A QUEUE WITH PARTICULAR BULK SERVICE HAVING DEPENDENT SERVICE TIMES

By

K. Alagaraja

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§ 1. Here a queuing process is considered where each customer is served by servers simultaneously, the service distributions not being independent. Earlier the same queuing process is considered by *Fujisawa* [2] with 'n' independent phases. As dependent phases seem to be more realistic the case of dependent phases is considered in this paper. Exact results for two phases are given and for more phases the method is pointed out.

The queuing process considered here is as follows. There is only one counter; inter arrival times have the negative exponential distribution and independent of service time. Only one customer is served at a time, queue discipline being first come first served and a customer is served in dependent phases, the service time distribution in each phase being negative exponential; infinite queue is allowed outside the counter.

§ 2. Let $g(x)$ be the p. d. f. of the service time distribution in each phase. Let the random variables $(x_1 \cdots x_n)$ describing the service times in n phases have a constant correlation ρ ; $\rho<1$. Then the joint p. d. f. $g(x_1 \cdots x_n)$ takes a form whose characteristic function [3].

$$\Phi(t_1, t_2 \cdots t_n)=\begin{vmatrix} 1-it_1\theta & -it_1\theta\rho & \cdots & it_1\theta\rho \\ -it_2\theta\rho & 1-it_2\theta & \cdots & it_2\theta\rho \\ \cdots & \cdots & & \cdots \\ -it_n\theta\rho & \cdots & & 1-it_n\theta \end{vmatrix}$$

where θ is the parameter of the service time distribution of a phase. Now (see appendix)

$$\text{1.} \qquad g(x_1, x_2)=\frac{R}{\theta^2(1-\rho^2)}\, exp\left\{-\frac{(x_1+x_2)}{\theta(1-\rho^2)}\right\}$$

where

$$\text{2.} \qquad R=\sum_{n=0}^{\infty}\left[\frac{x_1 x_2 \rho^2}{\theta^2(1-\rho^2)^2}\right]^n \frac{1}{(n\,!)^2}$$

Then the service time distribution for our queuing process $F(x)$ (say) is the distribution function of the maximum of (x_1, x_2).

3. $$F(x)=\int_0^x\int_0^x g(x_1, x_2)\,dx_1\,dx_2$$

4. $$=(1-\rho^2)\sum_{n=0}^{\infty}\frac{\rho^{2n}}{(n\,!)^2}\gamma^2\left(n+1,\frac{x}{\theta(1-\rho^2)}\right)$$

where

5. $$\gamma(n, x)=\int_0^x e^{-y}\,y^{n-1}\,dy$$

Thus $F(x)$ can be expressed as a mixture where the mixing distribution is given by the p. d. f.

6. $$p(n)=\rho^{2n}(1-\rho^2) \qquad n=0, 1, \cdots\cdots\cdots$$

and the kernel is

7. $$G(x, n)=\frac{1}{(n\,!)^2}\gamma^2\left(n+1,\frac{x}{\theta(1-\rho^2)}\right)$$

Similarly it can be seen that

$$g(x_1, x_2, x_3)=\frac{1}{\theta^3 b}\,exp\left\{-\frac{a}{\theta b}(x_1+x_2+x_3)\right\}\times$$

$$\sum_{n=0}^{\infty}\left(\frac{\rho^2 a^2 x_1 x_2}{b^2\theta^2}\right)^n\frac{1}{(n\,!)^2}\sum_{r=0}^{2n}(-1)^r\,{}_{2n}C_r\left(\frac{a^2-b}{a}\right)^r\times$$

$$\sum_{m=0}^{\infty}\left(\frac{x_3}{\theta b}\right)^{r+m}\frac{1}{(r+m)\,!}\left[\frac{(a^2-b)(x_1+x_2)}{\theta b}\right]^m$$

where $\qquad a=1-\rho^2$ and $b=1-3\rho^2+2\rho^3$

and the distribution function of the maximum is given by

$$F(x)=\frac{1}{\theta^3 b}\sum_{n=0}^{\infty}\left(\frac{\rho a}{\theta b}\right)^{2n}\frac{1}{(n\,!)^2}\sum_{r=0}^{2n}(-1)^r\,{}_{2n}C_r\left(\frac{a^2-b}{a}\right)^r\times$$

$$\sum_{m=0}^{\infty}(-1)^m\frac{1}{(\theta b)^{r+2m}}\frac{(a^2-b)^n}{(r+m)\,!\;m\,!}\left(\frac{\theta b}{a}\right)^{r+m+1}\times$$

$$\gamma\,[r+m+1,\lambda x]\left(\frac{d}{dx}\right)^m\left\{\frac{1}{\lambda^{2n+2}}\gamma^2(n+1,\lambda x)\right\}$$

where $\lambda=a/\theta b$. Thus this can be extended to the case of n phases.

In particular when $\rho=0$ the service time distributions in these phases become independent.

§ 3. Let

$$8. \qquad F(s)=\int_0^\infty e^{-sx}\,dF(x)\,; \qquad s\geqslant 0$$

Then for two phases

$$9. \qquad F(s)=2(1-\rho^2)\sum_{n=0}^{\infty}\frac{\rho^{2n}}{(n\,!)^2}\frac{(2n+1)\,!}{(n+1)}\Big/[2+s\theta(1-\rho^2)]^{2n+2}\times$$

$$ {}_2F_1\,[1,2(n+1),n+2,1/2+s\theta(1-\rho^2)]$$

where [4]

$$10. \qquad {}_2F_1(a,b,c,.\,z)=1+\frac{a\cdot b}{1\cdot c}\,z+\frac{a(a+1)\,b(b+1)}{1\cdot 2\cdot c\cdot\ (c+1)}\,z^2+\cdots\cdots$$

Now putting $\alpha=\theta(1-\rho^2)$ and rearranging we have

$$11. \qquad F(s)=2(1-\rho^2)\sum_{n=0}^{\infty}\frac{\rho^{2n}}{(2+\alpha s)^{n+1}}\Big[{}_{2n+1}C_{n+1}\frac{1}{(2+\alpha s)^{n+1}}+$$

$${}_{2n+2}C_{n+2}\frac{1}{(2+\alpha s)^{n+2}}+\cdots+{}_{2n+r}C_{n+r}\frac{1}{(2+\alpha s)^{n+r}}+\cdots\cdots\Big]$$

$$=2(1-\rho^2)\sum_{n=0}^{\infty}\frac{\rho^{2n}}{(2+\alpha s)^{n+1}}\sum_{r=1}^{\infty}{}_{2n+r}C_{n+r}\frac{1}{(2+\alpha s)^{n+r}}$$

Let λ and α_1 be the means of the inter arrival time distribution and the service time distribution. Let $C=1-\lambda\alpha_1$. If $W(s)$ is the Laplace transform of the waiting time distribution $W(x)$ then [5]

$$12. \qquad W(s)=C/\{1-\lambda\,[(1-F(s))/s]\}$$

and if $B(s)$ is the Laplace transform of the distribution of the busy period $B(x)$ then

$$13. \qquad B(s)=F\,[s+\lambda-\lambda B(s)] \qquad \text{for real } s\geqslant 0.$$

Equations (12) and (13) give unique solutions and $W(x)$ and $B(x)$ are proper distribution functions if $\lambda\alpha_1<1$ and the solution for $B(s)$ is given by

$$14. \qquad B(s)=\sum_{n=1}^{\infty}\frac{-\lambda^{n-1}}{n\,!}\Big[\frac{d^{n-1}}{ds^{n-1}}\,\{F(\lambda+s)\}^n\Big]$$

§ 4. As we are interested in the first two moments of each distribution considered earlier, we shall find them in this section in terms of the moments of service time distribution. Let α_i, β_i and γ_i be the i^{th} moments of the distributions of service time and busy period respectively. Then from (12)

15. $$E(W)=\beta_1=\frac{\lambda}{2c}\alpha_2$$

16. $$E(W^2)=\beta_2=\frac{\lambda}{3c}\alpha_3+\frac{\lambda^2}{2c^2}\alpha_2^2 \text{ (using L'hospitals' rule)}$$

Hence

17. $$V(W)=\beta_2-\beta_1^2=\frac{\lambda}{3c}\alpha_3+\frac{\lambda^2}{4c^2}\alpha_2^2$$

Similarly from (13)

18. $$E(b)=\gamma_1=\frac{\alpha_1}{1-\lambda\alpha_1}$$

19. $$E(b^2)=\gamma_2=\alpha_2\frac{(1+\lambda\gamma_1)^2}{1-\lambda\alpha_1}=\frac{\alpha_2}{(1-\lambda\alpha_1)^3}$$

20. $$V(b)=\gamma_2-\gamma_1^2=\frac{\alpha_2-\alpha_1^2(1-\lambda\alpha_1)}{(1-\lambda\alpha_1)^3}$$

Thus we have to evaluate α_1, α_2 and α_3. For this we shall use the following results.

21. $$\sum_{r=0}^{n} {}_{n+r}C_r \frac{1}{2^{n+r}}=1 \qquad \text{for } n=0, 1, 2\cdots\cdots$$

This can easily be proved by induction as below. It is easy to verify that (21) is true for $n=0$, 1 and 2. Let (21) be true for $n=N-1$

22. $$\sum_{r=0}^{N-1} {}_{N-1+r}C_r \frac{1}{2^{N-1+r}}=1$$

Then,

23. $$\sum_{r=0}^{N} {}_{N+r}C_r \frac{1}{2^{N+r}}$$

$$=1/2\left[\sum_{r=0}^{N} {}_{N+r-1}C_r \frac{1}{2^{N-1+r}}+\sum_{r=1}^{N} {}_{N+r-1}C_{r-1}\frac{1}{2^{N-1+r}}\right]$$

$$=\frac{1}{2}\left[1+{}_{2N-1}C_N\frac{1}{2^{2N-1}}+1-{}_{2N}C_N\frac{1}{2^{2N}}\right]$$

$$=1.$$

Hence (21).

From [1]

24. $$\sum_{n=0}^{\infty} {}_{2n}C_n \frac{\rho^{2n}}{2^{2n}} = \sum_{n=0}^{\infty} -\gamma_2 C_n (-\rho^2)^n = 1/\sqrt{1-\rho^2} \; ; \qquad |\rho|<1$$

We have

25. $$F(s) = 2(1-\rho^2) \sum_{n=0}^{\infty} \frac{\rho^{2n}}{(2+\alpha s)^{n+1}} C_n$$

where

26. $$C_n = \sum_{r=1}^{\infty} {}_{2n+r}C_{n+r} \frac{1}{(2+\alpha s)^{n+r}} = \frac{1}{\left(1-\dfrac{1}{2+\alpha s}\right)^{n+1}} - \sum_{r=0}^{n} {}_{n+r}C_r \frac{1}{(2+\alpha s)^r}$$

$$= \left(\frac{2+\alpha s}{1+\alpha s}\right)^{n+1} - \sum_{r=0}^{n} {}_{n+r}C_r \frac{1}{(2+\alpha s)^r}$$

Hence

27. $$F(s) = 2(1-\rho^2) \sum_{n=0}^{\infty} \rho^{2n} \left[\frac{1}{(1+\alpha s)^{n+1}} - \frac{1}{(2+\alpha s)} \sum_{r=0}^{n} {}_{n+r}C_r \frac{1}{(2+\alpha s)^{n+r}}\right]$$

Differentiating (27) once w. r. t. 's'

28. $$F'(s) = \frac{-2\alpha(1-\rho^2)}{(1-\rho^2+\alpha s)^2} + \alpha \sum_{r=0}^{n} (n+r+1)\, {}_{n+r}C_r \frac{\rho^{2n}}{(2+\alpha s)^{n+r+2}}$$

Using (24) and the fact that considering (24) as a furction of ρ, where $|\rho| < 1$, we can differentiate (24) w. r. t. ρ any number of times and get (see appendix)

29. $$F'(0) = -2\theta + \theta\,[1 + \tfrac{1}{2}\{\rho^2 \sqrt{1-\rho^2} + (1-\rho^2)^{3/2}\}]$$

$$= -\theta\,[1 + \tfrac{1}{2}\sqrt{1-\rho^2}\,].$$

Differentiating (28) once again w. r. t. s we get

30. $$F''(s) = 4\alpha^2(1-\rho^2) / (1-\rho^2+\alpha s)^3$$

$$-\alpha^2 \sum_{r=0}^{n} (n+r+1)(n+r+2)\, {}_{n+r}C_r \frac{\rho^{2n}}{(2+\alpha s)^{n+r+3}}$$

Hence

31. $$F''(0)=4\theta^2-2\theta^2(1+\rho^2)+\frac{\theta^2}{2}(1-\rho^2)^{\frac{3}{2}}(3-2\rho^2)$$

$$=\frac{\theta^2}{2}\left[4(1-\rho^2)+(1-\rho^2)^{\frac{3}{2}}(3-2\rho^2)\right]$$

$$=\frac{\theta^2}{2}(1-\rho^2)\left[4+(3-2\rho^2)\sqrt{1-\rho^2}\right]$$

Finally differentiating (30) once again with reference to 's'

32. $$F'''(s)=\frac{-12\alpha^3(1-\rho^2)}{(1-\rho^2+\alpha s)^3}+\alpha^3\sum_{r=0}^{n}\frac{(n+r+3)!}{n!\,r!}\frac{\rho^{2n}}{(2+\alpha s)^{n+r+4}}$$

Hence

33. $$F'''(0)=-12\theta^3+\frac{3\theta^3}{4}\left[8-\frac{14+\rho^2}{2}\sqrt{1-\rho^2}\right]$$

$$=-3\theta^3\left[2+\sqrt{1-\rho^2}(14+\rho^2)/8\right]$$

Hence for the service time distribution:

34. $$\alpha_1=\theta\left[1+\tfrac{1}{2}\sqrt{1-\rho^2}\right]$$

35. $$\alpha_2=\frac{\theta^2}{2}(1-\rho^2)[4+(3-2\rho^2)\sqrt{1-\rho^2}]$$

36. $$\alpha_3=3\theta^3\left[2+\sqrt{1-\rho^2}(14+\rho^2)/8\right]$$

37. $$\sigma_s^2=\alpha_2-\alpha_1^2=\theta^2\Big[(7/4)(1-\rho^2)-1+\frac{\sqrt{1-\rho^2}}{2}\{(1-\rho^2)(3-2\rho^2)-2\}\Big]$$

For waiting time distribution:

38. $$\beta_1=\frac{\lambda\theta^2}{4C}(1-\rho^2)\left[4+(3-2\rho^2)\sqrt{1-\rho^2}\right]$$

39. $$\beta_2=\frac{\lambda\theta^3}{C}\left[2+(14+\rho^2)\sqrt{1-\rho^2}/8\right]+\frac{\lambda^2\theta^4}{8c^2}(1-\rho^2)[4+(3-2\rho^2)\sqrt{1-\rho^2}]^2$$

and

40. $$\sigma_w^2=\beta_2-\beta_1^2=\frac{\lambda\theta^3}{C}\left[2+(14+\rho^2)\sqrt{1-\rho^2}/8\right]$$

$$+\left[\frac{\lambda\theta^2(1-\rho^2)}{4C}\right]^2\left[4+(3-2\rho^2)\sqrt{1-\rho^2}\right]^2$$

For the distribution of the busy period:

41. $$\gamma_1=\theta\left[1+\tfrac{1}{2}\sqrt{1-\rho^2}\right]/\left[1-\lambda\theta\left(1+\tfrac{1}{2}\sqrt{1-\rho^2}\right)\right]$$

42. $$\gamma_2=\frac{\theta^2(1-\rho^2)}{2}\,\frac{\left[4+(3-2\rho^2)\sqrt{1-\rho^2}\right]}{\left[1-\lambda\theta\left(1+\tfrac{1}{2}\sqrt{1-\rho^2}\right)\right]^3}$$

43. $$\sigma_b^2=\gamma_1-\gamma_1^2$$

$$=\frac{\theta^3\left[11-15\rho^2+4\sqrt{1-\rho^2}\,(2-5\rho^2+2\rho^4)+4\lambda\theta\left\{1+(1/2)\sqrt{1-\rho^2}\right\}^3\right]}{8\left[1-\lambda\theta\left\{1+(1/2)\sqrt{1-\rho^2}\right\}\right]^3}$$

Appendix

$$g(x_1,x_2)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(t_1x_1+t_2x_2)}\,\Phi(t_1,t_2)\,dt_1\,dt_2$$

$$=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}e^{-it_1x_1}\,dt_1\int_{-\infty}^{\infty}\frac{e^{-it_2x_2}}{c_1-c_2t_2}\,dt_2$$

where $c_1=1-it_1\theta,\ c_2=i\theta\left[1-it_1\theta(1-\rho^2)\right]$

Now

$$\int_{-\infty}^{\infty}\frac{e^{-it_2x_2}}{c_1-c_2t_2}\,dt_2=-\frac{1}{c_2}\int_{-\infty}^{\infty}\frac{e^{-it_2x_2}}{t_2-c_1/c_2}\,dt_2=-\frac{1}{c_2}\left[-2\pi i\,e^{-i(c_1/c_2)x_2}\right]$$

Hence

$$g(x_1,x_2)=\frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{exp\left\{-i\left[t_1x_1+(c_1/c_2)x_2\right]\right\}}{i\theta\left[1-it_1\theta(1-\rho^2)\right]}\,dt_1$$

Putting $z=1-it_1\theta(1-\rho^2)$, we have $c_2=i\theta z$

Thus

$$g(x_1,x_2)=\frac{1}{2\pi}\int_{1-i\infty}^{1+i\infty}\frac{exp\left\{-\frac{1}{\theta(1-\rho^2)}\left[(x_1+x_2)-\left(zx_1+\frac{\rho^2x_2}{z}\right)\right]\right\}}{i\theta_2 z(1-\rho^2)}\,dz$$

$$=\frac{1}{2\pi i\theta^2(1-\rho^2)}\,exp\left\{-\frac{(x_1+x_2)}{\theta(1-\rho^2)}\right\}\int_{1-i\infty}^{1+i\infty}\frac{exp\,\frac{1}{\theta(1-\rho^2)}\left(zx_1+\frac{\rho^2x_2}{z}\right)}{z}\,dz$$

$$=\frac{R}{\theta^2(1-\rho^2)}\, exp\left\{-\frac{(x_1+x_2)}{\theta(1-\rho^2)}\right\} \qquad \text{etc.}$$

Let
$$f(\rho)=\sum_{n=0}^{\infty} {}_{2n}C_n\frac{\rho^{2n}}{2^{2n}}=\frac{1}{\sqrt{1-\rho^2}} \ ; \ |\rho|<1$$

Differentiation on both sides with respect to ρ is admissible. Hence differentiating $f(\rho)$ once

$$f_1(\rho)=(2/\rho)\sum_{n=0}^{\infty} n\ {}_{2n}C_n\frac{\rho^{2n}}{2^{2n}}=\rho/(1-\rho^2)^{3/2}$$

Similarly from $f_2(\rho)$ we get

$$\sum_{n=0}^{\infty} n(n-1)\ {}_{2n}C_n\frac{\rho^{2n}}{2^{2n}}=(3/4)\frac{\rho^4}{(1-\rho^2)^{5/2}}$$

And from $f_3(\rho)$

$$\sum_{n=0}^{\infty} n(n-1)(n-2)\ {}_{2n}C_n\frac{\rho^{2n}}{2^{2n}}=15\,\rho^6/8\,(1-\rho^2)^{7/2} \ \text{etc.}$$

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Jute Agricultural Research Institute
Barrackpore, (W. Bengal), India