ON PIECEWISE LINEAR UNKNOTTING OF POLYHEDRA

By

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1. Introduction. A polyhedron P *unknots* in a PL (=piecewise linear) manifold M if for any two homotopic PL–embeddings, f, g, of P into M, there is a PL–isotopy, $h_t, t \in [0, 1]$, of M such that h_0=identity and $h_1 f=g$. If P is a compact k–dimensional polyhedron, it follows from general position that P unknots in Euclidean $(2k+2)$–space, E^{2k+2}, and, in general, P knots in E^{2k+1}. One problem in PL–topology has been to determine conditions on P so that P unknots in E^{2k+1}.

Wu [8] showed that a necessary and sufficient condition was the vanishing of certain obstructions in the integral $2k$–cohomology group of the reduced symmetric product of P. Some other sufficient conditions for the unknotting of P in E^{2k+1} are

1) P is a connected closed PL–manifold (Zeeman [9]);
2) $H^k(P)=0$ (Price [7]);
3) P is a connected homology manifold (Edwards [2]);
4) P collapses to a subpolyhedron which unknots in E^{2k+1} (Edwards) [2]).

Let K be a finite complex and let T be a subcomplex. Consider the complex K/T obtained from the first derived of K by removing the first derived neighborhood of T and adding the cone over the boundary of this neighborhood. A polyhedron P is called *reduced* if it is PL–homeomorphic to the underlying polyhedron of K/T where K is a finite complex and T is a maximal tree in K; i.e., T is a maximal contractible subcomplex of dimension one in K. Some examples of reduced polyhedra are closed connected PL–manifolds.

Theorem 1. *Reduced k–dimensional polyhedra unknot in* E^{2k+1}, $k>1$.

It is easily seen that if K and T are as above in the definition of a reduced polyhedron, then the underlying polyhedron of K and the underlying polyhedron of K/T have the same simple homotopy type [6].

Corollary 1. *If P is a connected k–dimensional polyhedron which knots in E^{2k+1}, then there exists a polyhedron of the same simple homotopy type of P which unknots in E^{2k+1}, $k > 1$.*

In proving this theorem we need a lemma which extends to give a generalization of Zeeman's unknotting theorem [9] for proper embeddings.

Theorem 2. *Let* $\{M_i\}$, $i=1, \cdots, n$, *be a collection of disjoint compact orientable* $(2m-q+1)$*-connected* PL*-manifolds with nonempty boundaries,* $m=$ *maximum* $\{m_i=$ *dimension* $M_i\}$, *and let* Q *be a* $(2m-q+2)$*-connected PL-q-manifold with nonempty boundary,* $3m+4\leq 2q$. *If* f *and* g *are two homotopic* (*relative* $\cup$ bdry M_i) *proper PL-embeddings of* $\cup M_i$ *into* Q *such that* $f| \cup$ bdry $M_i=g| \cap$ bdry M_i *and* $f(\cup$ bdry $M_i)$ *is contained in a single boundary component of* Q, *then there is a PL-isotopy* h_t, $t\in[0, 1]$, *of* Q *onto itself such that* $h_0=$*identity and* $h_1 f=g$.

Note that if we require that h_t be the identity on bdry Q or if we did not require that $\cup$ bdry M_i be mapped into a single boundary component of Q, then there are counterexamples to the theorem.

2. Preliminaries. We shall assume familiarity with either [4] or [9]. All maps will be assumed to be PL unless stated otherwise; hence, we shall drop the prefix PL. In the proof of Theorem 2 we shall assume that all manifolds considered are orientable and thus have some fixed orientation. The boundary of a manifold shall have its orientation induced from the manifold. Hence, if N is oriented, then $N\times[0,1]$ shall be oriented so that the natural map $N\longrightarrow N\times\{0\}$ is orientation preserving. All homeomorphisms, unless stated otherwise, shall be orientation preserving.

Cl, bdry, int will mean closure, boundary and interior respectively.

3. Proof of Theorem 2 when $n=2$. By Zeeman [9; Theorem 24], there exist isotopies $k^i: M_i\times[0, 1]\longrightarrow Q\times[0, 1], i=1,2,$ such that $k_0^i=f| M_i$, $k_1^i=g| M_i$, and $k_t^i|$ bdry $M_i=f$ for all t. Let $M_i^*=k^i(M_i\times[0, 1])$. By general position, we may assume that dim $(M_1^*\cup M_2^*)\leq 2m-q+1$. By the engulfing theorem [9; Theorem 20], there exist collapsible polyhedra C_i in int M_i^* such that $M_1^*\cap M_2^*\subseteq C_i$ and *dim* $C_i\leq 2m-q+2$. Again by the engulfing theorem, there is a collapsible polyhedron C_3 in int $(Q\times[0, 1])$ such that $C_1\cup C_2\subseteq C_3$ and *dim* $C_3\leq 2m-q+3$. By general position and the restriction $3m+4\leq 2q$, we may assume that $C_3\cap(M_1^*\cup M_2^*)=C_1\cup C_2$. Hence, by taking second derived neighborhoods, there is a $(q+1)$-cell N in int $(Q\times[0,1])$ such that $N\cup M_i^*$ is a properly embedded (m_i+1)-cell in N and $N\cup M_1^*$ and $N\cup M_2^*$ have disjoint boundaries, say S_1, S_2, respectively. Suppose $N\cup M_i^*$ and N are oriented compatibly with M_i^* and $Q\times[0, 1]$, respectively.

Propostion. *If the inclusion* i: $S_2\longrightarrow$ bdry $N-S_1$ *represents the trivial element of* π_{m_2} (bdry $N-S_1$), *then there is an isotopy* $k_t, t\in[0, 1]$, *of* N *such that* $k_0=$*identity,* $k_t|$ bdry $N=$*identity and* $k_1(M_2^*\cap N)\cap(M_1^*\cap N)=\phi$.

This proposition is well known. By Irwin [5] [9; Theorem 23], S_2 bounds a (m_2+1)-cell C in $N-M_1^*$; now apply Zeeman's unknotting theorem for codimension three cell pairs [10] [9; Cor. 1 to Theorem 9].

Hence, if the inclusion i represents the trivial element of π_{m_2} (bdry $N-S_1$), we can move M_2^* off of M_1^* and the theorem would follow from [3]. Therefore, suppose that i represents a non-trivial element of π_{m_2} (bdry $N-S_1$).

Let $p_i \in g$(bdry M_i), $q_i \in S_i$, $i=1,2$. There exist arcs α_i in M_i^* such that bdry $\alpha_i=\{p_i\times\{1\}, q_i\}$ and int $\alpha_i \subset$ int M_i^*—N. Let α be an arc in bdry Q such that bdry $\alpha=\{p_1, p_2\}$ and int $\alpha \subseteq$ bdry $Q-g$ (bdry $(M_1\cup M_2)$). Note that if $2m-q+1<0$, then the theorem is true; hence we can suppose that Q is simply connected. Hence there is a 2–cell D in $Q\times[0,1]$ such that bdry $D=\alpha_1\cup\alpha_2\cup\alpha\times\{1\}\cup\alpha_3$ where bdry $\alpha_3=\{q_1, q_2\}$ and int $\alpha_3\subseteq$ bdry $N-(S_1\cup S_2)$ and such that int $D\subseteq Q\times(0,1)-(N\cup M_1^*\cup M_2^*)$. Hence, by taking a second derived neighborhood of D, there is a $(q+1)$–cell N_0 in $Q\times[0,1]$ such that

i) $\bar{N}=N\cup N_0$ is a $(q+1)$–cell in $Q\times[0,1]$;
ii) $\bar{N}\cap M_i^*$ is a properly embedded (m_i+1)–cell in $\bar{N}$;
iii) $\bar{N}\cap Q\times\{1\}=$bdry $\bar{N}\cap Q\times\{1\}=\bar{\eta}$ is a regular meighborhood of $\alpha\times\{1\}$ in $Q\times\{1\}$;
iv) $\bar{\eta}\cap M_i^*$ is a properly embedded m_i–cell in $\bar{\eta}$;
v) $\bar{\eta}_0=\bar{\eta}\cap$(bdry $Q\times\{1\}$), $\bar{A}=\bar{\eta}\cap$(bdry $g(M_i)\times\{1\}$), $\bar{B}_i=\bar{\eta}\cap(g(M_i)\times\{1\})$ are regular neighborhoods of $\alpha\times\{1\}, p_i\times\{1\}$, respectively, in bdry $Q\times\{1\}$, bdry $g(M_i)\times\{1\}$, $g(M_i)\times\{1\}$, respectively;
vi) the triple (bdry $\bar{N}$, bdry $(\bar{N}\cap M_1^*)$, bdry $(\bar{N}\cap M_2^*)$) is homeomorphic to (bdry N, S_1, S_2).

Let $\rho: Q\times[1]\longrightarrow Q$ be the natural map defined by $\rho(x,1)=x$ and let $\eta=\rho(\bar{\eta})$, $\eta_0=\rho(\bar{\eta}_0)$, $A_i=\rho(\bar{A}_i)$, $B_i=\rho(\bar{B}_i)$. Consider the standard sphere pair (S^q, S^{m_1}). $S^q=\Delta^q_+\cup\Delta^q_-$ where Δ^q_+, Δ^q_- are q–cells such that bdry $\Delta^b_+\cap$ bdry $\Delta^q_-=\Delta^q_+\cap\Delta^q_-$. Similarly $S^{m_1}=\Delta^{m_1}_+\cup\Delta^{m_1}_-$ such that $(\Delta^p_+, \Delta^{m_1}_+)$ and $(\Delta^q_-, \Delta^{m_1}_-)$ are cell pairs.

There exists a homeomorphism $\lambda:(S^q, S^{m_1})\longrightarrow$(bdry $(\eta\times[1,2])$, bdry $(B_1\times[1,2])$) such that $\lambda(\Delta^q_+, \Delta^{m_1}_+)=(\eta_0\times[1,2], A_1\times[1,2])$. By Irwin [5], [9; Theorem 23], there exists an embedding $\varphi: S^{m_2}\longrightarrow S^q-S^{m_1}$ such that $\varphi(S^{m_2})$ represents any element of $\pi_{m_2}(S^q-S^{m_1})$. We'll make our choice of the element of this group later; suppose that some choice has been made.

We may assume that

vii) the m_2–cell $\tilde{B}=\lambda^{-1}(B_2\times\{1,2\}\cup Cl$(bdry $B_2-A_2)\times[1,2])\subseteq\varphi(S^{m_2})$;
viii) $Cl(\varphi(S^{m_2})-\tilde{B})$ is properly embedded in Δ^q_+;
ix) there exists a homeomorphism $\beta_0: A_2\times[1,2]\longrightarrow Cl(\varphi(S^{m_2})-B)$ such that $\lambda\beta_0: A_2\times[1,2]\longrightarrow Q\times[1,2]$ is a concordance and hence may be assumed to be an isotopy [3] such that $\lambda\beta_0(x,t)=(g(x),t)$ for $t=1,2$.

$\lambda\varphi(S^{m_2})$ bounds a properly embedded (m_2+1)–cell D_0 in $\eta\times[1,2]$. Again we find a homeomorphism $\beta_1: B_2\times[1,2]\longrightarrow D_0$ for which $\lambda\beta_1: B_2\times[1,2]$ is a concordance such that

x) $\lambda\beta_1(x,t)=(g(x),t), t=1,2$;
xi) $\lambda\beta_1(x,t)=(g(x),t), t\in[1,2], x\in Cl(\text{bdry } B_2-A_2)$;
xii) $\lambda\beta_1|A_2\times[1,2]=\lambda\beta_0$.

Define $H: M_2\times[1,2]\longrightarrow Q\times[1,2]$ by

$$\begin{aligned} H(x,t) &= (g(x),t) && x\in Cl(M_2-B_2) \\ &= \lambda\beta_1(x,t) && x\in B_2. \end{aligned}$$

Note that H is a concordance such that $H(x,2)=(g(x),2)$, $x\in M_2$,

Consider $M_1^{**}=M_1^*\cup g(M_1)\times[1,2]$ and $M_2^{**}=M_2^*\cup H(M_2\times[1,2])$ in $M\times[0,2]$. $M_1^{**}\cap M_2^{**}=\text{Int } M_1^{**}\cap \text{int } M_2^{**}=[(\bar{N}\cap M_1^*)\cup B_1\times[1,2])]\cap[(\bar{N}\cap M_2^*)\cup D_0]$, the intersection of a (m_1+1)–cell and a (m_2+1)–cell, both of which are properly embedded in the $(q+1)$–cell $N\cup(\eta\times[1,2])$. By choosing φ correctly above, the inclusion bdry $[(\bar{N}\cap M_2^*)\cup D_0]\longrightarrow$ bdry $[\bar{N}\cup(\eta\times[1,2])]$ – bdry $[(\bar{N}\cap M_1^*)\cup(B_1\times[1,2])]$ will be null-homotopic and we can proceed as before after the Proposition.

4. Proof of Theorem 2 when $n>2$. There exist isotopies $k^i: M_i\times[0,1]\longrightarrow Q\times[0,1]$, $i=1,\cdots,n$, such that $k_0^i=f|M_i$, $k_1^i=g|M_i$. Consider the following statement $I(j)$: *There exist isotopies* $\bar{k}^i: M_i\times[0,p]\longrightarrow Q\times[0,p]$, $i=1,\cdots,n$, *such that* $\bar{k}_0^i=f|M_i$, $\bar{k}_p^i=g|M_i$, $\bar{k}_t^r(\text{bdry } M_r)\cap\bar{k}_t^s(\text{bdry } M_s)=\phi$ for $r,s\in\{1,\cdots,n\}, t\in[0,p]$ and $\bar{k}^r(M_r\times[0,p])\cap\bar{k}^s(M_s\times[0,p])=\phi$ for $r,s\in\{1,\cdots,j\}$.

By §3, $I(2)$ is true. Suppose that $I(j)$ is true and that $\bar{k}^1(M_1\times[0,p])\cap\bar{k}^{j+1}(M_{j+1}\times[0,p])\neq\phi$. We shall attempt to move $\bar{k}^{j+1}(M_{j+1}\times[0,p])$ off $\bar{k}^1(M_1\times[0,p])$ so that $k^r(M_r\times[0,p])\cap\bar{k}^s(M_t\times[0,p])=\phi$ for $r,s\in\{1,\cdots,j\}$. Consider the proof in §3. In the first part of the proof, we worked in the interior of the cell N, where N is a regular neighborhood of C_3. Note that since $3m+4\leq 2q$ C_3 can be chosen so that $C_3\cap\bar{k}^r(M_r\times[0,p])=\phi$ for $r\in\{2,\cdots,j\}$ and hence N can be chosen so that $N\cap\bar{k}^r(M_r\times[0,p])=\phi$.

In the second part of the proof, we worked in $\eta\times[1,2]$. η can be chosen so that $\eta\cap g(M_r)=\phi$ for $r\neq 1, j+1$, Hence from the proof of §3, we can find isotopies $\bar{k}^i: M_i\times[0,p+1]\longrightarrow Q\times[0,p+1]$, $i=1, j+1$, such that $\bar{k}_0^i=f|M_i$, $\bar{k}_{p+1}^i=g|M_i$, $\bar{k}^1(M_1\times[0,p+1])\cap\bar{k}^{j+1}(M_{j+1}\times[0,p+1])=\phi$. For $i=2,\cdots,j$, define $\bar{k}^i(x,t)=(g(x),t)$ for $t\in[p,p+1]$. Now separate $\bar{k}^2(M_2\times[0,p+1])$ and $\bar{k}^{j+1}(M_{j+1}\times[0,p+1])$ as above, using the interval $[p+1,p+2]$ if necessary. By induction, $I(j+1)$ is true, hence, by induction, the theorem is true.

Note by choosing carefully the required $\eta' s$ in the two proofs above, we have the following.

Corollary 2. *Let M_i, Q, f and g be as in Theorem 2 except suppose that there exists a connectted nonempty open subset V of* bdry Q *such that* f(bdry M_i)$\cap V \neq \phi$ *for each i. The isotopy h_t can be chosen so that $h_t|$*bdry *$Q-V$ is the identity for all t and i.*

4. Proof of Theorem 1. Suppose $P=|K/T|$ where K and T are given in the definition that P is reduced. Let a be the cone point and let $p:|K| \longrightarrow P$ be the usual projection map. (Note that, in general, p is not PL.) Let $\bar{R}$ be the second derived neighborhood of the $(k-1)$-skeleton of K in K and let $R=p(\bar{R})$. Note that $Cl(|K|-\bar{R}) = Cl(P-R) = \cup D_i$ where $\{D_i\}$ is a collection of disjoint k-cells. Let f and g be two embeddings of P into E^{2k+1}. By either [2] or [7], there is an isotopy k_t of E^{2k+1} onto itself such that $k_0=$identity and $k_1 f|R=g|R$. Let N_0 be a regular neighborhood of $g(|\text{st}(a, K/T)|)$ mod $f(Cl(P-|\text{st}(a, K/T)|)) \cup g(Cl(P-|\text{st}(a, K/T)|))$ in E^{2k+1} [1] and let N be a regular neighborhood of $g(Cl(R-|\text{st}(a, K/T)|))$ mod $f(\cup D_i) \cup g(\cup D_i)$ in $Cl(E^{2k+1}-N_0)$. Let $Q=Cl(E^{2k+1}-N)$, $V=$int (bdry $N_0 \cap$ bdry N). Hence $k_1 f| \cup D_i$ and $g| \cup D_i$ are two embeddings of $\cup D_i$ into Q which satisfy the hypotheses of Corollary 2. Hence there is an isotopy $\bar{h}_t$ of Q onto itself such that $\bar{h}_0=$identity, $\bar{h}_t|$bdry $Q-V$ is the identity and $\bar{h}_1 k_1 f| \cup D_i = g| \cup D_i$. Define an isotopy h_t of E^{2k+1} onto itself by

$$\begin{aligned} h_t(x) &= \bar{h}_t(x) && x \in Q \\ &= x && x \in Cl(N-N_0) \\ &= j_t(x) && x \in N_0 \end{aligned}$$

where j_t is an isotopy of N_0 onto itself which is the conical extension of $h_t|$bdry $N_0 \cap$ bdry N and the identity on bdry N_0-bdry N. The composition $h_t k_t$ gives the desired isotopy.

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