$$\omega\eta > |q-\pi_p(q)| = |v-\pi_p v| \geqq \alpha |v|,$$

giving (2).

Next, if $q=\varphi(\sigma^r)$, $r > s$, then the definition (8.2) and linearity of g_p show that q and $g_p(q)$ are the same average of sets of points, each corresponding pair satisfying (2); hence (2) holds for $q=\varphi(\sigma^r)$. Finally, for any simplex of K_p, (2) holds for its vertices and hence for all its points.

We shall show that

(3) $\qquad K \cap U_{2\delta}(p) \subset K_p$.

For take any point q in a simplex $\tau=\varphi(\sigma_0)\cdots\varphi(\sigma_r)$ of K, $|q-p| < 2\delta$. Then $\sigma_r \subset U_{4\delta}(p)$, hence σ_r is in L_p^*, and τ and q are in K_p. Choose an orientation of Q_p, and orient all n-simplexes of K_p' accordingly. Now K_p' is an oriented n-dimensional pseudo-manifold and (1) and the definition of L_p^* show that

(4) $\qquad K_p' \subset U_{6\delta}(p), \partial K_p' \subset Q_p - \bar{U}_{4\delta}(p)$.

Define the mapping π_p^* of K_p into Q_p as follows.

Each $q\varepsilon K_p$ is in a unique $P_{p'}^*$; then $p'=\pi_{p'}(q)$.

By (1), $|q-p| \leqq 6\delta$, and by (9.2), $|p'-q| < 4\omega\eta < \delta$; hence $|p'-p| < \eta$.

By Lemma 6, $P_{p'}$ intersect Q_p in a unique point, which we call $\pi_p^*(q)$. We prove

(5) $$|\pi_p^*(q)-q| < 6\omega\eta, \quad q\varepsilon K_p.$$

Because of (9.2), we need merely prove $|v| < 2\omega\eta$, where $v=p'-\pi_p^*(q)$.

Since $\omega < 1/2$, (3.10) gives $|\pi_p v| \leqq |v|/2$. By (3.4), $|v-\pi_p v| < \omega\eta$.

Therefore $|v| < \omega\eta + |v|/2$, and the statement follows.

11. Proof of the theorem.

Given $p\varepsilon M$, choose an orientation of Q_p, and orient the n-simplexes of K_p' and K_p correspondingly.

Now K_p is an oriented pseudo n-manifold with boundary. The proof of Theorem results on the following lemmas.

Lemma 8. *π_p^* is a simplexwise positive mapping of K_p into Q_p.*

Proof. Take any n-simplex σ of K_i. Set

(1) $$g_{p,t}(q)=(1-t)\,q+tg_p(q) \text{ in } \sigma, \sigma_t=g_{p,t}(\sigma).$$

Since g_p is affine in σ, so is $g_{p,t}$, and σ_t is a simplex.

Say $\sigma=q_0\cdots q_n$. For any $t\,(0 \leqq t \leqq 1)$, set $q_{it}=g_{p,t}(q_i)$; then $\sigma_t=q_{0t}\cdots q_{nt}$.

By (10.2), $|q_{it}-q_i| < \omega\eta/\alpha$. By Lemma 7, π^* imbeds σ_t in M; hence (by the reasoning of that lemma) π_p^* imbeds σ_t in σ_p.

Since σ_1 is in Q_p, π_p^* is the identity in σ_1, and hence orientation preserving in σ_1. Since π_p^* is non-degenerate for all t, π_p^* is orientation preserving as required.

Lemma 9. *Let K be a pseudo n-manifold and $f: K \longrightarrow R^n$ be simplexwise positive in K. Then for any connected open subset R of $f(K)-f(\partial K)$, any two points of R not in $f(K^{n-1})$ are covered the same number of times. If this number is* 1, *then f, considered in the open subset $R'=f^{-1}(R)$ of K only, is one-one onto R.*

Proof. Since f is simplexwise positive in K, $f(\text{int}\,\sigma_i^n)$ is open for each σ_i^n. We show first that for any σ^{n-1} not in ∂K, $f(St(\sigma^{n-1}))$ is open. Let σ_1^n and σ_2^n be the n-simplexes of K with σ^{n-1} as face.

Take any $p\varepsilon$ int (σ^{n-1}); we need merely show that $f(St(\sigma^{n-1}))$ covers a neighborhood of $f(p)$. Since f is one-one in each σ_i^n, there is a neighborhood U of $f(p)$ not touching $f(\partial St(\sigma^{n-1}))$. Since f is linear in σ^{n-1}, we may choose U so that $f(\sigma^{n-1})$ cuts it into two connected parts, U_1 and U_2.

Let $\overline{pp_i}$ be a segment in σ_i^n, $p_i \varepsilon$ int (σ_i^n), mapping into an arc A_i in $U(i=1,2)$. If we orient σ^{n-1}, it is in $\partial\sigma_i^n$ $i=1,2$ with opposite signs; since f is orientation preserving each σ_i^n, we may suppose $f(p_1)\varepsilon U_1, f(p_2)\varepsilon U_2$. Now suppose there were a point $q\varepsilon U-f(\sigma^{n-1})$ not in $f(St(\sigma^{n-1}))$. There is an arc A in $U-f(\sigma^{n-1})$ joining q to either $f(p_1)$ or $f(p_2)$. There is a first point q^* in A which is in $f(St(\sigma^{n-1}))$; by the choice of U and A, $q^*\varepsilon U_j'=f(\text{int}\,\sigma_j^n)$ for $j=1$ or 2. But U_j' is open, contradicting the definition of q^*, and the statement is proved.

Suppose the first conclusion of the lemma were false. Then we may express $R-f(K^{n-1})$ as the union of two disjoint sets R_1 and R_2, such that for some h each point of R_1 is covered h times and each point of R_2 is covered a different number of times. We may choose an arc A from a point of R_1 to a point of R_2, lying in $R-f(K^{n-2})$, which crosses from R_1 to R_2 at a point q; then $q\varepsilon f(\sigma^{n-1})$ for some σ^{n-1}. Let $\sigma_1^{n-1},\cdots,\sigma_k^{n-1}$ be the $(n-1)$-simplexes of K whose images contain q; say σ_i^{n-1} is the face of the n-simplexes σ_i, σ_i'.

Since f is one-one in the n-simplexes, we see at once that these n-simplexes are distinct.

By the proof above, there is a neighborhood U of q such that for each i, each point of $U-f(K^{n-1})$ is in just one of $f(\sigma_i), f(\sigma_i')$. We may suppose U touches no $f(\sigma_j^{n-1})$ for any other j; then any other $f(\sigma_1^n)$ containing q contains U. Hence all points of $U-f(K^{n-1})$ are covered the same number of times, contradicting the choice of q.

Next we show that for any simplex σ^k of K, $f(St(\sigma^k))$ is open.

Given $p\ \varepsilon$ int (σ^k), we must show that $f(St(\sigma^k))$ covers a neighborhood U of $q=f(p)$. We may suppose U is connected and does not touch $f(\partial\, St(\sigma^k))$. Now $L=\overline{St}(\sigma^k)$ is an oriented n-ball, and the proof above shows that all points of U not in $f(L^{n-1})$ are covered the same number N of times by n-simplexes of L. Since some points near q are covered, $N\geqq 1$; hence all points of U are in $f(L)$.

We now prove the last part of the lemma. Since the number of times points are covered is 1, f maps R' onto R. Now suppose $f(p_1)=f(p_2)=q, p_1\neq p_2$.

Say $p_i\varepsilon$ int (σ_i) (dimension of σ_i unspecified).

Since f is one-one in all simplexes, $St(\sigma_1)\cap St(\sigma_2)=\phi$.

By the proof above, $f(St(\sigma_i))$ covers all points of some neighborhood U_i of q

not in $f(K^{n-1})$ a number of times $N_i>0$, for $i=1,2$.

But this shows that f, in K, covers points of $U=U_1\cap U_2$ at least twice, a contradiction, completing the proof of the lemma.

For each $p\varepsilon M$, let R_p be the set of those points $q\varepsilon K_p$ such that $\pi_p^*(q)\,\varepsilon Q_{p,3\delta}$.

Lemma 10. *For each* $p\varepsilon M, \pi_p^*$, *considered in* R_p *only, is one-one and onto* $Q_{p,3\delta}$.

Proof. First, let σ_1 be an n-simplex of K'_p containing p; say $\sigma_1=g_p(\sigma_0)$ (σ_0 in K_p), and let p_0 be the barycenter of σ_0.

By (4.4), (8.6), (8.5), (5.3), (5.6),

$$d(p_0,\partial\sigma_0)\geqq n\,!\,F_1\,b\,/\,(n+1)\geqq c.$$

Hence, by (10.2)

$$(2)\qquad d(g_p(p_0),\partial\sigma_1)>c-2\omega\eta/\alpha=c'.$$

Now take any $q\varepsilon K_p-\sigma_0$. Since g_p is an isomorphism, (2) shows that $|g_p(q)-g_p(p_0)|>c'$. By (5.4), (5.5) and (5.6)

$$4\omega\eta/\alpha+12\omega\eta<16\,\omega\eta/\alpha\leqq\gamma\delta=c.$$

Hence, by (10.2) and (10.5)

$$|\pi_p^*(q)-\pi_p^*(p_0)|>c'-2(\omega\eta/\alpha+6\,\omega\eta)>0,$$

proving $\pi_p^*(q)\neq\pi_p^*(p_0)$. This shows that $p^*=\pi_p^*(p_0)$ is covered exactly once, under π_p^*, by simplexes of K_p.

Also

$$|p^*-p|\leqq|\pi_p^*(p_0)-g_p(p_0)|+\operatorname{diam}(\sigma_1)<3\delta,\text{ and hence } p^*\varepsilon Q_{p,3\delta}.$$

By (10.4), (10.2) and (10.5), since $2(\omega\eta/\alpha+6\omega\eta)<\delta$,

$$(3)\qquad \pi_p^*(\partial K_p)\subset Q_p-\bar{U}_{3\delta}(p).$$

The lemma now follows from Lemma 8 and Lemma 9.

Proof of the theorem. First, given $p\varepsilon M$, the last lemma shows that $\pi_p^*(q)=p$ for some $q\varepsilon K_p$; hence $\pi^*(q)=p$, and π^* is onto.

Next suppose that $\pi^*(q')=p$ also, $q'\varepsilon K$. By (9.2), $|q'-p|<4\omega\eta<\delta$; hence, by (10.3), $q'\varepsilon K_p$, and since $|\pi_q^*(q')-p|\leqq|\pi_p^*(q')-q'|+|q'-p|\leqq 6\omega\eta+4\omega\eta<2\delta$ using (10.5), hence $q'\varepsilon R_p$. By lemma 10, $q'=q$. This proves that π^* is one-one and hence $\pi^*:|K|\longrightarrow M$ is a homeomorphism.

Since L_0 is a cubical subdivision of R^{n+k}, L is a combinatorial triangulation of

R^{n+k}. L^* is isomorphic to L by (6.1) and (6.2).

And K is a subcomplex of some subdivision of L^*. So K is a combinatorial triangulation of M^n.

Next if $n+k \leqq 4, M^n$ has a unique combinatorial triangulation by [5] because $k \geqq 1$. If $n+k \geqq 5, R^{n+k}$ has a unique combinatorial triangulation by [4]. Hence M^n has a unique combinatorial triangulation because K is induced by the triangulation of R^{n+k}.

This completes the proof.

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