SOME REMARKS ON A CLASS OF OPERATORS

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1. Let H be a Hilbert space and T a bounded linear operator on H. Denote by $\sigma(T)$ the spectrum, by $\rho(T)$ the resolvent set and by $W(T)=\{(Tx,x):\|x\|=1\}$ the numerical range of T. It is known that $W(T)$ is convex and conv $\sigma(T)\subseteq$ cl $W(T)$ (conv=convex hull, cl=closure). T is called convexoid if cl $W(T)=$conv $\sigma(T)$.

In this Note we give some conditions on a class of operators implying normality.

2. Consider the following conditions that an operator T may or may not satisfy:

(G_1) $\quad \|(T-\lambda I)^{-1}\|\geq[\text{dist}(\lambda,\sigma(T))]^{-1}$ for all $\lambda\in\rho(T)$

(B) T is reduced by each of its finite-dimensional eigen-spaces.

It is known [2] that if T satisfies (G_1) and λ is an isolated point of $\sigma(T)$ then λ is an eigenvalue of T, $\bar{\lambda}$ is an eigenvalue of T^*, and $\eta_T(\lambda)=\eta_{T^*}(\bar{\lambda})$ ($\eta_T(\lambda)=\{x\in H: Tx=\lambda x\}$) Also, from [8] T is a convexoid operator.

Theorem 1. *If T satisfies (G_1) and $\sigma(T)$ has only λ_0 as limit point, then it can be expressed uniquely as a direct sum $T=T_1\oplus T_2$ defined on a product space $H=H_1\oplus H_2$ where H_1 is spanned by all the eigenvectors of T such that: (a) T_1 is normal, (b) $\sigma(T_2)=\{\lambda_0\}$.*

Proof. Since $T-\lambda_0 I$ also satisfies (G_1) one can suppose $\lambda_0=0$ and we have that every $\lambda\in\sigma(T)-\{0\}$ is a normal eigenvalue of T. Let $H_1=\sum_{\lambda\in\sigma(T)-\{0\}}\oplus\,\eta_T(\lambda)$, then H_1 reduces $T_1=T|H_1$, the restriction of T onto H_1, is normal, and $\sigma(T)=\sigma(T_1)\cup\sigma(T|H_1^\perp)$ If $H_1^\perp=\{0\}$ we are through. If not, then $\sigma(T|H_1^\perp)=\{0\}$ since otherwise $0\neq\lambda\in\sigma(T|H_1^\perp)$ implies the the existence of an eigenvector which is a contradiction.

Corollary. *If T has property (G_1) on every reducing subspace and λ_0 is the single limit point of $\sigma(T)$ then T is normal.*

Remark. The corollary for the case when T is a restriction-convexoid operator is proved in [1, Lemma 4].

Recall that an operator $T=A+iB$ is said to be *hyponormal* if $\|Tx\|\geqslant\|T^*x\|$ or $AB-BA=iC$, $C\geq 0$.

Definition. An operator T is said to be *hyponormal* of order m if

$$A^mB-BA^m=iC_m,\qquad T=A+iB$$

where $C_m \geqslant 0$ for some nonnegative integer m [5].

Theorem 2. *If T is a hyponormal operator of order m which satisfies (B) and $T=S+C$ where S is a self-adioint operator and C compact then T is a normal operator.*

Proof . Let $\omega(T)=\cap\{\sigma(T+K): K$ compact$\}$ the Weyl spectrum of T. It is known [2] that $\sigma(T)-\omega(T)$ is either empty or consists of eigenvalues of finite multiplicity. Since $T=S+C$ it follows that $\omega(T)$ is real.

Since the finite-dimensional eigenspaces of T are mutually orthogonal then their orthogonal direct sum H_1 reduces T, $T_1=T|H_1$ is normal and $\sigma(T_2)=\omega(T_2)$ where $T_2=T|H_1^{\perp}$ [2].

It is easy to see that T is hyponormal of order m if and only if $T_m=A^m-iB$ is a hyponormal operator. Since H_1 reduces T we have that $H_1^{\perp}$ is invariant under A and B and therefore $T_m|H_1^{\perp}$ is hyponormal which implies T_2 is a hyponormal operator of order m. From the fact that $\omega(T)=\omega(T_1)\cup\omega(T_2)$ we have that $\sigma(T_2)$ is real and thus [4, Lemma 1] T_2 is a self-adjoint operator.

Corollary 1. *If T is hyponormal of order m which satisfies (B) and with compact imaginary part, then T is a normal operator.*

Remark. This result for T a hyponormal operator is proved in [7]. From this result we conclude that if T is a hyponormal operator of order m with compact imaginary part then $A^mB=BA^m$.

Corollary 2. *If T is a hyponormal operator of order m such that*

1) *satisfies (B)*

2) $T^p=ST^{*p}S^{-1}+C, p$ *an integer* $\geqslant 1$, $0\notin cl\ W(S)$, C *compact and* $1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2+\cdots+\left(\frac{\lambda}{\mu}\right)^{p-1}\neq 0$, *if* $\lambda,\mu\in\sigma(T)$ *then T is a normal operator.*

Proof. If we denote $I(H)$ the ideal of all compact operators on H then $\mathfrak{C}=\mathfrak{L}(H)/I(H)$ is a B^*-algebra with an involution induced by the natural involution on $\mathfrak{L}(H)$. For each operator T on H, let $\bar{T}$ denote its natural image in $\mathfrak{C}$. It is easy to see that $\overline{T^*}=\overline{T}^*$, $\overline{T^{-1}}=(\bar{T})^{-1}$. It follows that $\bar{T}^p=\bar{S}\bar{T}^{*p}\bar{S}^{-1}$ and by Theorem 1.1 [6] we conclude that $\sigma(\bar{T})$ is real. Since $\omega(T)=\sigma(\bar{T})$ we have that T is a normal operator.

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