NOTE ON BOUNDED SURFACES IN A 3-MANIFOLD

By

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(Received June 23, 1969)

1. Introduction

The following considerations are based upon the semi-linear point of view. By $(P \subset M^3)$ we denote a pair of manifolds such that M^3 is a triangulated 3-dimensional manifold and P is an embedded 2-dimensional manifold, which may not be connected, as a subcomplex in the interior[1] $\mathcal{I}M^3$ of M^3. Then, a simple polygonal loop J in P will be called a *co-unknotted loop*, iff J is the boundary of a disk $D(J)$ whose interior is contained in $M^3 - P$. Especially, a co-unknotted loop J is said to be *non-trivial* iff[2] $J \not\simeq 1$ on P. Let F_p resp. $F_{p,q}$ be a connected closed (=compact, without boundary) orientable surface of genus p resp. a connected bounded (=compact, with boundary) orientable surface of genus p with q boundary curves. R. H. Fox [1] and T. Homma [3] shown that for any pair $(F_p \subset S^3)$, there exists a non-trivial co-unknotted loop on F_p. S. Kinoshita [5] proved the similar theorem for some kinds of $(F_p \subset M^3)$, and L. P. Neuwirth [6] for $(F_{p,1} \subset S^3)$. The purpose of the paper is to generalize the theorems for $(F_{p,q} \subset M^3)$ as follows:

Theorem 1. *Let $(F_{p,q} \subset M^3)$ be a pair. If the homomorphism $\iota_\#$ of the fundamental group $\pi_1(\mathcal{I}F_{p,q})$ into $\pi_1(M^3 - \partial F_{p,q})$, induced by the natural inclusion ι: $\mathcal{I}F_{p,q} \longrightarrow M^3 - \partial F_{p,q}$, is not isomorphism, then there exists a non-trivial co-unknotted loop J on $F_{p,q}$.*

The proof will be given in § 3 by using the Loop Theorem and the Dehn's Lemma.

2. Boundary trivial surfaces

Let $J_1 \cup \cdots \cup J_q \subset M^3$ be an union of q pairwise disjoint simple loops in M^3. We will say that $J \subset M^3$ is *unknotted* iff there exists a disk $D(J)$ in M^3 such that $\partial D(J) = J$, and $J_1 \cup \cdots \cup J_q \subset M^3$ is *trivial* iff there exist q pairwise disjoint disks $D(J_1), \cdots, D(J_q)$ in M^3 such that $\partial D(J_1) = J_1, \cdots, \partial D(J_q) = J_q$. A pair $(F_{p,q} \subset M^3)$ is said to be *boundary trivial* iff $(\partial F_{p,q} \subset M^3)$ is trivial.

A boundary trivial pair $(F_{p,q} \subset M^3)$ has almost similar properties of $(F_p \subset M^3)$.

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1) $\mathcal{I}$=interior, ∂=boundary.

2) $\simeq$means homotopic to, $\sim$means homologous to.

Theorem 2. [Fox–Homma–Kinoshita]. *Let $(F_{p,q} \subset M^3)$ be a boundary trivial pair. If $\pi_1(M^3)$ is isomorphic to either a finite fundamental group or infinite cyclic one, then*

(i) *there exists a non-trivial co-unknotted loop on $F_{p,q}$,*

(ii) *there exist $p+q-1$ mutually disjoint unknotted loops on $F_{p,q}$ such that they are linearly independent in $H_1(F_{p,q})$.*

Proof. Let $k_1 \cup \cdots \cup k_q = \partial F_{p,q}$. If there are q pairwise disjoint disks $D(k_1), \cdots, D(k_q)$ in M^3 such that $\partial D(k_i) = k_i, D(k_i) \cap F_{p,q} = k_i, i=1, \cdots, q$, then the Theorem is obtained immediately from Theorem B in [5]. So, from now on we suppose that such disks do not exist.

Let c_i be a simple loop on $F_{p,q}$ lying in a small neighborhood of k_i and "parallel" to k_i, $i=1, \cdots, q$. Because $F_{p,q}$, is orientable, the linking number of c_i with k_i is the same as the sum of the linking numbers of the $k_1, \cdots, \check{k}_i, \cdots, k_q$ with $k_i, i=1, \cdots, q$. But each of these linking numbers is zero, since $k_1 \cup \cdots \cup k_q \subset M^3$ is trivial. Because the linking number of c_i with k_i is zero, we can have q pairwise disjoint disks $D(k_1), \cdots, D(k_q)$ in M^3 so that a small neighborhood of k_i in $F_{p,q}$ meets $D(k_i)$ only at k_i, while the total intersection of $F_{p,q}$ with $D(k_1) \cup \cdots \cup D(k_q)$ consists of a number of simple loops $J_1, \cdots, J_\nu$, where $\nu > 0$. Let $A(J_1), \cdots, A(J_\nu)$ be disks on $D(k_1) \cup \cdots \cup D(h_q)$ bounded by $J_1, \cdots, J_\nu$, respectively.

Let J_1 be a minimal, i.e. there is no other J_i in $A(J_1)$. Then the following three cases can occur:

1) $J_1 \simeq 1$ on $F_{p,q}$. Since J_1 is simple, J_1 bounds a disk $B(J_1)$ on $F_{p,q}$. We consider the intersection $B(J_1) \cap (D(k_1) \cup \cdots \cup D(k_q))$, which is a subset of $\{J_1, \cdots, J_\nu\}$. Let $J_1 \cup J_{t_1} \cup \cdots \cup J_{t_n} = B(J_1) \cap (D(k_1) \cup \cdots \cup D(k_q)))$. Then $J_{t_i} \simeq 1$ on $F_{p,q}$ and J_{t_i} bounds a disk $B(J_{t_i})$ on $F_{p,q}$, $i=1, \cdots, n$. Thus there exists a disk, say $B(J_{t_1})$, in $B(J_{t_n})$ such that $B(J_{t_1}) \cap (D(k_1) \cup \cdots \cup D(k_q)) = J_{t_1}$. Let $D(k_1) \cup \cdots \cup D(k_q) - A(J_{t_1}) \cup B(J_{t_1})$ be new disks bounded by $k_1 \cup \cdots \cup k_q$, and again denote these by $D(k_1), \cdots, D(k_q)$. We may deform $D(k_1) \cup \cdots \cup D(k_q)$ into general position in M^3, so that

$$F_{p,q} \cap (D(k_1) \cup \cdots \cup D(k_q)) = J_1 \cup \cdots \cup J_\nu - J_{t_1}.$$

By the repetition of the procedure we can get rid of all intersections $J_1, J_{t_1}, \cdots, J_{t_n}$ of $F_{p,q} \cap (D(k_1) \cup \cdots \cup (D(k_q))$.

From this argument, if all intersections $J_1, \cdots, J_\nu$ of $F_{p,q} \cap (D(k^1) \cup \cdots \cup D(k_q))$ are of this type, we can conclude that there are q pairwise disjoint disks $D'(k_1), \cdots, D'(k_q)$ in M^3 such that $F_{p,q} \cap D'(k_i) = k_i$. So, we may assume that at least one of $J_1, \cdots, J_\nu$ is not homotopic to 1 on $F_{p,q}$.

2) $J_1 \not\sim 0$ on $F_{p,q}$. Let $h_1 : D\times I \longrightarrow M^3$ be an embedding of a 3-cell $D\times I$ into M^3 such that:

$$h_1(D\times I)\cap(D(k_1)\cup\cdots\cup D(k_q))=h_1(D\times\{0\})=A(J_1),$$
$$h_1(D\times I)\cap F_{p,q}=h_1(\partial D\times I)=N(J_1; F_{p,q}),$$

where D is the unit disk, $I=[-1,1]$ and $N(J_1; F_{p,q})$ is a regular neighborhood of J_1 in $F_{p,q}$. Let $F(h_1)=F_{p,q}-N(J_1; F_{p,q})\cup h_1(D\times\partial I)$. If J_1 does not homologous to any combination of $k_1,\cdots,k_q$ on $F_{p,q}$, $F(h_1)$ is an orientable surface of genus $p-1$ with boundary curves $\partial F_{p,q}$. If J_1 is homologous to a combination of $k_1,\cdots,k_{p,q}$, $F(h_1)$ is two disjoint orientable surfaces, say M_1 and M_2, such that[3] $g(M_1)+g(M_2)=p$ and $\partial M_1\cup\partial M_2=\partial F_{p,q}$. Note that $F(h_1)\cap(D(k_1)\cup\cdots\cup D(k_q))=F_{p,q}\cap(D(k_1)\cup\cdots\cup D(k_q))-J_1$.

3) $J_1\sim 0$ but $J_1 \not\approx 1$ on $F_{p,q}$. Since J_1 is simple, J_1 bounds an orientable surface, say $F_{p',1}$, on $F_{p,q}$, where $p>p'>0$. With $h_1 : D\times I\longrightarrow M^3$ and $F(h_1)$ as in 2), $F(h_1)$ will be two disjoint orientable surfaces, so that we can denote these by $F_{p-p',q}$ and $F_{p'}$. Note that $F(h_1)\cap(D(k_1)\cup\cdots\cup D(k_q))=F_{pq}\cap(D(k_1)\cup\cdots\cup D(k_q))-J_1$.

In both 2) and 3), it is clear that J_1 is a non-trivial co-unknotted loop on $F_{p,q}$; hence the proof of (i) is complete.

By the repetition of the suitable procedure of 1) 2) and 3) for $F(h_1)\cap(D(k_1)\cup\cdots\cup D(k_q))$, and so on, we have a finite number of orientable surfaces $F(h_1,h_2,\cdots,h_\nu)$ in M^3 such that $\partial F(h_1,h_2,\cdots,h_\nu)=\partial F_{p,q}$ and $F(h_1,h_2,\cdots,h_\nu)\cap(D(k_1)\cup\cdots\cup D(k_q))=\partial F_{p,q}$. Note that $\Phi=F(h_1,h_2,\cdots,h_\nu)\cup D(k_1)\cup\cdots\cup D(k_q)$ is a system of mutually disjoint closed orientable surfaces in M^3. Then, according to Theorem B in [5], we have mutually disjoint unknotted loops $J'_1,\cdots,J'_\mu$ on Φ, where $\mu=g(\Phi)$. Especially, we may assume that $(J'_1\cup\cdots\cup J'_\mu)\cap(D(k_1)\cup\cdots\cup D(k_q))=\phi$ and $(J'_1\cup\cdots\cup J'_\mu)\cap h_i(D\times\partial I)=\phi, i=1,\cdots,\nu$. It is easily checked that we can select p loops in $J_1,\cdots,J_\nu,J'_1,\cdots,J'_\mu$ so that they and $k_1,\cdots,k_{q-1}$ are linearly indepenpent in $H_1(F_{p,})$.

This completes the proof of (ii), and of Theorem 2.

Theorem 2 (i) shows that the natural inclusion map $\iota : \mathcal{I}F_{p,q}\longrightarrow M^3-\partial F_{p,q}$ induces a homomorphism $\iota_\# : \pi_1(\mathcal{I}F_{p,q})\longrightarrow\pi_1(M^3-\partial F_{p,q})$ such that ker. $(\iota_\#)\neq\{1\}$. This can be weakened as follows:

Theorem 3. *Let $(F_{p,q}\subset M^3)$ be a pair, and assume that $q\geq 2$. If there are r mutually disjoint 3-cells $Q_1,\cdots,Q_r$ in M^3 such that $q\geq r\geq 2$ $Q_i\cap\partial F_{p,q}\neq\phi$ and $\partial Q_i\cap\partial F_{p,q}=\phi$. Then there exists a non-trivial co-unknotted loop on $F_{p,q}$, that is, the inclusion map $\iota : \mathcal{I}F_{p,q}\longrightarrow M^3-\partial F_{p,q}$ induces a homomorphism $\iota_\# : \pi_1(\mathcal{I}F_{p,q})\longrightarrow\pi_1$*

3) $g(P)$ denotes the genus of P.

$(M^3-\partial F_{p,q})$ *such that* ker. $(\iota_\#)\neq\{1\}$.

The proof will be easily given by using the intersection $F_{p,q}\cap(\partial Q_1\cup\cdots\cup\partial Q_r)$.

Remark 1. The genera of knots and links, [2], [10], show the necessity for the condition on $\partial F_{p,q}$ in Theorems 2 and 3.

Remark 2. Let $(F_{p_1,q_1}\cup\cdots\cup F_{p_n,q_n}\subset M^3)$ be a system of compact orientable surfaces, which may be closed or bounded, in $\mathcal{J}M^3$. Scrutiny of the proof of Theorem 2 shows that Theorem 2 is true for $(F_{p_1,q_1}\cup\cdots\cup F_{p_n,q_n}\subset M^3)$ with $\pi_1(M^3)$ and $\partial F_{p_1,q_1}\cup\cdots\cup\partial F_{p_n q_n},\subset M^3$ as in Theorem 2.

3. Proof of Theorem 1

It will be noticed that for $F_{p,q}\subset\mathcal{J}M^3$ and $q>0$, $M^3-F_{p,q}$ is connected. Let $X=\{M^3$ split along $F_{p,q}\}$.[4] Then X has on its boundary two copies of $F_{p,q}$, which denote F^1, F^2, noting that

$$\partial X=\partial M^3\cup(F^1\cup F^2),\ F^1\cap F^2=\partial F_{p,q}.$$

Let $\rho: X\longrightarrow M^3$ be a map defined by

$$\rho\,|\,X-(F^1\cup F^2)=\text{identity},$$
$$\rho(F^1)=\rho(F^2)=F_{p,q}.$$

In order to prove the Theorem 1, we shall apply the Loop Theorem [8], and the Dehn's Lemma [4], [9] for X. We shall need the following two lemmas.

Lemma 1. *The inclusion map* $\varphi^i: F^i\longrightarrow F^1\cup F^2$ *induces a monomorphism* $\varphi^i_\#:\pi_1(F^i)\longrightarrow\pi_1(F^1\cup F^2)$, $i=1,2$.

Proof. This Lemma is immediately from the following well-known facts. $\pi_1(F^i)$ is free with $2p+q-1$ generators $a_{i,1},\cdots,a_{i,p},b_{i,1},\cdots.b_{i,p},c_1,\cdots,c_{q-1}$, where $a_{i,1},\cdots,a_{i,p},b_{i,1},\cdots,b_{i,p}$ are associated with p handles of F^i, and $c_1,\cdots,c_{q-1}$ are $q-1$ components of q boundary curves $\partial F^i=\partial F_{p,q}$, $i=1,2$. On the other hand, $\pi_1(F^1\cup F^2)$ is generated by $a_{1,1},\cdots,a_{1,p},b_{1,1},\cdots,b_{1,p},a_{2,1},\cdots,a_{2,p},b_{2,1},\cdots b_{2,p},c_1,\cdots,c_{q-1},d_1,\cdots,d_{q-1}$ subject to the single relation

$$(*)\qquad \prod_{j=1}^{p}[a_{1,j},b_{1,j}]\prod_{j=1}^{p}[a_{2,j},b_{2,j}]\prod_{j=1}^{q-1}[c_j,d_j]\simeq 1,$$

where d_j is associated with the handle determined by c_j and c_q, $j=1,\cdots,q-1$.

Let η_i be an element in $\pi_1(F^i)$ such that $\eta_i\not\simeq 1$ in F^i, $i=1,2$. η_i is represented by a word $W(a_{i,1},\cdots,a_{i,p},b_{i,1},\cdots,b_{i,p},c_1,\cdots,c_{q-1})$. It is easily checked that the reduced element $W/(*)\not\simeq 1$ on $F^1\cup F^2$.

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4) For more precise construction see Neuwirth [7], Chap. III.

Lemma 2. *One of the inclusion maps* $\theta^i : F^i \longrightarrow X, i=1,2,$ *induces a homomorphism* $\theta^i_{\#} : \pi_1(F^i) \longrightarrow \pi_1(X)$ *such that* ker. $(\theta^i_{\#}) \neq \{1\}$.

Proof. By the assumption of the Theorem 1, there is a closed curve η in $\mathcal{J}F_{p,q}$ such that $\eta \not\simeq 1$ in $M^3-\partial F_{p,q}$. Let $\eta_1=\rho^{-1}(\eta)\cap F^1$ and $\eta_2=\rho^{-1}(\eta)\cap F^2$. According to Lemma 1, $\eta_i \not\simeq 1$ on $F^1 \cup F^2$, $i=1,2$. So, it suffices to show that $\theta^i(\eta_i)\simeq 1$ in X.

Let $f : D \longrightarrow M^3-\partial F_{p,q}$ be a continuous map such that $f(\partial D)=\eta$. Let U_f be the set of points $x \in D$ such that $f^{-1}f(x)$ contains at least two points, and let S_f be the closure of U_f in D. Then we may assume that $\dim. S_f \leq 1$ and S_f consists of i) double lines, ii) triple points which are crossing points of double lines and iii) branch points S_f-U_f. Since the number of triple points and branch points is finite, we may assume that triple points and branch points do not lie on $F_{p,q}$. Then $f^{-1}(f(\mathcal{J}D)\cap F_{p,q})$ is a finite number of simple arcs $\Gamma_1, \cdots, \Gamma_\lambda$ on D. From the above assumption we may assume that for every point $y \in \Gamma_1 \cup \cdots \cup \Gamma_\lambda \cup \partial D$, $f|N(y;D)$ is a homeomorphism. Therefore, by a slight modification of f, we conclude that $\Gamma_1, \cdots, \Gamma_\lambda$ are mutually disjoint, $\partial D \cap \Gamma_i=\partial\Gamma_i$, $i=1,\cdots,\lambda$, and $\Gamma_1 \cup \cdots \cup \Gamma_\lambda$ divides D into $\lambda+1$ regions $E_1, \cdots, E_{\lambda+1}$. Then, at least one of $E_1, \cdots, E_{\lambda+1}$ is a disk, say E_1. Note that $f(\partial E_1)$ is a loop on $F_{p,q}$ and homotopic 1 in $M^3-\partial F_{p,q}$. We will examine $f(\partial E_1)$ on $F_{p,q}$, and $\gamma_1'=\rho^{-1}(f(\partial E_1))\cap F^1$ and $\gamma_2'=\rho^{-1}(f(\partial E_1))\cap F^2$ on $F^1 \cup F^2$. The following two cases can occur:

1) $f(\partial E_1)\not\simeq 1$ on $F_{p,q}$. In this case, according to Lemma 1, each loop of γ_1' and γ_2' is not homotopic to 1 on $F^1 \cup F^2$. On the other hand, one of them bounds a singular disk $\rho^{-1}(f(E_1))$ in X. So, we have completed the proof of Lemma.

2) $f(\partial E_1)\simeq 1$ on $F_{p,q}$. In this case, it will be noticed that the loops γ_1' and γ_2' are homotopic to 1 in F^1 and F^2, respectively. Since $f(\mathcal{J}E_1)\cap F_{p,q}=\phi$ and one of γ_1' and γ_2', say γ_1', bounds a singular disk $\rho^{-1}(f(E_1))$ in X. We replace $\rho^{-1}(f(E_1)))$ by a singular disk Δ_1 in X which boundes by γ_2'. Then we have another continuous map $f_1 : D \longrightarrow M^3-\partial F_{p,q}$ such that $f_1|D-\mathcal{J}E_1=f$ and $\rho^{-1}(f_1(E_1))=\Delta_1$. Thus, we can eliminate $\partial E_1-\partial D$ from $\Gamma_1 \cup \cdots \cup \Gamma_2$ by a slight modification of f_1.

By the repetition of the procedure we can have a continuous map $f_n, n\leq\lambda$, of D into $M^3-\partial F_{p,q}$ such that $f_n(D)\cap \mathcal{J}F_{p,q}=f_n(\partial D)=\eta$. Therefore, one of loops $\eta_1' \cup \eta_2'=\rho^{-1}(\eta)$ comes to bound a singular disk $\rho^{-1}(f_n(D))$ in X.

This completes the proof of the Lemma 2.

We can now proceed the proof of the Theorem 1. By Lemma 2, there is a loop η' in $F^1 \cup F^2-(F^1 \cap F^2)$ such that $\eta' \not\simeq 1$ on $F^1 \cup F^2$ and $\eta' \simeq 1$ in X. Since X and η' satisfy the hypothesis of the Loop Theorem, there exists another simple loop J' in $F^1 \cup F^2-(F^1 \cap F^2)$ such that $J' \not\simeq 1$ on $F^1 \cup F^2$ and $J' \simeq 1$ in X. Then J' may bound a singular disk without singularity on the boundary. Hence by the Dehn's Lemma J' bounds a disk $D(J')$ in X. So, we have a non-trivial co-unknotted loop J on $F_{p,q}$ by setting $J=\rho(J')$, and we have completed the proof of Theorem 1.

Remark 3. Let $(F_{p_1,q_1}\cup\cdots\cup F_{p_n,q_n}\subset M^3)$ be as Remark 2 in § 2. If one of the homomorphisms $\iota^i_\#$ of the fundamental groups $\pi_1(\mathcal{J}F_{p_i,q_i})$ into $\pi_1(M^3-\partial F_{p_1,q_1}-\cdots-\partial F_{p_n,q_n})$, induced by the natural inclusions, is not an isomorphism, then there exists a non-trivial co-unknotted loop on $F_{p_1,q_1}\cup\cdots\cup F_{p_n,q_n}$.

Remark 4. Stallings' counter-example [11, § 6] shows that Theorem 1 is not true for a non-orientable surface G even with $\iota_\#:\pi_1(\mathcal{J}G)\longrightarrow\pi_1(M^3-\partial G)$ as in Theorem 1.

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