VARIOUS HAMILTON'S CANONICAL FORMALISMS AS NON-CONNECTION METHODS FOR VARIOUS CONNECTION GEOMETRIES IN THE LARGE. PART II.

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This Part II is a sequel to the previous one T. Takasu, [2], which was a detailed exposition of T. Takasu, [1], in which I had introduced non-connection geometries *in the large* based on canonical equations of Hamiltonian types of II-*geodesic curves* in my sense.

§ 1 consisted in "an extension of the *duality* exposed in the book H. Rund, [12] to the case depending on *special* higher order derivatives" adding new formulas to the Hamilton's canonical formalism of $H(y_\lambda, x^\lambda)$ (Hamiltonian) and $L(x^\lambda, \dot{x}^\lambda)$ (Lagrangian), where they were the Lagrangian $L(x^\lambda, \dot{x}^\lambda)$ and Hamiltonian $H(y^\lambda, x_\lambda)$ respectively.

Thereby it was *assumed* that the values of

(1)	$\displaystyle\int_{\tau_0}^{\tau} L(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda)\, d\tau$	$\displaystyle\int_{\tau_0}^{\tau} H(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}{}_\lambda)\, d\tau$

are *invariant under the arbitrary parameter transformation of the type* $\sigma=\sigma(\tau)$, the function σ being of class C^1 such that

$$\dot{\sigma} = d\sigma/d\tau > 0. \tag{2}$$

This assumption implies that *the theory is invariant under the transformations of the local coordinates*

x^λ	x_λ

as well as under transformations of the parameter τ *subject to the above condition.* It was also *assumed* that the

Lagrangian $L(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda)$,	*Hamiltonian* $H(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}{}_\lambda)$,

say, *is positively homogeneous of the first degree in* $\dot{x}$:

(3)	$(x^\lambda, \lambda\dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda)$ $=\lambda L(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda)>0,$	$H(x_\lambda, \lambda\dot{x}_\lambda, \cdots, \overset{(M)}{x}{}_\lambda)$ $=\lambda H(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}{}_\lambda)>0,$

so that

$$(4)\qquad \frac{\partial L(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})}{\partial \dot{x}^\mu}\dot{x}^\mu = L(x^\lambda, \dot{x}^\lambda, \cdots \overset{(M)}{x^\lambda}), \quad \Big| \quad \frac{\partial H(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})}{\partial \dot{x}_\mu}\dot{x}_\mu = H(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}),$$

it being said in the foot-note (p. 17) that "other cases shall be studied later".

The purpose of this Part II consists in $1^0, 2^0$ and 3^0 below.

1^0. *The meaning of*

$$(3.3)\qquad \frac{d}{ds}\frac{\omega}{ds} = \omega_\lambda(\ddot{x}^\lambda + \Lambda^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu) = \cdots\cdots$$

and

$$(3.4)\qquad \Lambda^\lambda_{\mu\nu} \overset{\text{def}}{=} \omega^\lambda \frac{\partial \omega_\mu}{\partial x^\nu}$$

of Part I *will be shown* (*cf.* (2.6.3)) *in particular to be*

$$(5)\qquad \frac{d}{ds}\frac{\omega}{ds} \equiv \omega_\lambda(\ddot{x}^\lambda + \Lambda^\lambda_{\mu\nu(0)}\dot{x}^\mu\dot{x}^\nu) \equiv \omega_\lambda(\ddot{x}^\lambda + \{{}^{\lambda}_{\mu\nu}\}_{(0)}\dot{x}^\mu\dot{x}^\nu)$$

and

$$(6)\qquad \Lambda^\lambda_{\mu\nu(0)} \equiv \omega^\lambda \frac{\partial \omega_\mu}{\partial x^\nu}. \qquad \text{(Cf. also (1.1.23) and (2.5.32).)}$$

2^0. To treat the "other cases" (depending on *general* higher order derivatives) spoken of above, where M runs over $1, 2, \cdots, M$, while in § 5 of Part I (p. 40), the case $M=1$ only is recapitulated.

3^0. As for the Kawaguchi spaces of order M, which was slightly touched under IV, p. 44 of Part I of this paper, it will in this paper be, in particular, brought to "Kawaguchi space *in the large*" exposed more in detail (cf. § 4 – § 13), "general metric space" being also considered (cf. § 14 – § 23).

N. B. *As for the degree of specializations of* (i. e. the conditions for) *the differentiable manifolds, see the comment* N. B. *under* Art. 1. 2.

§ 1. II–Geodesic Curves in the n–Dimensional Doubly Extended Differentiable Manifolds.

1.1. II–Geodesic Curves in the n–Dimensional Doubly Extended Differentiable Manifolds. We consider a general n–ary linear differential form

$$(1.1.1)\qquad \omega \overset{\text{def}}{=} \omega_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})\,dx^\mu, \qquad (\dot{x} = dx/d\tau, \text{ etc.}; \lambda, \mu, \cdots = 1, 2, \cdots, M),$$

which is *global* in the n–dimensional differentiable Manifold

$$\cup\, U_\alpha(x^\lambda_{(\alpha)}), \qquad (\lambda = 1, 2, \cdots, n; \alpha = 1, 2, \cdots)$$

(in current notation (cf. Part I, p. 16)) of class C^v,

v=positive integer,	$v=\infty$,	$v=\omega$,

the (1.1.1) being written in an invariant form.

Setting

$$ds \stackrel{\mathrm{def}}{=} \omega_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})\, \dot{x}^\mu\, d\tau, \tag{1.1.2}$$

$$\mathscr{L} \stackrel{\mathrm{def}}{=} \omega_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})\, \dot{x}^\mu, \tag{1.1.3}$$

as we have done in (1.2) and (1.3) of Part I, we solve the extremal problem

$$\delta s = \delta \int_{\tau_0}^{\tau_1} \mathscr{L} d\tau = 0 \tag{1.1.4}$$

in two ways I and II below.

I. The Euler–Lagrange equations are

$$\frac{\partial \mathscr{L}}{\partial x^\mu} - \frac{d}{d\tau}\left[\frac{\partial \mathscr{L}}{\partial \dot{x}^\mu} - \frac{d}{d\tau}\frac{\partial \mathscr{L}}{\partial \ddot{x}^\mu} + \cdots + (-1)^{M-1}\frac{d^{M-1}}{d\tau^{M-1}}\frac{\partial \mathscr{L}}{d\overset{(M)}{x^\mu}}\right] = 0. \tag{1.1.5}$$

If we define $y_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})$ by (this seems to be the initiative of the present author)

$$y_\mu \stackrel{\mathrm{def}}{=} \frac{\partial \mathscr{L}}{\partial \dot{x}^\mu} - \frac{d}{d\tau}\frac{\partial \mathscr{L}}{\partial \ddot{x}^\mu} + \cdots + (-1)^{M-1}\frac{d^{M-1}}{d\tau^{M-1}}\frac{\partial \mathscr{L}}{\partial \overset{(M)}{x^\mu}}, \tag{1.1.6}$$

then (1.1.3), ($\omega_\mu = y_\mu$) and (1.1.5) give

$$\frac{\partial \mathscr{L}}{\partial x^\mu} = \dot{y}_\mu, \qquad \frac{\partial \mathscr{L}}{\partial y_\mu} = \dot{x}^\mu \tag{1.1.7}$$ [1]

forming the canonical equations of Lagrangian types. When the points τ_0, τ_1 in (1.1.4) and the curve C passing through τ_0, τ_1 belong to one and the same U_α of local coordinates, *the* (1.1.7) *are the canonical equations of the ordinary local geodesic curve C, and otherwise the curve C is a* II*-geodesic curve corresponding to $y_\mu = \omega_\mu(x, \dot{x} \cdots, \overset{(M)}{x})$, which is global and an extremal of* (1.1.4).

In order to render (1.1.7) into the corresponding Hamiltonian canonical equations, we proceed as follows. From (1.1.7) we obtain

$$\delta \mathscr{L} = \frac{\partial \mathscr{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial \mathscr{L}}{\partial y_\mu}\delta y_\mu = \dot{y}_\mu\, \delta x^\mu + \dot{x}^\mu\, \delta y_\mu = 2\dot{x}^\mu \delta y_\mu + (\dot{y}_\mu\, \delta x^\mu - \dot{x}^\mu \delta y_\mu).$$

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1) This differs from (I.8), p. 41 of Part I in that $M \geqq 1$.

Taking $\mathcal{H}$ such that

$$\delta\mathcal{H} = \dot{x}^\mu\,\delta y_\mu - \delta x^\mu\,\dot{y}_\mu \tag{1.1.8}$$

for the Hamiltonian corresponding to $\mathcal{L}$, we obtain the Hamilton's canonical equations:

$$\frac{\partial\mathcal{H}}{\partial y_\mu} = \dot{x}^\mu, \qquad \frac{\partial\mathcal{H}}{\partial x^\mu} = -\dot{y}_\mu \tag{1.1.9}$$

of the extremals (II-*geodesic curves*):

$$\delta s = \delta\int_{\tau_0}^{\tau_1} \dot{s}\,d\tau = \int_{\tau_0}^{\tau_1} \delta\dot{s}\,d\tau = \int_{\tau_0}^{\tau_1} \frac{d}{d\tau}\,\delta s\,d\tau = [\delta s]_{\tau_0}^{\tau_1} = 0. \tag{1.1.10}$$

Theorem. *For that the extremals* $\delta\mathcal{H}(x, \dot{x}. \cdots, y_\lambda) = 0$ *and* $\delta\mathcal{L}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) = 0$ *coincide, it is necessary and sufficient that*

$$\delta\,(y_\mu\,\dot{x}^\mu) = 0. \tag{1.1.11}$$

Proof. From (1.1.3), we obtain

$$\delta\mathcal{L} = \delta\,(y_\mu\,\dot{x}^\mu) = \delta y_\mu\,\dot{x}^\mu + y_\mu\,\delta\dot{x}^\mu = \dot{y}_\mu\,\delta\tau\,\dot{x}^\mu + y_\mu\,\delta\dot{x}^\mu = \dot{y}_\mu\,\delta x^\mu + y_\mu\,\delta\dot{x}^\mu,$$

$$\delta\mathcal{L}(x, \dot{x}, \cdots, \overset{(M)}{x}) = \delta\,(y_\mu\,\dot{x}^\mu) - \delta\mathcal{H}, \tag{1.1.12}$$

where

$$\delta\mathcal{H} = \delta y_\mu\,\dot{x}^\mu - \delta x^\mu\,\dot{y}_\mu. \tag{1.1.13}$$

The (1.1.12) gives our theorem.

Cor. $\mathcal{H} = \text{const.}$ *along the extremal* $\delta\mathcal{H} = 0$.

Proof. From (1.1.12), it follows that

$$\frac{d\mathcal{L}}{d\tau} = \frac{d\,(y_\mu\dot{x}^\mu)}{d\tau}, \qquad \frac{d\mathcal{H}(x^\mu, y_\mu)}{d\tau} = 0.$$

II. We define the parameters of teleparallelism types

$$\Lambda^\lambda_{\mu\nu(s)}, \quad (s = 0, 1, \cdots, M) \tag{1.1.14}$$

by

$$d\omega_\mu - \sum_{s=0}^{M} \Lambda^\lambda_{\mu\nu(s)}\,\omega_\lambda\,\overset{(s)}{dx^\nu} = 0 \qquad \Bigg| \qquad d\omega^\lambda + \sum_{s=0}^{M} \Lambda^\lambda_{\mu\nu(s)}\,\omega^\mu\,\overset{(s)}{dx^\nu} = 0 \tag{1.1.15}$$

for the given ω_λ and ω^λ defined by

$$\omega^\lambda\,\omega_\mu = \delta^\lambda_\mu, \tag{1.1.16}$$

so that

$$\Lambda^{\lambda}_{\mu\nu(s)} = \omega^{\lambda} \frac{\partial \omega_{\mu}}{\partial \overset{(s)}{x}{}^{\nu}} \equiv -\omega_{\mu} \frac{\partial \omega^{\lambda}}{\partial \overset{(s)}{x}{}^{\nu}} \,. \tag{1.1.17}$$

A straight forward calculation gives the identity:

$$\frac{d}{d\tau} \frac{\omega}{d\tau} = \{\ddot{x}^{\lambda} + \sum_{s=0}^{M} \Lambda^{\lambda}_{\mu\nu(s)} \dot{x}^{\mu} \overset{(s+1)}{x}{}^{\nu}\}. \tag{1.1.18}$$

Indeed, we have

$$\begin{aligned} \frac{d}{d\tau} \frac{\omega}{d\tau} &= \frac{d}{d\tau} (\omega_{\mu} \dot{x}^{\mu}) = \frac{d\omega^{\mu}}{d\tau} \dot{x}^{\mu} + \omega_{\lambda} \ddot{x}^{\lambda} \\ &= \omega_{\lambda} \left\{ \ddot{x}^{\lambda} + \sum_{s=0}^{M} \omega^{\lambda} \frac{\partial \omega_{\mu}}{\partial \overset{(s)}{x}{}^{\nu}} \dot{x}^{\mu} \overset{(s+1)}{x}{}^{\nu} \right\} = \omega_{\lambda} \left\{ \ddot{x}^{\lambda} + \sum_{s=0}^{M} \Lambda^{\lambda}_{\mu\nu(s)} \dot{x}^{\mu} \overset{(s+1)}{x}{}^{\nu} \right\}. \end{aligned} \tag{1.1.19}$$

By virtue of (1.1.18), we can deduce

$$\omega^{\lambda} \frac{d}{d\tau} \frac{\omega}{d\tau} = \ddot{x}^{\lambda} + \sum_{s=0}^{M} \Lambda^{\lambda}_{\mu\nu(s)} \dot{x}^{\mu} \overset{(s+1)}{x}{}^{\nu}. \tag{1.1.20}$$

The equations of the

global paths	*local paths*
(1.1.21) $\frac{d}{d\tau} \frac{\omega}{d\tau} = 0$	$\ddot{x}^{\lambda} + \sum_{s=0}^{M} \Lambda^{\lambda}_{\mu\nu(s)} \dot{x}^{\mu} \overset{(s+1)}{x}{}^{\nu} = 0$

are projections of the

local ones $(1.1.21)_2$	*global ones* $(1.1.21)_1$
by the globalizing function factor $\omega_{\lambda}(x, \dot{x}, \cdots, \overset{(M)}{x})$.	*by the localizing function factor* $\omega^{\lambda}(x, \dot{x}, \cdots, \overset{(M)}{x})$.

The (11.20) may be rewritten as

$$\omega^{\lambda} \frac{d}{d\tau} \frac{\omega}{d\tau} = \ddot{x}^{\lambda} + \overline{\Lambda}^{\lambda}_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}, \tag{1.1.22}$$

where

$$\overline{\Lambda}^{\lambda}_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x}) \overset{\text{def}}{=\!=} \omega^{\lambda}(x, \dot{x}, \cdots, \overset{(M)}{x}) \sum_{s=0}^{M} \frac{\overset{(s+1)}{x}{}^{\sigma}}{\dot{x}^{\nu}} \frac{\partial \omega_{\mu}}{\partial \overset{(s)}{x}{}^{\sigma}} = \sum_{s=0}^{M} \Lambda^{\lambda}_{\mu\sigma(s)} \frac{\overset{(s+1)}{x}{}^{\sigma}}{\dot{x}^{\nu}} \tag{1.1.23}$$

with

$$d\omega_{\mu}(x, \dot{x}, \cdots, \overset{(M)}{x}) \overset{\text{def}}{=\!=} \omega^{\lambda}(x, \dot{x}, \cdots, \overset{(M)}{x}) \sum_{s=0}^{M} \frac{\overset{(s+s)}{x}{}^{\sigma}}{\dot{x}^{\nu}} \frac{\partial \omega_{\mu}}{\partial \overset{(s)}{x}{}^{\sigma}} \overset{(s+1)}{d x}{}^{\nu} = \sum_{s=0}^{M} \Lambda^{\lambda}_{\mu\sigma(s)} \frac{\overset{(s+1)}{x}{}^{\sigma}}{\dot{x}^{\nu}} dx^{\nu} = \overline{\Lambda}^{\lambda}_{\mu\nu} \omega_{\lambda} \, dx^{\nu} \tag{1.1.24}$$

or

$$d\omega_\mu - \overline{\Lambda}^\lambda_{\mu\nu}\,\omega_\lambda\, dx^\nu = 0, \tag{1.1.24}$$

where

$$\overline{\Lambda}^\lambda_{\mu\nu} \overset{\text{def}}{=\!=} \omega^\lambda \left(\frac{\partial \omega_\mu}{\partial x^\sigma}\dot{x}^\sigma + \frac{\partial \omega_\mu}{\partial \dot{x}^\sigma}\ddot{x}^\sigma + \cdots + \frac{\partial \omega_\mu}{\partial \overset{(M)}{x^\sigma}} \overset{(M+1)}{x^\sigma} \right) = \sum_{s=0}^{M} \Lambda^\lambda_{\mu\sigma(s)} \overset{(s+1)}{x^\sigma} \Big/ \dot{x}_\nu. \tag{1.1.26}$$

The global paths $(1.1.21)_1$ *are evidently the extremals*

$$\delta \frac{\omega}{d\tau} = 0, \tag{1.1.27}$$

which coincides with (1.1.7).

1.2. Finite Equations of the II–Geodesic Curves.

Our

ω	ω_μ	ω^μ

are vectors with components

(ω^l),	(ω_μ),	(ω^μ),	(ω^l_μ),	(ω^μ_l),

$(\mu, l = 1, 2, \cdots, n)$, so that all formulas in

ω	ω_μ	ω^μ

may be written in

ω^l.	ω_μ.	ω^μ.	ω^l_μ.	ω^μ_l.

Thus it should be emphasized that *we are considering the case, where there exist* n *linearly independent* 1*-forms* $\omega^l, (l=1, 2, \cdots, n)$, *such that* $\det(\omega^l_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})) \neq 0$.

N. B. Since $g_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x}) = \omega^l_\mu\, \omega^l_\nu$, $\det(g_{\mu\nu}) = \{\det(\omega^l_\mu)\}\ \{\det(\omega^l_\nu)\} = \{\det(\omega^l_\mu)\}^2$, *the condition* $\det(\omega^l_\mu) \neq 0$ *is comparable with the condition* $\det(g_{\mu\nu}) \neq 0$, *where the matrices* $(g_{\mu\nu})$ *exist.* Conversely, *when the matrices* (ω^l_μ) *exist, we can construct the matrices* $(g_{\mu\nu})$. Thus *the condition spoken of above is comparable with the condition for the existence of the doubly extended*

affine	*equiform*	*Euclidean*

geometry in the differentiable manifolds under consideration, what enables us to establish doubly extended geometries (in F. Klein's sense) corresponding to the branches enlisted in the schemata given in Art. 3.1. *later.*

The differential equation $(1.1.18)_1$ is readily integrable:

$$\omega^l = a^l d\tau, \quad (a^l : \text{const.}). \tag{1.2.1}$$

We set

$$d\xi^l \overset{\text{def}}{=} a^l d\tau, \tag{1.2.2}$$

so that

$$\xi^l = a^l \tau + c^l, \quad (c^l : \text{const.}). \tag{1.2.3}$$

This represents *a curve in the large, which behave as for meet and join as well as for the extremal*

$$\delta\tau = 0 \tag{1.2.4}$$

like straight lines. We have called such curves II-*geodesic curves* (T. Takasu, [1]) *corresponding to* $(\omega^l_\lambda(x, \dot{x}, \cdots, \overset{(M)}{x}))$ and the (ξ^l) II-*geodesic rectangular coordinates referred to the II-geodesic* ξ^l*-axes.* The (ξ^l) *are global.*

The (1.1.1) becomes

$$d\xi^l = \omega^l = \omega^l_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})\, dx^\mu. \tag{1.2.5}$$

The form of (1.1.15) tells us that *the local paths of the teleparallelism* $(1.1.18)_2$ *are projected piece-wise into the global paths*

$$\frac{d^2\xi^l}{d\tau^2} = \frac{d}{d\tau}\frac{\omega^l}{d\tau} = 0 \tag{1.2.6}$$

by the transformation (1.2.5):

$$d\xi^l = \omega^l_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})\, dx^\mu \tag{1.2.7}$$

continuingly and smoothly.

The identity (1.1.17) tells us that *the global paths* $(1.1.18)_1$ *or* (1.2.6) *are projected piece-wise onto the local paths* $(1.1.18)_2$ *by the inverse transformation*

$$dx^\lambda = \omega^\lambda_l\, \omega^l = \omega^\lambda_l\, d\xi^l \tag{1.2.8}$$

of (1.2.7). Indeed, the $(1.1.18)_2$ is a *linear combination of* $(1.1.18)_1$ as (1.1.16) tells us.

Multiplying (1.2.2) with ω^λ_l, we see that *the relation*

$$\frac{dx_\lambda}{d\tau} = a^l\, \omega^\lambda_l \tag{1.2.9}$$

holds along the II-geodesic line-elements.

§ 2. Doubly Extended Affine Geometry and Doubly Extended Euclidean Geometry. [2]

2.1. Doubly Extended Affine Transformations. We consider the case, where (ξ^l) stands for (x^λ). In this case, (1.2.7) becomes to the form

$$d\bar{\xi} = a^l_h(\dot{\xi}, \xi, \cdots, \overset{(M)}{\xi})\, d\xi^h, \qquad (\det(a^l_h) \neq 0) \tag{2.1.1}$$

for the II-geodesic line-elements corresponding to $(a^l_h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}))$. In order that the II-geodesic curve $\xi^l(\tau)$ may be transformed by (2.1.1) into II-geodesic curves $(\bar{\xi}^l(\tau))$ corresponding to $(a^l_h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}))$, we must have

$$da^l_h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\, d\xi^h = 0 \tag{2.1.2}$$

along the II-*geodesic line-elements.* For, from (2.1.1), we obtain

$$\frac{d^2\bar{\xi}^l}{d\tau^2} = \frac{d}{d\tau} a^l_h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) \frac{d\xi^h}{d\tau} + a^l_h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) \frac{d^2\xi^h}{d\tau^2}. \tag{2.1.3}$$

Integrating (2.1.1) along the $\bar{\xi}^l$-axes, we obtain

$$\bar{\xi}^l = a^l_h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\, \xi^h - \int \xi^h (da^l_h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})/d\tau)\, d\tau.$$

Now

$$\int \xi^h (da^l_h/d\tau)\, d\tau = \int (da^l_h/d\tau)\, d\tau \int d\xi^h = \int\int \{da^l_h/d\tau)\, d\tau\, d\xi^h\} = \text{const.} = -a^l_o,$$

say, by (2.1.2), the sufficient condition for that the repeated integral may be converted into the double integral (that is, that the integrand is continuous) being evidently satisfied. Thus we have

$$\bar{\xi}^l = a^l_h(\xi, \cdots, \overset{(M)}{\xi})\, \xi^h + a^l_o, \qquad (\det(a^l_h) \neq 0,\ a^l_o = \text{const.}). \tag{2.1.4}$$

I have called (2.1.4) a *doubly extended affine transformation* (T. Takasu, [6], (26), p. 872; [5]. (3.2), p. 63).

From (2.1.1) and (2.1.4), we see that the *necessary condition*

$$da^l_h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\, \xi^h = 0 \tag{2.1.5}$$

is satisfied for the II-geodesic line-elements under doubly extended affine transformations.

2.2 Doubly Extended Affine Transformation Group. Let us prove the

Theorem. *The totality of the doubly extended affine transformations*

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2) In § 5 of Part I (p. 40), *the case* $M=1$ *only* is recapitulated. In this § 2. M shall run over $1, 2, \cdots, M$.

$$\bar{\xi}^h = a^h_k(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\xi^k + a^h_o, \qquad (a^h_o = \text{const.}, \ \det(a^h_k) \neq 0), \tag{2.2.1}$$

whose inverse transformations are

$$\xi^k = a^k_h(\bar{\xi}, \dot{\bar{\xi}}, \cdots, \overset{(M)}{\bar{\xi}})\bar{\xi}^h + a^k_o, \qquad (a^k_o = \text{const.}, \ \det(a^k_h) \neq 0), \tag{2.2.2}$$

forms a group.

Proof. The combination of (2.2.1) with

$$\bar{\bar{\xi}}^l = \bar{a}^l_h(\bar{\xi}, \dot{\bar{\xi}}, \cdots, \overset{(M)}{\bar{\xi}})\bar{\xi}^h + \bar{a}^l_o, \qquad (\bar{a}^l_o = \text{const.}, \ \det(\bar{a}^l_h) \neq 0) \tag{2.2.4}$$

gives

$$\bar{\bar{\xi}}^l = b^l_k(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\xi^k + b^l_o, \qquad (b^l_o = \text{const.}, \ \det(b^l_k) \neq 0), \tag{2.2.5}$$

where

$$\begin{cases} b^l_k(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) = \bar{a}^l_m(a^q_h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\xi^h + a^q_o)\, a^m_k(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}), \\ b^l_o = \bar{b}^l_m(\xi, \dot{\xi}, \cdots \overset{(M)}{\xi})\, a^m_o + \bar{a}^l_o, \end{cases} \tag{2.2.6}$$

$$\bar{b}^l_m(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) = \bar{a}^l_m(a^q_k(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\xi^k + a^q_o). \tag{2.2.7}$$

We shall see that

$$b^l_o = \bar{b}^l_m(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\, a^m_o + \bar{a}^l_o = \text{const.} \tag{2.2.8}$$

owing to the summation with respect to m, for which it suffices to prove that

$$a^m_o \, d\bar{b}^l_m(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) = 0 \tag{2.2.9}$$

on summation with respect to m. For (2.2.4), the condition (2.1.8) for that the $\bar{\xi}^l$–axes may be II–geodesic curves corresponding to $\bar{a}^l_m(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})$ becomes

$$\bar{\xi}^m \, d\bar{a}^l_m(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) = 0. \tag{2.2.10}$$

(2.2.9) follows from the law (2.2.10) for $\bar{a}^l_h$. Indeed, (2.2.10) becomes

$$\begin{aligned} &\{a^m_k(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\xi^k + a^m_o\}\, d\bar{a}^l_m(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) \\ &\qquad = \{a^m_k(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\xi^k + a^m_o\}\, d\bar{b}^l_m(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) = 0, \end{aligned}$$

so that

$$\begin{aligned} a^m_o \, d\bar{a}^l_m(\bar{\xi}, \dot{\bar{\xi}}, \cdots, \overset{(M)}{\bar{\xi}}) &= a^m_o \, d\bar{b}^l_m(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) \\ &= -a^m_k(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\, d\bar{b}^l_m(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\xi^k \\ &= -a^m_k(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\, d\bar{b}^l_m(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\xi^k \\ &\quad - \{\xi^k \, da^m_k(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\}\, \bar{b}^l_m(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) \end{aligned} \tag{2.2.11}$$

by the differential equation

$$\xi^k \, da_k^m (\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) = 0 \tag{2.2.12}$$

for the II-geodesic curves corresponding to $a_k^m (\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})$.

Thus we have

$$\begin{aligned} a_o^m \, d\bar{a}_m^l (\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) &= -\xi^k \, d\{a_k^m (\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) \, \bar{b}_m^l (\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\} \\ &= -\xi^k \, d\{a_k^m (\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) \, \bar{a}_m^l \, \dot{\xi}, (\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\} \\ &= -\xi^k \, db_k^l (\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) = 0 \end{aligned} \tag{2.2.13}$$

by the differential equation

$$\xi^k \, db_k^l (\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi}) = 0$$

for the II-geodesic curves corresponding to $b_k^l (\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})$. The (2.2.13) shows us (2.2.9).

Definition. The group stated in the last theorem will be called the *doubly extended affine group* and the geometry belonging to it the *doubly extended affine geometry*. (Cf. Part I, § 5, I, p. 40 in the case $M=1$).

N. B. *The detail* may be exposed quite as in T. Takasu, [16].

2.3 Doubly Extended Euclidean Transformations. We consider

$$ds^2 = d\xi^l \, d\xi^l = \omega^l \, \omega^l = \omega\omega = \omega_\mu^l \, dx^\mu \, \omega_\nu^l \, dx^\nu = g_{\mu\nu} \, dx^\mu \, dx^\nu, \tag{2.3.1}$$

$$g_{\mu\nu} (x, \dot{x}, \cdots, \overset{(M)}{x}) = \omega_\mu^l \, \omega_\nu^l = \omega_\mu \, \omega_\nu, \qquad (\det (\omega_\mu^l) \neq 0), \tag{2.3.2}$$

for which the *global orthogonality conditions*

$$a_h^l \, a_k^l = \delta_{hk} \quad \Longleftrightarrow \quad a_l^h \, a_l^k = \delta^{hk} \tag{2.3.3}$$

hold for (2.1.4). In this case, we call (2.1.4) a *doubly extended Euclidean transformation.*

The condition (2.1.8)

$$da_h^l \, \xi^h = 0$$

is satisfied for the II-*geodesic line-elements under doubly extended Euclidean transformations still.*

The $\bar{\xi}$ and ξ^l in (2.1.4) will be called the II-*geodesic rectangular coordinates referred to the* II-*geodesic rectangular* $\xi^{\bar{l}}$– *resp.* ξ^l– *axes.*

The results of Artt. 2.1 and 2.2 hold still and we have

$$\omega^2 = \omega^l \, \omega^l = d\xi^l \, d\xi^l = ds^2 = (c^l \, c^l) \, \omega^2, \qquad (\omega^l = c^l \, \omega), \tag{2.3.4}$$

so that

(2.3.5) $$c^l c^l = 1.$$

2.4. Doubly Extended Euclidean Transformation Group. *Since the doubly extended Euclidean transformation group is obtained evidently as a subgroup of the doubly extended affine transformation group, the following theorem holds still.*

Theorem. *The totality of the doubly extended Euclidean transformattons*

(2.4.1) $$\bar{\bar{\xi}}^h = a^h_k(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\,\xi^k + a^h_o, \qquad (a^h_o = \text{const.},\ \det(a^l_h) \neq 0),$$

whose inverse transformations are

(2.4.2) $$\xi^k = a^k_h(\bar{\xi}, \dot{\bar{\xi}}, \cdots, \overset{(M)}{\bar{\xi}})\,\bar{\xi}^h + a^k_o, \qquad (a^k_o = \text{const.},\ \det(a^k_h) \neq 0)$$

accompanied by (2.3.1) *forms a subgroup of the doubly extended affine transformation group.*

Definition. The geometry belonging to the doubly extended Euclidean transformation group will be called the *doubly extended Euclidean geometry.* (Cf. Part I, § 5, p. 40, II, p. 43. This is the case $M=1$).

N. B. *The detail* may be exposed quite as in T. Takasu, [17], [18].

Duality of Hamiltonian Canonical Formalisms in the Doubly Extended Euclidean Geometry. Since we have introduced the metric tensor $g_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x})$ in (2.3.2) into our *doubly extended Euclidean geometry,* we can establish a *duality of Hamilton's canonical formalisms* quite as in Art. (1.16)′ on p. 22 of Part I of our paper as well as in Art. 6.6. in such a way that the Hamiltonian $\mathscr{H}$ and the Lagrangian $\mathscr{L}$ for (x^λ, y_λ) are the Lagrangian and Hamiltonian for (x_λ, y^λ) respectively.

2.5. The Differential Equations of the Extremals $\delta s = 0$ in terms of $\{{}^{\lambda}_{\mu\nu}\}_{(s)}$, $(s=0, 1, \cdots, M)$ in the Doubly Extended Euclidean Geometry. In the *doubly extended affine geometry,* we have had (1.1.5):

(2.5.1) $$\frac{d^2\xi^l}{d\tau^2} = \frac{d}{d\tau}\frac{\omega^l}{d\tau} = \omega^l_\lambda\left\{\frac{d^2x^\lambda}{d\tau^2} + \sum_{s=0}^{M} \Lambda^\lambda_{\mu\nu(s)}\frac{dx^\mu}{d\tau}\frac{\overset{(s)}{dx^\nu}}{d\tau}\right\} = 0.$$

Now for the *doubly extended Euclidean geometry,* we shall prove the equations:

(2.5.2) $$\begin{aligned}\frac{d^2\xi^l}{ds^2} &\equiv \frac{d}{ds}\frac{\omega^l}{ds} \equiv \omega^l_\lambda\left\{\frac{d^2x^\lambda}{ds^2} + \sum_{s=0}^{M} \Lambda^\lambda_{\mu\nu(s)}\frac{dx^\mu}{ds}\frac{\overset{(s)}{dx^\nu}}{ds}\right\}\\ &\equiv \omega^l_\lambda\left\{\frac{d^2x^\lambda}{ds^2} + \sum_{s=0}^{M} \{{}^{\lambda}_{\mu\nu}\}_{(s)}\frac{dx^\mu}{ds}\frac{\overset{(s)}{dx^\nu}}{ds}\right\} = 0,\end{aligned}$$

where

(2.5.3) $$\left\{ {}^{\sigma}_{\mu\nu} \right\}_{(s)} \overset{\text{def}}{=} \frac{1}{2} g^{\mu\sigma} \left\{ \frac{\partial g_{\lambda\mu}}{\partial \overset{(s)}{s^{\nu}}} + \frac{\partial g_{\lambda\nu}}{\delta \overset{(s)}{x^{\mu}}} - \frac{\partial g_{\mu\nu}}{\partial \overset{(s)}{x^{\lambda}}} \right\}, \qquad (s=0, 1, \cdots, M)$$

with

(2.5.4) $$g^{\lambda\sigma} = \omega^{\lambda}_{l}\, \omega^{\sigma}_{l}, \quad g_{\lambda\sigma} = \omega^{l}_{\lambda}\, \omega^{l}_{\sigma}.$$

Proof of (2.5.2). Let

(2.5.5) $$\boldsymbol{e}_{\mu} = (\omega^{l}_{\mu}\,(x, \dot{x}, \cdots, \overset{(M)}{x})), \qquad (l=1, 2, \cdots, n)$$

be a natural frame and $\boldsymbol{V}$ a vector in the differentiable manifold $\boldsymbol{M}(x, \dot{x}, \cdots, \overset{(M)}{x}) = \bigcup_{\alpha} U_{\alpha}(x_{(\alpha)})$, so that we may put

(2.5.6) $$\delta\, \boldsymbol{e}_{\mu} = \sum_{s=0}^{M} \Gamma^{\lambda}_{\mu\nu(s)}\,(x, \dot{x}, \cdots, \overset{(M)}{x})\, \boldsymbol{e}_{\lambda}.$$

We have

$$\boldsymbol{V} = V^{\mu}\, \boldsymbol{e}_{\mu},$$

$$\boldsymbol{V} + \delta\, \boldsymbol{V} = (V^{\mu} + dV^{\mu})\,(\boldsymbol{e}_{\mu} + \delta \boldsymbol{e}_{\mu}),$$

(2.5.7) $$\delta\, \boldsymbol{V} = dV^{\mu}\, \boldsymbol{e}_{\mu} + V^{\mu}\, \delta \boldsymbol{e}_{\mu}.$$

Introducing $\delta \boldsymbol{e}_{\mu}$ from (2.5.6) into (2.5.7), we have

(2.5.8) $$\delta V^{\lambda}\, \boldsymbol{e}_{\lambda} = \delta\, \boldsymbol{V} = (dV^{\lambda} + \sum_{s=0}^{M} \Gamma^{\lambda}_{\mu\nu(s)}\, d\overset{(s)}{x^{\nu}})\, \boldsymbol{e}_{\lambda}.$$

We set

(2.5.9) $$\delta V^{\lambda} \overset{\text{def}}{=} dV^{\lambda} + \sum_{s=0}^{M} \Gamma^{\lambda}_{\mu\nu(s)}\, V^{\mu}\, d\overset{(s)}{x^{\nu}}.$$

For $\delta V=0$ a parallelism $\delta \boldsymbol{V}=0$ arises.

If we solve the differential equations

(2.5.10) $$\frac{dV^{\lambda}}{ds} + \sum_{s=0}^{M} \Gamma^{\lambda}_{\mu\nu(s)}\, V^{\mu} \frac{d\overset{(s)}{x^{\nu}}}{ds} = 0$$

with the initial condition $V^{\lambda} = V^{\lambda}$ at a point P $(s=s_0)$, the solution $V^{\lambda}(s)$ has *displaced parallel* along the curve $x^{\lambda}(s)$ to another point $Q\,(s=s_1)$ acquiring $V^{\lambda}(s_1)$.

For

(2.5.11) $$V^{\lambda} = dx^{\lambda}/ds,$$

the (2.5.10) becomes to

(2.5.12) $$\frac{d^2x^{\lambda}}{ds^2} + \sum_{s=0}^{M} \Gamma_{\mu\nu(s)} \frac{dx^{\mu}}{ds} \frac{d\overset{(s)}{x^{\nu}}}{ds} = 0.$$

The curves $x^\lambda(s)$ as solutions of (1.5.12) will be called the II-*geodesic curves.*

For the extremal of (2.5.1), we have

$$\frac{\delta V^\lambda}{\delta s}=\frac{\delta}{\delta s}\frac{dx^\lambda}{ds}=\frac{d^2x^\lambda}{ds^2}+\sum_{s=0}^{M}\Gamma^\lambda_{\mu\nu(s)}(x,\dot{x},\cdots,\overset{(M)}{x})\frac{dx^\mu}{ds}\frac{d\overset{(s)}{x^\nu}}{ds}=0. \tag{2.5.13}$$

We are in the situation to determine $\Gamma^\lambda_{\mu\nu(s)}$.

Now we have

$$e_\lambda e_\mu=\omega^l_\lambda(x,\dot{x},\cdots,\overset{(M)}{x})\,\omega^l_\mu(x,\dot{x},\cdots,\overset{(M)}{x})=g_{\lambda\mu}(x,\dot{x},\cdots,\overset{(M)}{x}) \tag{2.5.14}$$

$$g_{\lambda\mu}+\delta g_{\lambda\mu}=(e_\lambda+\delta e_\lambda)(e_\mu+\delta e_\mu)$$

giving

$$\delta e_\lambda\, e_\mu+e_\lambda\,\delta e_\mu=\delta g_{\lambda\mu}. \tag{2.5.15}$$

Introducing (2.5.6), we obtain

$$\sum_{s=0}^{M}\{\Gamma^\sigma_{\lambda\nu(s)}\,d\overset{(s)}{x^\nu}\}\,e_\sigma\, e_\mu+\sum_{s=0}^{M}\{\Gamma^\sigma_{\mu\nu(s)}\,d\overset{(s)}{x^\nu}\}\,e_\lambda\, e_\sigma=\delta g_{\lambda\mu}=\sum_{s=0}^{M}\frac{\partial g_{\lambda\mu}}{\partial\overset{(s)}{x^\nu}}\,d\overset{(s)}{x^\nu}$$

for all values of $d\overset{(s)}{x^\nu}$, so that

$$g_{\sigma\mu}\Gamma^\sigma_{\lambda\nu(s)}+g_{\lambda\sigma}\Gamma^\sigma_{\mu\nu(s)}=\frac{\partial g_{\lambda\mu}}{\partial\overset{(s)}{x^\nu}}. \tag{2.5.16}$$

Interchanging μ and ν in (2.5.16),

$$g_{\sigma\nu}\Gamma^\sigma_{\lambda\mu(s)}+g_{\lambda\sigma}\Gamma^\sigma_{\nu\mu(s)}=\frac{\partial g_{\lambda\nu}}{\partial\overset{(s)}{x^\mu}}. \tag{2.5.17}$$

Interchanging λ and μ,

$$g_{\sigma\nu}\Gamma^\sigma_{\mu\lambda(s)}+g_{\mu\sigma}\Gamma^\sigma_{\nu\lambda(s)}=\frac{\partial g_{\mu\nu}}{\partial\overset{(s)}{x^\lambda}}. \tag{2.5.18}$$

Forming (2.5.16)+(2.5.17)−(2.5.18) and taking $\Gamma^\sigma_{\lambda\mu(s)}=\Gamma^\sigma_{\mu\lambda(s)}$ into account,

$$2g_{\sigma\lambda}\Gamma^\sigma_{\mu\nu(s)}=\frac{\partial g_{\lambda\mu}}{\partial\overset{(s)}{x^\nu}}+\frac{\partial g_{\lambda\nu}}{\partial\overset{(s)}{x^\mu}}-\frac{\partial g_{\mu\nu}}{\partial\overset{(s)}{x^\lambda}}. \tag{2.5.19}$$

Multiplying this with $\frac{1}{2}g_{\lambda\sigma}$ and replacing $\Gamma^\sigma_{\mu\nu(s)}$ by $\{{}^{\sigma}_{\mu\nu}\}_{(s)}$, we obtain

$$\left\{{}^{\sigma}_{\mu\nu}\right\}_{(s)}=\tfrac{1}{2}g^{\lambda\sigma}\left(\frac{\partial g_{\lambda\mu}}{\partial\overset{(s)}{x^\nu}}+\frac{\partial g_{\lambda\nu}}{\partial\overset{(s)}{x^\mu}}-\frac{\partial g_{\mu\nu}}{\partial\overset{(s)}{x^\lambda}}\right). \tag{2.5.20}$$

Thus (2.5.13) gives

$$\frac{\delta}{\delta s}\frac{dx^\lambda}{ds}=\frac{d^2x^\lambda}{ds^2}+\sum_{s=0}^{M}\{{}^{\lambda}_{\mu\nu}\}_{(s)}(x,\dot{x},\cdots,\overset{(M)}{x})\frac{dx^\mu}{ds}\frac{\overset{(s)}{dx^\nu}}{ds}. \tag{2.5.21}$$

In (2.5.1), we have already seen

$$\frac{d}{ds}\frac{\omega^l}{ds}\equiv\omega^l_\lambda\left\{\frac{d^2x^\lambda}{ds^2}+\sum_{s=0}^{M}\Lambda^\lambda_{\mu\nu(s)}\frac{d^2x^\lambda}{ds^2}+\sum_{s=0}^{M}\Lambda^\lambda_{\mu\nu(s)}\frac{dx^\mu}{ds}\frac{\overset{(s)}{dx^\nu}}{ds}\right\}. \tag{2.5.22}$$

Now in (2.5.4), we have seen

$$g_{\mu\nu}(x,\dot{x},\cdots,\overset{(M)}{x})=\omega^l_\mu(x,\dot{x},\cdots,\overset{(M)}{x})\,\omega^l_\nu(x,\dot{x},\cdots,\overset{(M)}{x}), \tag{2.5.23}$$

$$g^{\mu\nu}(x,\dot{x},\cdots,\overset{(M)}{x})=\omega^\mu_l(x,\dot{x},\cdots,\overset{(M)}{x})\,\omega^\nu_l(x,\dot{x},\cdots,\overset{(M)}{x}). \tag{2.5.24}$$

Hence

$$\begin{aligned}\{{}^{\lambda}_{\mu\nu}\}_{(s)}&=\tfrac{1}{2}g^{\lambda\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial\overset{(s)}{x^\nu}}+\frac{\partial g_{\sigma\nu}}{\partial\overset{(s)}{x^\mu}}-\frac{\partial g_{\mu\nu}}{\partial\overset{(s)}{x^\sigma}}\right)\\&=\tfrac{1}{2}\omega^\lambda_h\,\omega^\sigma_h\left(\frac{\partial\omega^l_\mu}{\partial\overset{(s)}{x^\nu}}\omega^l_\sigma+\omega^l_\mu\frac{\partial\omega^l_\sigma}{\partial\overset{(s)}{x^\nu}}+\frac{\partial\omega^l_\sigma}{\partial\overset{(s)}{x^\mu}}\omega^l_\nu+\omega^l_\sigma\frac{\partial\omega^l_\nu}{\partial\overset{(s)}{x^\mu}}\right.\\&\qquad\left.-\frac{\partial\omega^l_\mu}{\partial\overset{(s)}{x^\sigma}}\omega^l_\nu-\omega^l_\mu\frac{\partial\omega^l_\nu}{\partial\overset{(s)}{x^\sigma}}\right),\end{aligned}$$

$$\begin{aligned}\{{}^{\lambda}_{\mu\nu}\}_{(s)}&=\tfrac{1}{2}(\Lambda^\lambda_{\mu\nu(s)}+\Lambda^\lambda_{\nu\mu(s)})+\tfrac{1}{2}\omega^\lambda_h\,\omega^\sigma_h\left\{\omega^l_\mu\left(\frac{\partial\omega^l_\sigma}{\partial\overset{(s)}{x^\nu}}-\frac{\partial\omega^l_\nu}{\partial\overset{(s)}{x^\sigma}}\right)\right.\\&\qquad\left.+\omega^l_\nu\left(\frac{\partial\omega^l_\sigma}{\partial\overset{(s)}{x^\mu}}-\frac{\partial\omega_\mu}{\partial\overset{(s)}{x^\sigma}}\right)\right\},\quad(s=0,1,\cdots,M).\end{aligned} \tag{2.5.25}$$

Let us show

$$\omega^\sigma_h\left\{\omega^l_\mu\left(\frac{\partial\omega^l_\sigma}{\partial\overset{(s)}{x^\nu}}-\frac{\partial\omega^l_\nu}{\partial\overset{(s)}{x^\sigma}}\right)+\omega^l_\nu\left(\frac{\partial\omega^l_\sigma}{\partial\overset{(s)}{x^\mu}}-\frac{\partial\omega^l_\mu}{\partial\overset{(s)}{x^\sigma}}\right)\right\}dx^\mu\,\overset{(s)}{dx^\nu}=0. \tag{2.5.26}$$

(2.5.27) The left-hand side $=\omega^\sigma_h\,\omega^l_\nu\left(\frac{\partial\omega^l_\alpha}{\partial\overset{(s)}{x^\nu}}-\frac{\partial\omega^l_\nu}{\partial\overset{(s)}{x^\sigma}}\right)\overset{(s)}{dx^\nu}$

$$+\omega^\sigma_h\,\omega^l_\nu\left(\frac{\partial\omega^l_\sigma}{\partial\overset{(s)}{x^\mu}}-\frac{\partial\omega^l_\mu}{\partial\overset{(s)}{x^\sigma}}\right)dx^\mu\,\overset{(s)}{dx^\nu}.$$

Before all, we see

$$\overset{ls)}{\omega^h}\overset{\text{def}}{=}\omega^h_\nu\,\overset{(s)}{dx^\nu}\rightarrow\overset{(s)}{dx^\nu}=\omega^\nu_h\,\overset{(s)}{\omega^h}, \tag{2.5.28}$$

$$\frac{\partial\omega^l_\nu}{\partial\overset{(s)}{x^\sigma}}\omega^\sigma_h=\frac{\partial\omega^l_\nu}{\overset{(s)}{\omega^h}}, \tag{2.5.29}$$

where the last equality is verified by multiplication of the numerator and the denominator of the left–hand side with ω^h_σ.

The first term on the right–hand side of (2.5.27) is

$$\omega^\sigma_h \omega^l_\nu \left(\frac{\partial \omega^l_\sigma}{\partial \overset{(s)}{x}{}^\nu} - \frac{\partial \omega^l_\nu}{\partial \overset{(s)}{x}{}^\sigma}\right)$$

$$= \omega^l \left(\omega^\sigma_h \frac{\partial \omega^l_\sigma}{\partial \overset{(s)}{x}{}^\nu} d\overset{(s)}{x}{}^\nu - \frac{\partial \omega^l_\nu}{\partial \overset{(s)}{x}{}^\sigma} \omega^\sigma_h d\overset{(s)}{x}{}^\nu\right)$$

$$= \omega^l \left(\omega^\sigma_h \frac{\partial \omega^l_\sigma}{d\overset{(s)}{x}{}^\nu} d\overset{(s)}{x}{}^\nu - \frac{\partial \omega^l_\nu}{\partial \overset{(s)}{x}{}^\sigma} \omega^\sigma_h \omega^\nu_h \overset{(s)}{\omega}{}^h\right)$$

$$= \omega^l \left(\omega^\sigma_h \frac{\partial \omega^l_\sigma}{\partial \overset{(s)}{x}{}^\nu} d\overset{(s)}{x}{}^\nu - \omega^\nu_h \frac{\partial \omega^l_\nu}{\partial \overset{(s)}{x}{}^\sigma} d\overset{(s)}{x}{}^\sigma\right)$$

The second term on the right–hand side of (2.5.27) is

$$\omega^\sigma_h \omega^l_\nu \left(\frac{\partial \omega^l_\sigma}{\partial \overset{(s)}{x}{}^\mu} \frac{\partial \omega^l_\mu}{\partial \overset{(s)}{x}{}^\sigma}\right) dx^\mu d\overset{(s)}{x}{}^\nu$$

$$= \overset{(s)}{\omega}{}^h \omega^l \left(\omega^\sigma_h \omega^\mu_h \frac{\partial \omega^l_\sigma}{\partial \overset{(s)}{x}{}^\mu} - \omega^\mu_h \omega^\sigma_h \frac{\partial \omega^l_\mu}{\partial \overset{(s)}{x}{}^\sigma}\right)$$

$$= \overset{(s)}{\omega}{}^h \omega^l \left(\frac{\partial \omega^l_\sigma}{\overset{(s)}{\omega}{}^h} \omega^\nu_h - \omega^\mu_h \omega^\sigma_h \frac{\partial \omega^l_\mu}{\partial \overset{(s)}{x}{}^\sigma}\right)$$

$$= \overset{(s)}{\omega}{}^h \omega^l \left(\frac{\partial \omega^l_\sigma}{\overset{(s)}{\omega}{}^h} \omega^\sigma_h - \frac{\partial \omega^l_\mu}{\overset{(s)}{\omega}{}^h} \omega^\mu_h\right)$$

$$= 0.$$

From (2.5.25) and (2.5.26), we see that

$$\text{(2.5.30)} \qquad \{{}^{\lambda}_{\mu\nu}\}_{(s)}\, dx^\mu d\overset{(s)}{x}{}^\nu = \Lambda^\lambda_{\mu\nu(s)}\, dx^\mu d\overset{(s)}{x}{}^\nu, \qquad (s=0,1,\cdots,M).$$

Introducing (2.5.30) into (2.5.21), we obtain

$$\text{(2.5.31)} \quad \frac{\delta}{\delta s}\frac{dx^\lambda}{ds} = \frac{d^2x^\lambda}{ds^2} + \sum_{s=0}^{M} \Lambda^\lambda_{\mu\nu(s)}(x, \dot{x}, \cdots, \overset{(M)}{x}) \frac{dx^\mu}{ds}\frac{d\overset{(s)}{x}{}^\nu}{ds} = \frac{d^2x^\lambda}{ds^2} + \overline{\Lambda}{}^\lambda_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x})\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

by (1.1.23).

Comparing (2.5.21), (2.5.31) with (2.5.22), we obtain (2.5.2). Q. E. D.

If we set

$$\text{(2.5.32)} \qquad \overline{\{{}^{\lambda}_{\mu\sigma}\}} \overset{\text{def}}{=} \sum_{s=0}^{M} \{{}^{\lambda}_{\mu\sigma}\}_{(s)} \overset{(s+1)}{x}{}^\sigma/\dot{x}^\nu,$$

we have

$$\text{(2.5.33)} \qquad \{{}^{\lambda}_{\mu\nu}\}\, dx^\mu dx^\nu = \sum_{s=0}^{M} \{{}^{\lambda}_{\mu\sigma}\}_{(s)} (\overset{(s+1)}{x}{}^\sigma/\dot{x}^\nu)\, dx^\mu dx^\nu,$$

$$\text{(2.5.34)} \qquad \frac{\delta}{\delta s}\frac{dx^\lambda}{ds} = \frac{d^2x^\lambda}{ds^2} + \sum_{(s=0)}^{M} \{{}^{\lambda}_{\mu\nu}\}_{(s)}(x, \dot{x}, \cdots, \overset{(M)}{x})\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$$

$$= \frac{d^2 x^\lambda}{ds^2} + \overline{\{{}^{\lambda}_{\mu\nu}\}}(x, \dot{x}, \cdots, \overset{(M)}{x})\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}.$$

Second proof for (2.5.26). We proceed as follows.

$$\omega_k^\lambda\, \delta_k^p = \omega_p^\lambda,$$

$$\omega_k^\lambda\, (\overset{(s)}{\omega}{}^p / \overset{(s)}{\omega}{}^k) = \omega_p^\lambda,$$

$$\omega_k^\lambda\, (\overset{(s)}{\omega}{}^p / \overset{(s)}{\omega}{}^k)\, \overset{(s)}{\omega}{}^h = \omega_p^\lambda\, \overset{(s)}{\omega}{}^h,$$

$$\omega_k^\lambda\, (\overset{(s)}{\omega}{}^h / \overset{(s)}{\omega}{}^k)\, \overset{(s)}{\omega}{}^p = \omega_p^\lambda\, \overset{(s)}{\omega}{}^h,$$

$$\omega_k^\lambda\, \delta_k^h\, \overset{(s)}{\omega}{}^p = \omega_p^\lambda\, \overset{(s)}{\omega}{}^h.$$

$$\omega_h^\lambda\, \overset{(s)}{\omega}{}^p = \omega_p^\lambda\, \overset{(s)}{\omega}{}^h,$$

$$\omega_h^\lambda\, \overset{(s)}{\omega}{}^p\, (\omega_h^\sigma\, \omega_p^\nu) = \omega_p^\lambda\, \overset{(s)}{\omega}{}^h\, (\omega_h^\sigma\, \omega_p^\nu).$$

$$\omega_h^\lambda\, \omega_h^\sigma\, \omega_p^\nu\, \overset{(s)}{\omega}{}^p = \omega_p^\lambda\, \omega_p^\nu\, \omega_h^\sigma\, \overset{(s)}{\omega}{}^h,$$

$$(*) \qquad \omega_h^\lambda\, \omega_h^\sigma\, d\overset{(s)}{x}{}^\nu = \omega_h^\lambda\, \omega_h^\nu\, d\overset{(s)}{x}{}^\sigma,$$

$$\begin{aligned} \omega_h^\lambda\, \omega_h^\sigma\, (\partial \omega_\nu^l / \partial \overset{(s)}{x}{}^\sigma)\, d\overset{(s)}{x}{}^\nu &= \omega_h^\lambda\, \omega_h^\nu\, (\partial \omega_\nu^l / \partial \overset{(s)}{x}{}^\sigma)\, d\overset{(s)}{x}{}^\sigma \\ &= \omega_k^\lambda\, \omega_k^\sigma\, (\partial \omega_\sigma^l / \partial \overset{(s)}{x}{}^\nu)\, d\overset{(s)}{x}{}^\nu. \end{aligned}$$

Hence, by multiplication with $\omega^l \omega_\lambda^h$,

$$\omega_h^\sigma\, \omega^l \{(\partial \omega_\nu^l / \partial \overset{(s)}{x}{}^\sigma)\, d\overset{(s)}{x}{}^\nu - (\partial \omega_\sigma^l / \partial \overset{(s)}{x}{}^\sigma)\, d\overset{(s)}{x}{}^\nu\} = 0,$$

so that

the first term on the right-hand side of (2.5.27)$=0$,

and similarly,

the second term on the right-hand side of (2.5.27) $= \omega_h^\sigma\, \omega_\nu^l \left(\dfrac{\partial \omega_\sigma^l}{\partial x^\mu} - \dfrac{\partial \omega_\mu^l}{\partial x^\sigma} \right) dx^\mu\, d\overset{(s)}{x}{}^\nu = 0,$

for, from (*), by multiplication with ω_λ^h, we have

$$\omega_h^\sigma\, d\overset{(s)}{x}{}^\nu = \omega_h^\nu\, d\overset{(s)}{x}{}^\sigma,$$

$$\omega_h^\sigma\, (\partial \omega_\nu^l / \partial \overset{(s)}{x}{}^\sigma)\, d\overset{(s)}{x}{}^\nu = \omega_h^\nu\, (\partial \omega_\nu^l / \partial \overset{(s)}{x}{}^\sigma)\, d\overset{(s)}{x}{}^\sigma = \omega_h^\sigma\, (\partial \omega_\sigma^l / \partial \overset{(s)}{x}{}^\nu)\, d\overset{(s)}{x}{}^\nu,$$

what proves (2.5.26).

2.6. The Differential Equations of the Extremals $ds=0$ in terms of $\{{}_{\mu\nu}^{\lambda}\}_{(s)}$, $(s=0,1,\cdots,M)$ in the Doubly Extended Euclidean Geometry, in Part I.

For this case, we will show before all that

$$\text{(2.6.1)} \qquad \{{}_{\mu\nu}^{\sigma}\}_{(s)} \frac{dx^\mu}{ds} = 0, \qquad (s=1,2,\cdots,M).$$

Proof.
$$\begin{aligned} \{{}_{\mu\nu}^{\sigma}\}_{(s)}\, \dot{x}^\mu &= \tfrac{1}{2} g^{\lambda\sigma} \left(\frac{\partial g_{\lambda\mu}}{\partial \overset{(s)}{x}{}^\nu} + \frac{\partial g_{\lambda\nu}}{\partial \overset{(s)}{x}{}^\mu} - \frac{\partial g_{\mu\nu}}{\partial \overset{(s)}{x}{}^\lambda} \right) \dot{x}^\mu \\ &= \tfrac{1}{2} g^{\lambda\sigma} \left(\dot{x}^\mu \frac{\partial g_{\lambda\mu}}{\partial \overset{(s)}{x}{}^\nu} + \frac{\partial g_{\lambda\nu}}{\partial \overset{(s)}{x}{}^\mu}\, \dot{x}^\mu - \frac{\partial g_{\mu\nu}}{\partial \overset{(s)}{x}{}^\lambda}\, \dot{x}^\mu \right). \end{aligned}$$

Now, when $g_{\mu\nu}$ are positively homogeneous of zero degree in $\dot{x}$, we have

$$\text{(2.6.2)} \qquad \dot{x}^{\mu}\frac{\partial g_{\mu\nu}}{\underset{(s)}{\partial x^{\lambda}}}=0, \qquad \Bigg| \qquad \dot{x}^{\mu}\frac{\partial g_{\lambda\nu}}{\underset{(s)}{\partial x^{\mu}}}=0,$$

$$(s=1, 2, \cdots, M),$$

the partial derivatives themselves being positively homogeneous of zero degree in $\dot{x}$, whence follows (2.6.1). Q. E. D.

Next let us prove

$$\text{(2.6.3)} \qquad \frac{d^2\xi^l}{ds^2}=\omega^l_{\lambda}\left\{\frac{d^2x^{\lambda}}{ds^2}+\Lambda^{\lambda}_{\mu\nu(0)}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}\right\}\equiv\omega^l_{\lambda}\left\{\frac{d^2x^{\lambda}}{ds^2}+\left\{{}^{\lambda}_{\mu\nu}\right\}_{(0)}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}\right\}$$

in the case, where $g_{\mu\nu}$ *and* ω^l_{ν} *are homogeneous of zero degree in* $\dot{x}^{\mu}$ *and* $M\geqq 1$ i. e. $\mathcal{L}(x, \dot{x}, \cdots, \overset{(M)}{x})$ *is homogeneous of one degree in* $\dot{x}^{\mu}$.

From (2.6.1) and (2.5.21), it follows that

$$\text{(2.6.4)} \qquad \frac{\delta}{\delta s}\frac{dx^{\lambda}}{ds}=\frac{d^2x^{\lambda}}{ds^2}+\left\{{}^{\sigma}_{\mu\nu}\right\}_{(0)}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds},$$

which becomes by (2.5.27) to

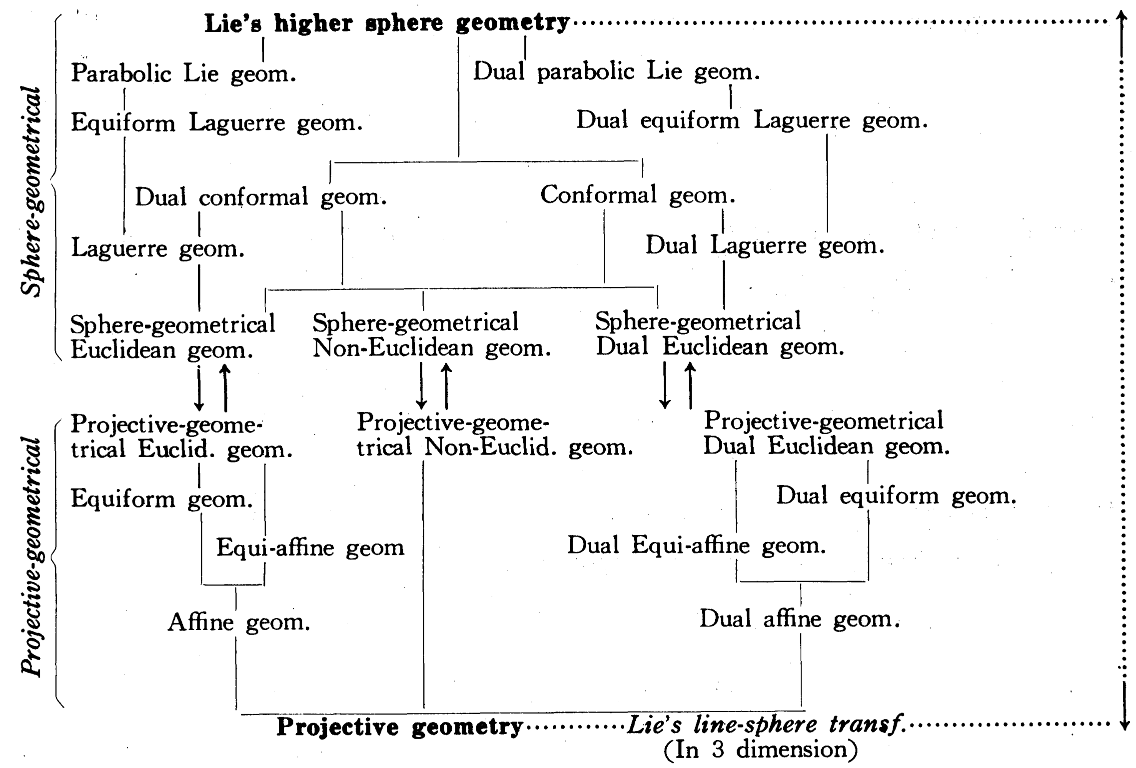
$$\text{(2.6.5)} \qquad \frac{\delta}{\delta s}\frac{dx^{\lambda}}{ds}=\frac{d^2x^{\lambda}}{ds^2}+\Lambda^{\lambda}_{\mu\nu(0)}\frac{dx^{\mu}}{ds}\frac{dx^{\nu}}{ds}.$$

From (2.6.4) and (2.6.5), we obtain (2.6.3). Q. E. D.

§ 3. Other (Doubly) Extended Geometries by Hamilton's Canonical Formalisms as Non-Connection Methods.

3.1. Other (Doubly) Extended Geometries by Hamilton's Canonical Formalisms as Non-Connection Methods. *The* 22 *branches of the classical geometry in the sense of "Erlanger Programm" of F. Klein enlisted below* may be doubly extended in two ways (cf. I and II, Art. 1.1.) quite as in the case of simply extending, directly or indirectly (cf. T. Takasu, [16–28]) even in the case[3] of Art. 1.1:

3) The branches obtained in this § 3 are different from those of III, p. 44 of Part I in that here ω_{μ} are not always homogeneous of zero degree in $\dot{x}$.



Thus we discovered 44 *new branches of geometry*. The details are under preparation.

§ 4. Local Geometry of Finsler–Craig–Synge–Kawaguchi Spaces.

4.1. Synge's Vectors and Metric Tensors in the Kawaguchi Space. Synge and Craig called the metric apace in which the arc length s along a parametrized curve $x^\lambda = x^\lambda(\tau)$, $(\lambda = 1, 2, \cdots, n)$ is given by the integral

$$s = \int_{\tau_0}^{\tau_1} F(x, \dot{x}, \cdots, \overset{(M)}{x})\, d\tau, \qquad (\dot{x} = dx/d\tau, \text{ etc.}) \tag{4.1.1}$$

the Kawaguchi space of order M as was stated under IV, p. 44 of Part I of this paper. Thereby it was assumed that (4.1.1) shall satisfy the so-called Zermero's conditions:

$$\begin{cases} \Delta_1 F = \sum_{s=1}^{M} s \overset{(s)}{x}{}^\lambda F_{(s)\lambda} = F, \\ \Delta_K F = \sum_{s=K}^{M} \binom{s}{K} \overset{(s-K+1)}{x}{}^\lambda F_{(s)\lambda} = 0, \qquad (K = 2, 3, \cdots; M) \end{cases} \tag{4.1.2}$$

for the invariancy (intrinsicality) under the parameter transformations, where

$$F_{(s)\lambda} = \partial F / \partial \overset{(s)}{x}{}^\lambda. \tag{4.1.3}$$

Owing to A. Kawaguchi [1] and H. Hombu, (4.1.2) can be reduced to the three conditions

(4.1.4) $$\Delta_1 F = F, \quad \Delta_2 F = 0, \quad \Delta_3 F = 0.$$

The metric tensor $g_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x})$ for (4.1.1) was written out on p. 44 of our Part I as (IV. 2) (cf. M. Kawaguchi, [1], p. 724):

(4.1.5)[4] $$*g_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x}) = MF^{2m} F_{(M)\mu} F_{(M)\nu} + *\overset{M}{\mathfrak{C}} *\overset{M}{\mathfrak{C}} + *\overset{1}{\mathfrak{C}} *\overset{1}{\mathfrak{C}},$$

to which I should have supplemented A. Kawaguchi's another form (cf. ibid, (4.4)):

(4.1.6) $$g_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x}) = MF^{2M-1} F_{(M)\mu} F_{(M)\nu} + \overset{M}{\mathfrak{C}}_\mu \overset{M}{\mathfrak{C}}_\nu + \overset{1}{\mathfrak{C}}_\mu \overset{1}{\mathfrak{C}}_\nu.$$

The quadratic form

(4.1.7)	$ds^2 = g_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x})\, dx^\mu\, dx^\nu$	$d\bar{s}^2 = *g_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x})\, dx^\mu\, dx^\nu$

is always expressible in the form

(4.1.8)	$ds^2 = \omega^l\, \omega^l,$	$d\bar{s}^2 = *\omega^l\, *\omega^l,$

$$(l = 1, 2, \cdots\cdots, n)$$

but for undergoing doubly extended transformations of the types:

(4.1.9)	$(a^l_h(x, \dot{x}, \cdots, \overset{(M)}{x}), a^l_o);$	$(*a^l_h(x, \dot{x}, \cdots \overset{(M)}{x}), *a^l_o),$

where

(4.1.10)	$(a^l_h(x, \dot{x}, \cdots, \overset{(M)}{x}))$	$(*a^l_h(x, \dot{x}, \cdots, \overset{(M)}{x}))$

is an orthogonal matrix and

(4.1.11)	$\omega^l = \omega^l_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})\, dx^\mu,$	$(*\omega^l = *\omega^l_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})\, dx^\mu,$

the (ω^l)	the (ω^l_μ)	the (ω_μ)	the $(*\omega^l)$	the $(*\omega^l_\mu)$	the $(*\omega_\mu)$

being the components of the vector

ω	ω_μ	ω^l	ω	$*\omega$	$*\omega_\mu$	$*\omega^l$	$*\omega_\mu$

and thus *we could have proceeded quite as in* § 2.

But, in this paper, I will give *a present author's new procedure* (cf. §5– §19, §13– §23 being in preparation).

It should be noted that the expressibility of ds^2 in the forms (4.1.7), (4.1.8), (6.3.1),

4) This is a corrected form. Another quadratic form is found on p. 6 of A. Kawaguchi, [1].

(6.3.5) and that of the foot-note[4] of Art. 4.1. is possible but for undergoing doubly extended transformations of the types (4.1.9), *thus the above different quadratic forms must be transformable into one another by a transformation of the type* (4.1.9).

§ 5. TT-Geometry as a Global Geometry of Finsler-Craig-Synge-Kawaguchi Spaces in the Large in Differentiable Manifolds.

5.1. Finsler-Craig-Synge-Kawaguchi Spaces in the Large in Differrentiable Manifolds. *The Finsler-Craig-Synge-Kawaguchi space is originarily of local nature.* I am now in the situation *to bring it into one in the large.* For this purpose, we consider an *n-dimensional differentiable manifold*

$$\bigcup_{\alpha} U_{\alpha}(x^{\lambda}{}_{(\alpha)}), \qquad (\lambda=1,2,\cdots,n;\ \alpha=1,2,\cdots) \tag{5.1.1}$$

(in current notation (cf. Art. 1.1.) *of class* C^v,

v=positive integer	$v=\infty$,	$v=\omega$,

where U_α is an open domain of the local coordinates $(x^{\lambda}_{(\alpha)})$.

Let (4.1.1):

$$s=\int_{\tau_0}^{\tau} F(x_{(\alpha)},\dot{x}_{(\alpha)},\cdots,\overset{(M)}{x}_{(\alpha)})\,d\tau=\int_{\tau_0}^{\tau} F(x_{(\beta)},\dot{x}_{(\beta)},\cdots,\overset{(M)}{x}_{(\beta)})\,d\tau \tag{5.1.2}$$

be defined in the domain $U_\alpha \cap U_\beta$.

The *Euler-Lagrange equations* of the extremal problem

$$\delta s=\delta\int_{\tau_0}^{\tau} F(x_{(\alpha)},\dot{x}_{(\alpha)},\cdots,\overset{(M)}{x}_{(\alpha)})\,d\tau=\dot{0} \tag{5.1.3}$$

are

$$\sum_{s=0}^{M}(-1)^s\frac{d^s}{d\tau^s}F_{(s)\mu}=\frac{\partial F}{\partial x^{\mu}_{(\alpha)}}-\frac{d}{d\tau}\Big[\frac{\partial F}{\partial \dot{x}^{\mu}_{(\alpha)}}-\frac{d}{d\tau}\frac{\partial F}{\partial \ddot{x}^{\mu}_{(\alpha)}}+\cdots +(-1)^M\frac{d^{M-1}}{d\tau^{M-1}}\frac{\partial F}{\partial \overset{(M)}{x}{}^{\mu}_{(\alpha)}}\Big]=0. \tag{5.1.4}$$

Define a covariant vector $\omega_\mu(x_{(\alpha)},\dot{x}_{(\alpha)},\cdots,\overset{(M)}{x}_{(\alpha)})$ by (this point seems to be initiative of the present author)

$$\omega_\mu \overset{\text{def}}{=} \sum_{s=1}^{M}(-1)^{s-1}F_{(s)\mu}\equiv\frac{\partial F}{\partial \dot{x}^{\mu}}-\frac{d}{d\tau}\frac{\partial F}{\partial \ddot{x}^{\mu}}+\cdots+(-1)^{M-1}\frac{d^{M-1}}{d\tau^{M-1}}\frac{\partial F}{\partial \overset{(M)}{x}{}^{\mu}}, \tag{5.1.5}$$

and let its components be

(5.1.6) $$\omega^l_\mu(x_{(\alpha)}, \dot{x}_{(\alpha)}, \cdots, \overset{(M)}{x}_{(\alpha)}) = \omega^l_\mu(x_{(\beta)}, \dot{x}_{(\beta)}, \cdots, \overset{(M)}{x}_{(\beta)}), \ (l=1, 2, \cdots, n) \text{ in } U_\alpha \cap U_\beta.$$

Let us now proceed in two ways I and II below.

I. Define the *Lagrangian* by

(5.1.7) $$\dot{\sigma} = \mathscr{L}(x, \dot{x}, \cdots, \overset{(M)}{x}) \overset{\text{def}}{=} \omega_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})\, \dot{x}^\mu$$

in $\bigcup_\alpha U_\alpha$. *Since* (5.1.7) *is written in an invariant form,* (5.1.7) *is global* in $\bigcup_\alpha U_\alpha$.

From (5.1.7), we obtain

(5.1.8) $$\frac{\partial \mathscr{L}}{\partial \omega_\mu} = \dot{x}_\mu.$$

The *Euler–Lagrange equations* (5.1.4) *for the extremals*

(5.1.9) $$\delta\sigma = \delta \int_{\tau_0}^{\tau} \mathscr{L}\, d\tau = 0$$

become now to

(5.1.10) $$\frac{\partial \mathscr{L}}{\partial x^\mu} = \dot{\omega}_\mu = \frac{d}{d\tau}\left(\frac{\partial F}{\partial \dot{x}^\mu} - \frac{d}{d\tau}\frac{\partial F}{\partial \ddot{x}^\mu} + \cdots + (-1)^{M-1}\frac{d^{M-1}}{d\tau^{M-1}}\frac{\partial F}{\partial \overset{(M)}{x}{}^\mu}\right)$$

on one hand and the *Euler–Lagrange equations* for the extremals (5.1.3), become

(5.1.11) $$\frac{\partial F}{\partial x^\mu} = \dot{\omega}_\mu$$

on the other hand. Thus we have

(5.1.12) $$\frac{\partial \mathscr{L}}{\partial x^\mu} = \dot{\omega}_\mu = \frac{\partial F}{\partial x^\mu}.$$

The (5.1.8) and the (5.1.10) constitute the *Lagrange's canonical equations for the extremals* (5.1.9):

(5.1.13) $$\frac{\partial \mathscr{L}}{\partial \omega_\mu} = \dot{x}^\mu, \quad \frac{\partial \mathscr{L}}{\partial x^\mu} = \dot{\omega}_\mu.$$

In order to establish $(5.1.14)_1$ below, which, together with $(5.1.11) = (5.1.14)_2$, constitutes *Lagrange's canonical equations*

(5.1.14) $$\frac{\partial F}{\partial \omega^\mu} = \dot{x}^\mu, \quad \frac{\partial F}{\partial x^\mu} = \dot{\omega}_\mu,$$

we consider (cf. (4.1.5), (4.1.6)):

(5.1.15) $$\dot{s}^2=F^2={}^*g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=y_\nu\dot{x}^\nu,$$

where

(5.1.16) $$y_\nu(x,\dot{x},\cdots,\overset{(M)}{x})={}^*g_{\mu\nu}(x,\dot{x},\cdots,\overset{(M)}{x})\dot{x}^\mu.$$

For $\delta\int_{\tau_0}^{\tau}F^2\,d\tau=0$, the *Lagrange's canonical equations* are

(5.1.17) $$\frac{\partial F^2}{\partial y_\mu}=\dot{x}^\mu,\ \frac{\partial F^2}{\partial x^\mu}=\dot{y}_\mu=\sum_{s=1}^{M}(-1)^{s-1}\frac{d^{s-1}}{d\tau^{s-1}}F^2_{(s)\mu}.$$

Now, by virtue of (5.1.11) = $(5.1.14)_2$, we have

$$\dot{y}_\mu=\frac{\partial F^2}{\partial x^\mu}=2F\frac{\partial F}{\partial x^\mu}=2F\,\dot{\omega}_\mu,$$

(5.1.18) $$\dot{y}_\mu=2F\,\dot{\omega}_\mu=2\dot{s}\,\dot{\omega}_\mu.$$

Hence $(5.1.17)_1$ gives

$$\frac{\partial F^2}{\partial y_\mu}=\frac{1}{2F}\frac{\partial F^2}{\omega_\mu}=\frac{\partial F}{\partial\omega_\mu}=\dot{x}^\mu,$$

what establishes $(5.1.14)_1$. Q. E. D.

If we set

(5.1.19) $$F=\mathcal{L}+\phi=\omega_\mu\dot{x}^\mu+\phi,$$

then (5.1.13) and (5.1.12) tell us that

(5.1.20) $$\frac{\partial p}{\partial\omega_\mu}=0,\quad \frac{\partial\phi}{\partial x^\mu}=0,$$

so that

(5.1.21) $$\phi=\phi(\dot{x},\ddot{x},\cdots,\overset{(M)}{x}).$$

The (5.1.19) *is a kind of canonical transformation* (cf. Def., p. 31 of Part I. Here we have $d\Psi=\phi(\dot{x},\ddot{x},\cdots,\overset{(M)}{x})\,d\tau$.)

Proceeding quite as in the case of (1.1.7) and (1.1.9), from

(5.1.13),	(5.1.14),

we obtain the *Hamilton's canonical equations*

(5.1.22) $$\frac{\partial\mathcal{H}}{\partial\omega_\mu}=\dot{x}^\mu,\quad \frac{\partial\mathcal{H}}{\delta x^\mu}=-\dot{\omega}_\mu,$$

adopting $\mathscr{H}$ such that

(5.1.23) $$\delta\mathscr{H}=\dot{x}^{\mu}\,\delta\omega_{\mu}-\delta x^{\mu}\,\dot{\omega}_{\mu}$$

for the extremals (II-geodesic curves) $\delta s=0$. Thus *to* (5.1.13) *and* (5.1.14), *there corresponds one and the same set of Hamilton's canonical equations* (5.1.22), what justifies the following theorem, which is obtained also by comparing (5.1.13) and (5.1.14).

Theorem. (5.1.9) *and* (5.1.3) *are one and the same extremal problem*: $\delta s=\delta\sigma=0$.

Quite as in the case of (1.1.12), we obtain

(5.1.24) $$\delta\mathscr{L}+\delta\mathscr{H}=\delta\,(\omega_{\mu}\,\dot{x}^{\mu})=\delta\dot{\sigma}.$$

II. We define the parameters of teleparallelism types

(5.1.25) $$\Lambda^{\lambda}_{\mu\nu(s)},\qquad (s=0,1,\cdots,M)$$

by

(5.1.26) $$d\omega_{\mu}-\sum_{s=0}^{M}\Lambda^{\lambda}_{\mu\nu(s)}\,\omega_{\lambda}\,\overset{(s)}{dx^{\nu}}=0 \qquad\Bigg|\qquad d\omega^{\lambda}+\sum_{s=0}^{M}\Lambda^{\lambda}_{\mu\nu(s)}\,\omega^{\mu}\,\overset{(s)}{dx^{\nu}}=0$$

for the ω_{μ} above and ω^{λ} defined by

(5.1.27) $$\omega^{\lambda}\,\omega_{\mu}=\delta^{\lambda}_{\mu},$$

so that

(5.1.28) $$\Lambda^{\lambda}_{\mu\nu(s)}=\omega^{\lambda}\frac{\partial\omega_{\mu}}{\partial\overset{(s)}{x^{\nu}}}\equiv-\omega_{\mu}\frac{\partial\omega^{\lambda}}{\partial\overset{(s)}{x^{\nu}}}\,.$$

A straight forward calculation gives the identity

(5.1.29) $$\frac{d}{d\tau}\frac{\omega}{d\tau}=\omega_{\lambda}\,\{\ddot{x}^{\lambda}+\sum_{s=0}^{M}\Lambda^{\lambda}_{\mu\nu(s)}\,\dot{x}^{\mu}\,\overset{(s+1)}{x^{\nu}}\}$$

quite as in the case of (1.1.18).

From (5.1.29) follows:

(5.1.30) $$\omega^{\lambda}\frac{d}{d\tau}\frac{\omega}{d\tau}=\ddot{x}^{\lambda}+\sum_{s=0}^{M}\Lambda^{\lambda}_{\mu\nu(s)}\,\dot{x}^{\mu}\,\overset{(s+1)}{x^{\nu}}.$$

The equations of the

global paths	*local paths*
(5.1.31) $\dfrac{d}{d\tau}\dfrac{\omega}{d\tau}=0$	$\ddot{x}^{\lambda}+\sum_{s=0}^{M}\Lambda^{\lambda}_{\mu\nu(s)}\,\dot{x}\,\overset{(s+1)}{x^{\nu}}=0$

show us that *these paths are projections of the*

local paths (5.1.31)$_2$ *by the globalizing function factor* $\omega_\lambda(x,\dot{x},\cdots,\overset{(M)}{x})$.	*global paths* (5.1.31)$_1$ *by the localizing funciton factor* $\omega^\lambda(x,\dot{x},\cdots,\overset{(M)}{x})$.

The (5.1.30) may be rewritten as

$$\omega^\lambda \frac{d}{d\tau}\frac{\omega}{d\tau} = \ddot{x}^\lambda + \overline{\Lambda}_{\mu\nu}\,\dot{x}^\mu\,\overset{(s+1)}{\dot{x}^\nu}, \tag{5.1.32}$$

where

$$\overline{\Lambda}^\lambda_{\mu\nu}(x,\dot{x},\cdots,\overset{(M)}{x}) \overset{\mathrm{def}}{=\!=} \omega^\lambda(x,\dot{x},\cdots,\overset{(M)}{x}) \sum_{s=0}^{M} \frac{\overset{(s+1)}{x^\sigma}}{\dot{x}^\nu}\frac{\partial\omega_\mu}{\partial \overset{(s)}{x^\sigma}} \tag{5.1.33}$$

with

$$d\omega_\mu(x,\dot{x},\cdots,\overset{(M)}{x}) \overset{\mathrm{def}}{=\!=} \omega^\lambda(x,\cdots,\overset{(M)}{x}) \sum_{s=0}^{M} \frac{\overset{(s+1)}{x^\sigma}}{\dot{x}^\nu}\frac{\partial\omega_\mu}{\partial \overset{(s)}{x^\sigma}}\,\overset{(s+1)}{dx^\nu}, \tag{5.1.34}$$

or

$$d\omega_\mu - \overline{\Lambda}^\lambda_{\mu\nu}\,\omega_\lambda\,dx^\nu = 0, \tag{5.1.35}$$

where

$$\begin{aligned}\overline{\Lambda}^\lambda_{\mu\nu} &\overset{\mathrm{def}}{=\!=} \omega^\lambda\left(\frac{\partial\omega_\mu}{\partial x^\sigma}\dot{x}^\sigma + \frac{\partial\omega_\mu}{\partial\dot{x}^\sigma}\ddot{x}^\sigma + \cdots + \frac{\partial\omega_\mu}{\partial\overset{(M)}{x^\sigma}}\overset{(M+1)}{x^\sigma}\right)\\ &= \sum_{s=0}^{M}\Lambda^\lambda_{\mu\nu(s)}\,\overset{(s+1)}{x^\sigma}/\dot{x}^\nu.\end{aligned} \tag{5.1.36}$$

Evidently *the global paths* (5.1.31)$_1$ *are the extremals*

$$\delta\frac{\omega}{ds} = 0, \tag{5.1.37}$$

which coincide with (5.1.29)$=0$.

5.2. Finite Equations of II-Geodesic Curves in the Finsler-Craig-Synge-Kawaguchi Space in the Large. Quite as in Art. 1.2., our

ω	ω_μ	ω^μ

are vectors with components

(ω^l),	(ω_μ),	(ω^μ),	(ω^l_μ),	(ω^μ_l),

$(\mu, l = 1, 2, \cdots, n)$,

so that all functions in

ω	ω_μ	ω^μ

may be written in

$\omega^l.$	$\omega_{\mu}.$	$\omega^{\mu}.$	$\omega^l_{\mu}.$	$\omega^{\mu}_{l}.$

The differential equation $(5.1.31)_1$ is readily integrable:

(5.2.1) $$\omega^l = a^l \, d\tau, \qquad (a^l : \text{const.}).$$

We set

(5.2.2) $$d\xi^l \overset{\text{def}}{=\!=} a^l \, d\tau = \omega^l,$$

so that

(5.2.3) $$\xi^l = a^l \, \tau + c^l, \qquad (c^l : \text{const.}).$$

This represents a curve *in the large, which behave as for meet and join as well as for the extremal*

(5.2.4) $$\delta s = 0$$

like straight lines. We will call such curves II–*geodesic curves* (T. Takasu, [1]) *corresponding to* $(\omega^l_\lambda(x, \dot{x}, \cdots, \overset{(M)}{x}))$ and the (ξ^l) II–*geodesic rectangular coordinates refered to the* II–*geodesic ξ^l–axes.* The (ξ^l) are *global.*

The (1.1.1) becomes

(5.2.5) $$d\xi^l = \omega^l = \omega^l_\mu (x, \dot{x}, \cdots, \overset{(M)}{x}) \, dx^\mu.$$

The form of (5.1.29) tells us that *the local paths of teleparallelism* $(5.1.31)_2$ *are projected piece–wise into the global paths*

(5.2.6) $$\frac{d^2 \xi^l}{d\tau^2} = \frac{d}{d\tau} \frac{\omega^l}{d\tau} = 0$$

by the transformation (5.2.5):

(5.2.7) $$d\xi^l = \omega^l_\mu (x, \dot{x}, \cdots, \overset{(M)}{x}) \, dx^\mu$$

continuingly and smoothly.

The identity (5.1.32) tells us that *the global paths* $(5.1.31)_1$ *or* (5.2.6) *are projected piece–wise onto the local paths* $(5.1.31)_2$ *by the inverse transformation*

(5.2.8) $$dx^\lambda = \omega^\lambda_l \, \omega^l = \omega^\lambda_l \, d\xi^l$$

of (5.2.5). Indeed, the $(5.1.31)_2$ is *a linear combination of* $(5.1.31)_1$ as (5.1.30) tells us.

Multiplying (5.2.5) with ω_l^λ, we see that *the relation*

$$\frac{dx^\lambda}{d\tau}=a^l\,\omega_l^\lambda \tag{5.2.9}$$

holds along the II*-geodesic line-elements.*

§ 6. TT-Affine Geometry as Doubly Extended Global Affine Geometry and TT-Euclidean Geometry as Doubly Extended Global Euclidean Geometry in the Finsler-Craig-Synge-Kawaguchi Spaces in the Large in Differentiable Manifolds.

6.1. Doubly Extended TT-Affine Transformations. We consider the case, where (ξ^l) stands for (x^λ). In this case, (5.2.7) becomes to the form

$$d\bar{\xi}^l=a_h^l\,(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi})\,d\xi^h, \qquad (\det(a_h^l)\neq 0) \tag{6.1.1}$$

for the II-geodesic line-elements corresponding to $(a_h^l\,(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi}))$. In order that the II-geodesic curve $(\xi^l(s))$ may be transformed by (6.1.1) into II-geodesic curves $(\bar{\xi}^l(s))$ corresponding to $(\bar{a}_h^l\,(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi}))$, we must have

$$da_h^l\,(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi})\,d\xi^h=0 \tag{6.1.2}$$

along the II*-geodesic line-elements.* For, from (6.1.1), we obtain

$$\frac{d^2\bar{\xi}^l}{ds^2}=\frac{d}{ds}\,a_h^l\,(\dot{\xi},\xi,\cdots,\overset{(M)}{\xi})\,\frac{d\xi^h}{ds}+a_h^l\,(\dot{\xi},\xi,\cdots,\overset{(M)}{\xi})\,\frac{d^2\xi^h}{ds^2}\,. \tag{6.1.3}$$

Integrating (6.1.1) along the $\bar{\xi}^l$-axes, we obtain

$$\bar{\xi}^l=a_h^l\,(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi})\,\xi^h-\int\xi^h\,(da_h^l\,(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi})/ds)\,ds.$$

Now

$$\begin{aligned}\int\xi^h\,(da_h^l/ds)\,ds&=\int(da_h^l/ds)\,ds\int d\xi^h\\&=\iint(da_h^l/ds)\,ds\,d\xi^h=\text{const.}=-a_o^l,\end{aligned}$$

say, by (6.1.2) the sufficient condition for that the repeated integral may be converted into the double integral (that is, that the integrand is continuous) being evidently satisfied. Thus we have

$$\bar{\xi}^l=a_h^l\,(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi})\,\xi^h+a_o^l, \qquad (\det(a_h^l)\neq 0,\ a_o^l=\text{const.}). \tag{6.1.4}$$

Let us call (6.1.4) a *doubly extended TT-affine Transformation.* From (6.1.1) and (6.1.4), we see that *the necessary condition*

$$(6.1.5) \qquad da_h^l(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\,\xi^h = 0$$

is satisfied for the II*-geodesic line-elements under doubly extended affine transformations.*

6.2. Doubly Extended TT-Affine Transformation Group. We can prove the following theorem quite parallel to the analogous theorem in Art. 2.2.

Theorem. *The totality of the doubly extended TT-affine transformations*

$$(6.2.1) \qquad \bar{\xi}^h = a_k^h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\,\xi^k + a_o^h, \; (a_o^h = \text{const.}, \; \det(a_k^h) \neq 0),$$

whose inverse transformations are

$$(6.2.2) \qquad \xi^k = a_h^k(\bar{\xi}, \dot{\bar{\xi}}, \cdots, \overset{(M)}{\bar{\xi}})\,\xi^h + a_o^k, \qquad (a_o^k = \text{const.}, \; \det(a_h^k) \neq 0),$$

forms a group.

Definition. The group stated in the last theorem will be called the *doubly extended TT-affine transformation group* and the geometry belonging to it the *doubly extended TT-affine geometry* (cf. Art. 2.2), which is *doubly extended global affine geometry in the Finsler-Craig-Synge-Kawaguchi space in the large in differentiable manifolds.*

N. B. *The detail* may be exposed as in T. Takasu, [16].

6.3. Doubly Extended TT-Euclidean Transformations. We consider

$$(6.3.1) \qquad ds^2 = d\bar{\xi}^l\,d\xi^l = \omega^l\,\omega^l = \omega\omega = \omega_\mu^l\,dx^\mu\,\omega_\nu^l\,dx^\nu = g_{\mu\nu}\,dx^\mu\,dx^\nu,$$

$$(6.3.2) \qquad g_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x}) = \omega_\mu^l\,\omega_\nu^l = \omega_\mu\,\omega_\nu$$

in the *Finsler-Craig-Synge-Kawaguchi space in the large in differentiable manifolds,* for which the *global orthogonality conditions*

$$(6.3.3) \qquad a_h^l\,a_k^l = \delta_{hk} \quad \Longleftrightarrow \quad a_l^h\,a_l^k = \delta^{hk}$$

hold for (6.1.4). In this case we call (6.1.4) a *doubly extended TT-Euclidean transformation.*

The condition (6.1.5)

$$(6.3.4) \qquad da_h^l(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\,\xi^h = 0$$

is satisfied for the II*-geodesic line-elements under doubly extended TT-Euclidean transformations still.*

The $\bar{\xi}^l$ and ξ^l in (6.1.4) will be called the II*-geodesic rectangular coordinates refered to the* II*-geodesic rectangular* $\bar{\xi}^l$*- resp.* ξ^l*-axes.*

The results of Artt. 6.1. and 6.2. hold still and we have

$$\omega^2 = \omega^l \omega^l = d\tilde{\xi}^l d\tilde{\xi}^l = ds^2 = (c^l c^l)\,\omega^2, \qquad (\omega^l = c^l \omega). \tag{6.3.5}$$

so that

$$c^l c^l = 1. \tag{6,3,6}$$

6.4. Doubly Extended TT–Euclidean Transformation Group. *Since the doubly extended TT–Euclidean transformation group is obtained evidently as a subgroup of the doubly extended TT–affine transformation group, the following theorem holds still.*

Theorem. *The totality of the doubly extended TT–Euclidean transformations forms a subgroup of the doubly extended TT–affine transformation group.*

Definition. The group stated in the last theorem will be called the *doubly extended TT–Euclicean transformation group* and the geometry belonging to it the *doubly extended TT–Euclidean geometry,* which is *doubly extended global Euclidean geometry in the Finsler– Craig–Synge Kawaguchi space in the large in differentiable manifolds.* (Cf. Art. 2.4.)

N. B. *The detail* may be exposed quite as in T. Takasu, [17], [18].

The following two sections may be exposed quite as in sections 2.4. resp, 2.6.

6.5. The Differential Equations of the Extremals $\delta s = 0$ in terms of $\{^{\lambda}_{\mu\nu}\}_{(s)}$, $(s = 0, 1, \cdots, M)$ in the Doubly Extended TT–Euclidean Geometry.

6.6. Duality of Hamilton's Canonical Formalisms in the Doubly Extended TT–Euclidean Geometry. Since we have introduced the metric tensor $g_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x})$ into our *doubly extended* TT–*Euclidean geometry,* we can establish a *duality of Hamilton's canonical formalisms* quite as in (1.16)′ on p. 22 of Part I of our paper in such a way that the Hamiltonian $\mathscr{H}$ and the Lagrangian $\mathscr{L}$ for (x^λ, y_λ) are the Lagrangian and the Hamiltonian for (x_λ, y^λ) respectively.

6.7. The Differential Equations of the Extremals $\delta s = 0$ in termas of $\{^{\lambda}_{\mu\nu}\}_{(s)}$, $(s = 0, \cdots, M)$ in the Doubly Extended TT–Euclidean Geometry, in the Case, where $g_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x})$ are Homogeneous of Zero Degree in $\dot{x}$ of Part I.

N. B. *The following* 17 *sections are in preparation.*

PART III.

§ 7. TT–Equiform (=TT–Conformal) Geometry as Doubly Extended Global Equiform Geometry in the Finsler–Craig–Synge–Kawaguchi Spaces in the Large in Differentiable Manifolds.

§ 8. TT–Projective Geometry as Doubly Extended Global Projective Geometry in the Finsler–Craig–Synge–Kawaguchi Spaces in the Large in Differentiable Manifolds.

§ 9. TT–Non–Euclidean Geometry as Doubly Extended Global Non–Euclidean Geometry in the Finsler–Craig–Synge–Kawaguchi Spaces in the Large in Differentiable Manifolds.

§ 10. TT–Laguerre Geometry as Doubly Extended Global Laguerre Geometry in the Finsler–Craig–Synge–Kawaguchi Spaces in the Large in Differentiable Manifolds.

§ 11. TT–Equiform Laguerre Geometry as Doubly Extended Global Equiform Laguerre Geometry in the Large in the Finsler–Craig–Synge–Kawaguchi Spaces in the Large in Differentiable Manifolds.

§ 12. TT–Lie Geometry as Doubly Extended Global Lie Geometry in the Finsler–Craig–Synge–Kawaguchi Spaces in the Large in Differentiable Manifolds.

§ 13. TT–Parabolic Lie Geometry as Doubly Extended Global Parabolic Lie Geometry in the Finsler–Craig–Synge–Kawaguchi Spaces in the Large in Differentiable Manifolds.

PART IV.

§ 14. Local General Affine Geometry and Local General Metric Geometry.

§ 15. Global General Affine Geometry in Differentiable Manifolds and Global General Metric (General Euclidean) Geometry in Differentiable Manifolds.

§ 16. Global General Equiform Geometry in Differentiable Manifolds.

§ 17. Global General Conformal Geometry in Differentiable Manifolds.

§ 18. Global General Projective Geometry in Differentiable Manifolds.

§ 19. Global General Non–Euclidean Geometry in Differentiable Manifolds.

§ 20. Global General Laguerre Geometry in Differentiable Manifolds.

§ 21. Global General Equiform Laguerre Geometry in Differentiable Manifolds.

§ 22. Global General Lie Geometry in Differentiable Manifolds.

§ 23. Global General Parabolic Lie Geometry in Differentiable Manifolds.

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Corrigenda To

VARIOUS HAMILTON'S CANONICAL FORMALISMS AS NON-CONNECTION METHODS FOR VARIOUS CONNECTION GEOMETRIES IN THE LARGE. PART I.

Page	line	for	read
15	↓ 7	*th*	*the*
,,	↓ 11	[12]	[2]
,,	↓ 17	*eqnation*	*equation*
,,	↓ 19	oj	of
,,	↓ 20	Hamiltonian	Hamilton–Jacobi
16	↓ 4	n	$\cdots, n$
,,	↑ 7	724.	724.)
17	↓ 6		
18	↓ 11	$L\cdots H$	$L^2\cdots H^2$
,,	↑ 1	$\partial g_{\eta\nu}\cdots\cdots\ddot{x}^\mu$	$\partial g_{\mu\nu}\cdots\cdots\dot{x}^\mu$
20	↓ 13	$g_{\mu\upsilon}$	$g_{\mu\lambda}$
22	↓ 4	$g^{\nu\mu}$	$g^{\mu\nu}$
,,	↓ 5	26	16.
,,	↓ 7	$x^\lambda\cdots x_\lambda)\,\dot{x}^\nu\cdots x^\lambda)\,\dot{x}_\sigma$	$x_\lambda\cdots \overset{(M)}{x}_\lambda)\,\dot{x}_\nu\cdots \overset{(M)}{x}{}^\lambda)\,\dot{x}^\sigma$
,,	,,	$y_\lambda \cdots, x_\lambda)$	$x_\lambda, \cdots, \overset{(M)}{x}_\lambda$
,,	↓ 9	$(x^\lambda, \dot{x}_\lambda, \cdots$	$(x_\lambda, \dot{x}_\lambda, \cdots$
23	↑ 6	$\overset{(M)}{x}$	$\overset{(M)}{x}{}^\lambda$
26	↑ 13	H^ι	H^l
27	↓ 8	$d\omega_\mu$	$\partial\omega_\mu$
,,	↓ 9	$\dot{x}^\lambda.$	$\dot{x}^\lambda,$
,,	↑ 5	$g_{\mu\nu},$	$g_{\mu\nu}$
28	↑ 7	$\partial y_\mu\cdots\partial x_\mu$	$\partial y^\mu\cdots\partial x_\mu$
,,	↑ 1	$\dot{x}^\mu.$	$\dot{x}^\mu,$
30	↓ 7	$g_{\nu\nu}$	$g_{\mu\nu}$
,,	↓ 9	$g_{\eta\nu}$	$g_{\mu\nu}$
33	↑ 1	(3.3)$\cdots$	cf. (2.6.3) of Part II
34	↓ 2	$d\dot{x}^\nu$	$\dot{x}^\nu$
,,	↓ 3	(3.4)$\cdots$	cf. (2.6.3) of Part II
,,	↑ 3	(3.13)$\cdots$	cf. (2.6.3) of Part II

Page	line	for	read
35	↑ 7	$\delta H=$	$=$
, ,	↑ 3	$\int_{s_0}^{s} ds$	$\int_{s_0}^{s} \mathcal{L} ds$
40	↑ 14	the)…(2.33)	the…(2.33)).
, ,	↓ 10	becomes	becomes (cf. (2.35))
41	↑ 8	(2.19)	(5.5)
, ,	↑ 3	*and*	*with* (1.11) *and*
43	↓ 12	d_h^l	a_h^l
44	↑ 2	$F_{(\mu)\nu}F_{(\mu)\nu}$	$F_{(M)\mu}F_{(M)\nu}$
46	↓ 2	Ψ	(6.6) Ψ
, ,	↓ 3	*funciton*	*function*
47	↑ 3	∂q^j	dq^j
. .	↑ 2	∂q_j	dq_j
48	↓ 3	$\partial q^j \cdots d\bar{p}^i$	$\partial q_j \cdots \partial \bar{p}^i$
, ,	↓ 13	∂p^i	$\partial \bar{p}^i$
49	↑ 1	dq^j,	dq^j
56	↓ 4	tty	ty
60	↑ 5	M. Kawaguchi	(p. 61, lines 2…27) M. Kawaguchi
61	↓ 1	: 4	; 4