A RECURRENCE FORMULA FOR A SET OF POLYNOMIALS GENERATED BY $e^t\Psi(xt)$

By

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1. **Introduction** : *Rainville* [1] considered the set of polynomials $g_n(x)$ defined by

(1.1) $$e^t\,\Psi(xt)=\sum_{n=0}^{\infty} g_n(x)\frac{t^n}{n!},$$

where $\Psi(u)$ admits of the power-series expansion

(1.2) $$\Psi(u)=\sum_{n=0}^{\infty} a_n\frac{u^n}{n!}.$$

The interest of considering such set of polynomials consists in observing that the *Laguerre* polynomials $L_n^{(\alpha)}(x)$ possess the generating function

(1.3) $$e^t\,{}_0F_1(-\,;\,1+\alpha\,;\,-xt)=\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(x)\,t^n}{(1+\alpha)_n}.$$

or that a more general class of hypergeometric polynomials possesses the analogous generating function [2] :

(1.4) $$e^t\,{}_pF_q\left[\begin{matrix}\alpha_1,\cdots\cdots\cdots\cdots,\alpha_p\,;\\ \beta_1,\cdots\cdots\cdots\cdots,\beta_q\,;\end{matrix}\,-xt\right]$$
$$=\sum_{n=0}^{\infty}{}_{p+1}F_q\left[\begin{matrix}-n,\alpha_1,\cdots\cdots\cdots\cdots,\alpha_p\,;\\ \beta_1,\cdots\cdots\cdots\cdots,\beta_q\,;\end{matrix}\,x\right]\frac{t^n}{n!}.$$

Our object is to derive a general recurrence formula from which the corresponding formula for the sets given by (1.1) will follow easily. Moreover, we shall show some applications of our result in the case of polynomials given by (1.3) and (1.4). Lastly it is interesting to note that the well-known Kummer's transformation

(1.5) $$e^x\,{}_1F_1(c-a\,;c\,;-x)={}_1F_1(a\,;c\,;x)$$

yields the obvious identity

(1.6) $${}_2F_1(-n,c-a+1\,;c+1\,;1)=\frac{(a)_n}{(c+1)_n}.$$

2. General recurrence formula :

For our purpose, we shall first prove the following theorem.

Theorem : *If* $$f(x)=\sum_{n=0}^{\infty} A_n x^n$$

and $$e^x f(x)=\sum_{n=0}^{\infty} B_n x^n,$$

(2.1) *then* $$(n+1) B_{n+1}=B_n+\sum_{m=0}^{n} \frac{m+1}{(n-m)!} A_{m+1}.$$

Proof :

Let $$y=e^x f(x)$$

then y is a solution of the differential equation

(2.2) $$y'=e^x [f(x)+f'(x)]$$

Now we have

$$\sum_{n=0}^{\infty} (n+1) B_{n+1} x^n=\sum_{n=0}^{\infty} B_n x^n+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{m+1}{n!} A_{m+1} x^{m+n}.$$

$$=\sum_{n=0}^{\infty} B_n x^n+\sum_{n=0}^{\infty}\sum_{m=0}^{n} \frac{m+1}{(n-m)!} A_{m+1} x^n.$$

Comparing terms on both sides we obtain

$$(n+1) B_{n+1}=B_n+\sum_{m=0}^{n} \frac{m+1}{(n-m)!} A_{m+1},$$

which is (2.1).

3. Recurrence formula for the set of Rainville :

From (1.1) we notice that the polynomial $g_n(t)$ is generated by

(3.1) $$e^x \Psi(xt)=\sum_{n=0}^{\infty} g_n(t) \frac{x^n}{n!},$$

where $$\Psi(t)=\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}.$$

Thus we have for A_n and B_n, the following relations

(3.2) $$A_n=\frac{a_n t^n}{n!}, \quad B_n=\frac{g_n(t)}{n!}.$$

It follows therefore from (2.1) the following relation

$$g_{n+1}(t)=g_n(t)+\sum_{m=0}^{n}\binom{n}{m}a_{m+1}\,t^{m+1}. \tag{3.3}$$

But we know that

$$g_n(t)=\sum_{m=0}^{n}\binom{n}{m}a_m\,t^m. \tag{3.4}$$

Thus it follows from (3.3) and (3.4)

$$(n+1)\left[g_{n+1}(t)-g_n(t)\right]=tg'_{n+1}(t) \tag{3.5}$$

where
$$g'_n(t)=\frac{d}{dt}\{g_n(t)\};$$

the result (3.5) was obtained by *Rainville* in a different manner.

4. Some applications of the result (3.3):

(A) From (1.3) we notice that

$$e^x\,{}_0F_1(-;1+\alpha;-xt)=\sum_{n=0}^{\infty}\frac{L_n^{(\alpha)}(t)\,x^n}{(1+\alpha)_n}. \tag{4.1}$$

Thus we obtain

$$g_n(t)=\frac{n!}{(1+\alpha)_n}L_n^{(\alpha)}(t);\quad a_n=\frac{(-1)^n}{(1+\alpha)_n}. \tag{4.2}$$

It follows therefore from (3.3) that

$$(n+1)\,L_{n+1}^{(\alpha)}(t)-(n+\alpha+1)\,L_n^{(\alpha)}(t)=\sum_{m=0}^{n}\frac{(-1)^{m+1}(\alpha+2)_n\,t^{m+1}}{m!\,(n-m)!\,(\alpha+2)_m}. \tag{4.3}$$

Now we know that

$$L_n^{(\alpha)}(t)=\sum_{k=0}^{n}\frac{(1+\alpha)_n\,(-t)^k}{k!\,(n-k)!\,(1+\alpha)_k}. \tag{4.4}$$

Thus we derive from (4.3) and (4.4) the well-known formula for the *Laguerre* polynomials

$$(n+1)\,L_{n+1}^{(\alpha)}(t)+t\,L_n^{(\alpha+1)}(t)=(n+\alpha+1)\,L_n^{(\alpha)}(t). \tag{4.5}$$

(B) Next noticing the generating function (1.4) we obtain from our result (3.3)

(4.6) $${}_{p+1}F_q\left[\begin{matrix}-(n+1), & \alpha_1, \cdots\cdots\cdots, \alpha_p; \\ & \beta_1, \cdots\cdots\cdots, \beta_q;\end{matrix} t\right] - {}_{p+1}F_q\left[\begin{matrix}-n, \alpha_1, \cdots\cdots\cdots, \alpha_p; \\ \beta_1, \cdots\cdots\cdots, \beta_q;\end{matrix} t\right]$$

$$= \sum_{m=0}^{n} \binom{n}{m} \frac{(\alpha_1)_{m+1} \cdots\cdots\cdots (\alpha_p)_{m+1}}{(\beta_1)_{m+1} \cdots\cdots\cdots (\beta_q)_{m+1}} (-t)^{m+1}$$

5. Kummer's transformation:

Applying the result (2.1) of our main theorem on the following *Kummer's* tansformation

(5.1) $$e^x \, {}_1F_1(c-a; c; -x) = {}_1F_1(a; c; x),$$

we derive

(5.2) $$A_n = \frac{(-1)^n (c-a)_n}{(c)_n \, n!}, \qquad B_n = \frac{(a)_n}{(c)_n \, n!};$$

so that we have

(5.3) $$\frac{(a)_n}{(c+1)_n} = \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{(c-a+1)_m}{(c+1)_m},$$

which is equivalent to

(5.4) $${}_2F_1(-n, c-a+1; c+1; 1) = \frac{(a)_n}{(c+1)_n}.$$

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REFERENCES

[1] Rainville, E. D. : *Special Functions*, Macmillan Company, New York, 1960, p. 132.

[2] Erdélyi, A. et al. : *Higher Transcendental Functions* Vol. 3, MacGraw-Hill Book Co., New York, 1953, p. 267.

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