ON RECURRENT SPACES OF SECOND ORDER IN FINSLER SPACES

By

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(Received Feb. 20, 1970)

Summary. *Chaki and Roychowdhary* have studied the Ricci recurrent spaces of second order in the Riemannian Geometry [1] [1]. The object of present paper is to define the recurrent curvature tensor fields of second order and to study the properties of recurrence tensor field and the curvature tensor fields in the Finsler spaces.

1. Introduction. We consider an n-dimensional Finsler space F_n in which the relative curvature tensor field is defined as

$$\tilde{K}^i_{jkh}=\left(\frac{\partial\Gamma^{*i}_{jk}}{\partial x^h}+\frac{\partial\Gamma^{*i}_{jk}}{\partial\dot{x}^l}\frac{\partial\xi^l}{\partial x^h}\right)-\left(\frac{\partial\Gamma^{*i}_{jh}}{\partial x^k}+\frac{\partial\Gamma^{*i}_{jh}}{\partial\dot{x}^l}\frac{\partial\xi^l}{\partial x^k}\right)+\Gamma^{*i}_{mh}\Gamma^{*m}_{jk}-\Gamma^{*i}_{mk}\Gamma^{*m}_{jh} \tag{1.1}$$

where Γ^{*i}_{jk} is connection parameter.

The relative curvature tensor field satisfies the identities

$$\tilde{K}^i_{jkh}+\tilde{K}^i_{khj}+\tilde{K}^i_{hjk}=0 \tag{1.2}$$

and

$$\tilde{K}^i_{jkh;l}+\tilde{K}^i_{jhl;k}+\tilde{K}^i_{jlk;h}=0. \tag{1.3}$$

The commutation formulae involving the relative curvature tensor field are as follows:

$$T_{ij;kh}-T_{ij;hk}=-T_{ir}\tilde{K}^r_{jkh}-T_{rj}\tilde{K}^r_{ikh} \tag{1.4}$$

and

$$T^i_{j;kh}-T^i_{j;hk}=T^r_j\tilde{K}^i_{rkh}-T^i_r\tilde{K}^r_{jkh}. \tag{1.5}$$

The relative curvature tensor field $\tilde{K}^i_{jkh}$ is said to be recurrent relative curvature tensor field of first order, if it satisfies the condition

$$\tilde{K}^i_{jkh;m}=v_m\tilde{K}^i_{jkh} \tag{1.6}$$

where v_m is a recurrence vector field [3].

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1) The numbers in the brackets refer to the references at the end of the paper.

Cartan regards the line-element ξ^i as the element of support and define the curvature tensor field as follows

$$K^i_{jkh}=\left(\frac{\partial\Gamma^{*i}_{jk}}{\partial x^h}-\frac{\partial\Gamma^{*i}_{jk}}{\partial\dot{x}^l}\frac{\partial G^l}{\partial\dot{x}^h}\right)-\left(\frac{\partial\Gamma^{*i}_{jh}}{\partial x^k}-\frac{\partial\Gamma^{*i}_{jh}}{\partial\dot{x}^l}\frac{\partial G^l}{\partial\dot{x}^k}\right) \tag{1.7}$$
$$+\Gamma^{*i}_{mh}\Gamma^{*m}_{jk}-\Gamma^{*i}_{mk}\Gamma^{*m}_{jh}.$$

The identities satisfied by the curvature tensor field K^i_{jkk} are

$$K^i_{jkh}+K^i_{khj}+K^i_{hjk}=0 \tag{1.8}$$

and

$$(K^i_{jkh|l}+K^i_{jhl|k}+K^i_{jlk|h})\,l^j+(A^i_{km|s}K^m_{jhl}+A^i_{hm|s}K^m_{jlk}+A^i_{lm|s}K^m_{jkh})\,l^j l^s=0, \tag{1.9}$$

where $A^i_{jk}=F\,C^i_{jk}$ (symmetric tensor) and l^j is a unit tensor field.

The commutation formulae involving the curvature tensor field K^i_{jkh} are given by

$$T_{|kh}-T_{|hk}=-\frac{\partial T}{\partial\dot{x}^l}K^l_{rkh}\dot{x}^r, \tag{1.10}$$

$$T^i_{j|kh}-T^i_{j|hk}=-\frac{\partial T^i_j}{\partial\dot{x}^l}K^l_{rkh}\dot{x}^r+T^r_j K^i_{rkh}-T^i_r K^r_{jkh} \tag{1.11}$$

and

$$\left(\frac{\partial T^i_j}{\partial\dot{x}^k}\right)_{|h}-\frac{\partial T^i_{j|h}}{\partial\dot{x}^k}=\frac{\partial T^i_j}{\partial\dot{x}^l}C^l_{kh|r}\dot{x}^r-T^r_j\frac{\partial\Gamma^{*i}_{rh}}{\partial\dot{x}^k}+T^i_r\frac{\partial\Gamma^{*r}_{jh}}{\partial\dot{x}^k}. \tag{1.12}$$

The curvature tensor field K^i_{jkh} is said to be recurrent of first order if it satisfies the relation

$$K^i_{jkh|l}=v_l K^i_{jkh}, \tag{1.13}$$

where v_l is recurrence vector field [3].

Contracting (1.13) with respect to the indices i and h, we have

$$K_{jk|l}=v_l K_{jk}. \tag{1.14}$$

2. Recurrent relative curvature tensor field of second order.

Definition 2.1. A n-dimensional Finsler space F_n, in which the relative curvature tensor field satisfies the relation

$$\tilde{K}^i_{jkh;lm}=a_{lm}\tilde{K}^i_{jkh}, \tag{2.1}$$

where

$$\tilde{K}^i_{jkh}\neq 0 \tag{2.2}$$

is said to be a recurrent Finsler space of second order and a_{lm} is a recurrence tensor field. The relative curvature tensor field of this space is defined as the recurrent relative curvature tensor field of second order.

Theorem 2.1. *The recurrence tensor field a_{lm} is non-symmetric.*

Proof. Commutating (2.1) with respect to the indices l and m we have

$$\tilde{K}^i_{jkh;lm} - \tilde{K}^i_{jkh;ml} = (a_{lm} - a_{m'})\, \tilde{K}^i_{jkh}. \tag{2.3}$$

From the commutation formula (1.5), it gives

$$\tilde{K}^i_{rlm}\, \tilde{K}^r_{jkh} - \tilde{K}^i_{rkh}\, \tilde{K}^r_{jlm} - \tilde{K}^i_{jrh}\, \tilde{K}^r_{klm} - \tilde{K}^i_{jkr}\, \tilde{K}^r_{hlm} = (a_{lm} - a_{ml})\, \tilde{K}^i_{jkh}, \tag{2.4}$$

which proves the result.

Theorem 2.2. *Every recurrent Finsler space for which the recurrence vector field v_m satisfies*

$$v_{m;l} + v_m\, v_l \not\equiv 0 \tag{2.5}$$

is a recurrent Finsler space of second order but the converse is not true in general.

Proof. The covariant differentiation of (1.6) yields

$$\tilde{K}^i_{jkh;ml} = (v_{m;l} + v_m\, v_l)\, \tilde{K}^i_{jkh}. \tag{2.6}$$

From (2.1) and (2.2), we have

$$a_{ml} = (v_{m;l} + v_m\, v_l), \tag{2.7}$$

which proves the statement.

From hereafter we shall consider such a recurrent Finsler space of second order and denote it by $\bar{F}_n$.

Theorem 2.3. *In $\bar{F}_n$, the recurrence tensor field a_{lm} satisfies the relation*

$$\begin{aligned}(a_{lm} - a_{ml})_{;n} + (a_{mn} - a_{nm})_{;l} + (a_{nl} - a_{ln})_{;m} \\ = (a_{lm} - a_{ml})\, v_n + (a_{mn} - a_{nm})\, v_l + (a_{nl} - a_{ln})\, v_m.\end{aligned} \tag{2.8}$$

Proof. The covariant differentiation of (2.4), with respect to (1.6) and (2.4), yields

$$(a_{lm} - a_{ml})_{;n}\, \tilde{K}^i_{jkh} = (a_{lm} - a_{ml})\, v_n\, \tilde{K}^i_{jkh}. \tag{2.9}$$

From (2.2), it becomes

$$(a_{lm} - a_{ml})_{;n} = (a_{lm} - a_{m\cdot})\, v_n. \tag{2.10}$$

Adding the expressions obtained by cyclic change of (2.10) with respect to the indices l, m and n, we have Theorem 2.3.

Theorem 2.4. *In $\bar{F}_n$, we have*

$$(a_{mr}-a_{rm})\tilde{K}^{r}_{lns}+(a_{nr}-a_{rn})\tilde{K}^{r}_{lsm}+(a_{sr}-a_{rs})\tilde{K}^{lmn}_{r}$$
$$=(a_{lm}-a_{ml})(v_{n;s}-v_{s;n})+(a_{ln}-a_{nl})(v_{s;m}-v_{m;s})$$
$$+(a_{ls}-a_{sl})(v_{m;n}-v_{n;m}). \tag{2.11}$$

Proof. The covariant differentiation of (2.10), yields

$$(a_{lm}-a_{ml})_{;ns}=(a_{lm}-a_{ml})\,v_{n;s}+(a_{lm}-a_{ml})\,v_nv_s. \tag{2.12}$$

Subtracting the result obtained by interchanging the indices n and s in (2.12), we get

$$(a_{lm}-a_{ml})_{;ns}-(a_{lm}-a_{ml})_{;sn}=(a_{lm}-a_{ml})(v_{n;s}-v_{s;n}). \tag{2.13}$$

From the commutation formula (1.5), it becomes

$$(a_{rl}-a_{lr})\tilde{K}^{r}_{mns}+(a_{mr}-a_{rm})\tilde{K}^{r}_{lns}=(a_{lm}-a_{ml})(v_{n;s}-v_{s;n}). \tag{2.14}$$

Adding the expressions obtained by cyclic change of the indices m, n and s in (2.14) and using (1.2), it establishes the result.

Theorem 2.5 *In $\bar{F}_n$, the Bianchi identity satisfied by the relative curvature tensor field, takes the form*

$$a_{lm}\tilde{K}^{i}_{jkh}+a_{km}\tilde{K}^{i}_{jhl}+a_{hm}\tilde{K}^{i}_{jkl}=0 \tag{2.15}$$

Proof. By covariant differentiation of (1.3), the result follows from (2.1).

3. Recurrent curvature tensor field of second order in the sense of Cartan.

Definition 3.1. A n-dimensional Finsler space F_n in which the curvature tensor field K^{i}_{jkh} satisfies the condition

$$K^{i}_{jkh|lm}=a_{lm}K^{i}_{jkh} \tag{3.1}$$

where

$$K^{i}_{jkh}\neq 0 \tag{3.2}$$

is said to to be recurrent Finsler space of second order and a_{lm} is recurrence tensor field. This curvature tensor field under above condition is known as recurrent curvature tensor field of second order.

Contracting (3.1) with respect to the indices i and h, we have

$$K_{jk|lm}=a_{lm}K_{jk} \tag{3.3}$$

We state below the theorems which are true for the recurrent curvature tensor field K^{i}_{jkh} of second order and the recurrence tensor field a_{lm}.

Theorem 3.1. *The recurrence tensor field a_{lm} is non-symmetric.*

Theorem 3.2. *Every recurrent Finsler space for which the recurrence vector field v_m satisfies.*

$$v_{m|l}+v_m v_l \not\equiv 0 \tag{3.4}$$

is a recurrent Finsler space of second order but the converse is not true in general.

We shall denote such a Finsler space by $\bar{F}_n$.

Theorem 3.3. *In $\bar{F}_n$, if the curvature tensor field K^i_{jkh} is independent of $\dot{x}^i$, the recurrence tensor field a_{lm} satisfies the relation.*

$$\begin{aligned}(a_{lm}-a_{ml})_{|n}+(a_{mn}-a_{nm})_{|l}+(a_{nl}-a_{ln})_{|m}\\ =(a_{lm}-a_{ml})v_n+(a_{mn}-a_{nm})\,v_l+(a_{nl}-a_{ln})\,v_m.\end{aligned} \tag{3.5}$$

Theorem 3.4. *In $\bar{F}_n$, if the curvature tensor field K^i_{jkh} is independent of $\dot{x}^i$, the relation*

$$\begin{aligned}&\left[\frac{\partial a_{lm}}{\partial \dot{x}^r}-\frac{\partial a_{ml}}{\partial \dot{x}^r}\right]K^r_{tns}\,\dot{x}^t+\left[\frac{\partial a_{ln}}{\partial \dot{x}^r}-\frac{\partial a_{nl}}{\partial \dot{x}^r}\right]K^r_{tsm}\,\dot{x}^t\\ &+\left[\frac{\partial a_{ls}}{\partial \dot{x}^r}-\frac{\partial a_{sl}}{\partial \dot{x}^r}\right]K^r_{tmn}\,\dot{x}^t+K^r_{lns}\,(a_{mr}-a_{rm})+\\ &K^r_{lsm}\,(a_{nr}-a_{rn})+K^r_{lmn}\,(a_{sr}-a_{rs})=(a_{lm}-a_{ml})\\ &(v_{n|s}-v_{s|n})+(a_{ln}-a_{nl})\,(v_{s|m}-v_{m|s})+(a_{ls}-a_{sl})\,(v_{m|n}-v_{n|m})\end{aligned} \tag{3.6}$$

is true.

The above theorems can be proved on the lines of Theorems 1, 2, 3 and 4 of article 2.

Theorem 3.5. *In $\bar{F}_n$ the Bianchi identity satisfied by the curvature tensor field K^i_{jkh}, takes the form.*

$$\begin{aligned}&[(a_{ln}-v_l\,v_n)\,K^i_{jkh}+(a_{kn}-v_k\,v_n)\,K^i_{jhl}+(a_{hn}-v_h\,v_n)\,K^i_{jlh}]\;l_j\\ &+[A^i_{km|sn}\,K^m_{jhl}+A^i_{hm|sn}\,K^m_{jlk}+A^i_{lm|sn}\,K^m_{jkh}]\;l^j\,l^s=0\end{aligned} \tag{3.7}$$

Proof. Differentiating (1.9) covariantly and using (1.13) and (1.9), we have

$$\begin{aligned}&(K^i_{jkh|ln}+K^i_{jhl|kn}+K^i_{jlk|hn})\,l^j-(v_l\,K^i_{jkh}+v_k\,K^i_{jhl}+v_h\,K^i_{jlk})\,l^j\,v_n\\ &+(A^i_{km|sn}\,K^m_{jhl}+A^i_{hm|sn}\,K^m_{jlk}+A^i_{lm|sn}\,K^m_{jkh})\,l^j\,l^s=0.\end{aligned} \tag{3.8}$$

By virtue of (3.1), it establishes Theorem 3.5.

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