SOME INVERSION FORMULAE

By

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1. **Introduction**: Bateman has proved that

$$2f(s)=\int_s^\infty dx\int_{-\infty}^\infty J_0(x-s)\frac{J_1(x-t)}{x-t}f(t)\,dt, \tag{1.1}$$

where $J_\nu(x)$ denotes the Bessel function of order ν.

A generalisation of (1.1) has been given by Hardy [2], who has obtained a formula envolving Bessel functions of order ν and $1-\nu$.

Fox [3] has obtained the following result on somewhat different lines:

Theorem IA:

If $f(s)$ is defined by either of the formulae

$$f(s)=\int_a^b \phi(\mu)\begin{matrix}\cos\\ \sin\end{matrix}(\mu s)\,d\mu, \tag{1.2}$$

where

$$\int_a^b |\phi(\mu)|\,d\mu \tag{1.3}$$

exists and $-1< a < b < 1$, then

$$2f(s)=\int_s^\infty dx\int_{-\infty}^x \left[(\nu-1)\frac{J_{\nu-1}(x-t)}{x-t}J_\nu(x-s)+\nu\frac{J_\nu(x-t)}{x-t}J_{\nu-1}(x-s)\right]f(t)\,dt, \tag{1.4}$$

for $\nu>1$ *and*

$$2f(s)=\int_s^\infty dx\int_{-\infty}^\infty \left[(\nu-1)\frac{J_{\nu-1}(x-t)}{x-t}J_\nu(x-s)+\nu\frac{J_\nu(x-t)}{x-t}J_{\nu-1}(x-s)\right]f(t)\,dt, \tag{1.5}$$

where ν is an integer or zero.

From (1.4) and (1.5), on subtraction, we get

$$\int_s^\infty dx\int_x^\infty \left[(\nu-1)\frac{J_{\nu-1}(x-t)}{x-t}J_\nu(x-s)+\nu\frac{J_\nu(x-t)}{x-t}J_{\nu-1}(x-s)\right]f(t)\,dt=0, \tag{1.6}$$

where ν is any integer other than 0 and 1.

Fox, at the end of his paper [3] has stated that eqns. (1.4), (1.5) and (1.6) hold good for a function satisfying the equation

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin(x-t)}{x-t}f(t)\,dt. \tag{1.7}$$

Brij Mohan [4] has further generalised this result, which runs as follows:

Theorem IB: *If $f(s)$ is defined by either of the formulae* (1.2), *then*

$$\int_{s}^{\infty}dx\int_{-\infty}^{x}\Big[J_{\lambda_1+1}(x-s)\,J_{\lambda_1-1}(x-t)+J_{\lambda_2}(x-s)J_{\lambda_2}(x-t)+J_{\lambda_3}(x-s)J_{\lambda_3}(x-t)$$
$$+J_{\lambda_4-1}(x-s)J_{\lambda_4+1}(x-t)\Big]f(t)\,dt=4f(s), \tag{1.8}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are all positive.

In this paper an attempt has been made to give a generalization of (1.4) and (1.5) together with a more generalised theorem (IB) and to obtain some new inversion formulae. The method employed is mainly that of Hardy.

2. Theorem 1:

If $f(s)$ is defined by either of the formulae given in (1.2)

$$\textit{or}\qquad f(s)=\int_{a}^{b}\phi(c)\begin{array}{c}\sin\\ \cos\end{array}\mu(s-c)\,dc, \tag{2.1}$$

where $\displaystyle\int_{a}^{b}|\phi(\mu)|\,d\mu$ *exists and* $-1<a<b<1$,

then

$$4f(s)=\int_{s}^{\infty}dx\int_{-\infty}^{x}\Big[(\lambda_1+1)J_{\lambda_1}(x-s)\frac{J_{\lambda_1+1}(x-t)}{x-t}+(\lambda_2-1)J_{\lambda_2}(x-s)$$
$$\cdot\frac{J_{\lambda_2-1}(x-t)}{x-t}+\lambda_3 J_{\lambda_3+1}(x-s)\frac{J_{\lambda_3}(x-t)}{x-t}+\lambda_4 J_{\lambda_4-1}(x-s)\frac{J_{\lambda_4}(x-t)}{x-t}\Big]f(t)\,dt, \tag{2.2}$$

where $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are all positive.

Before proceeding to the proof of this theorem, let us note that this equation leads to the following inversion formula.

$$\text{If}\qquad F_\lambda(x)=\int_{-\infty}^{x}\lambda\frac{J_\lambda(x-t)}{x-t}f(t)\,dt, \tag{2.3}$$

then

$$4f(s)=\int_s^\infty [J_{\lambda_1}(x-s)\,F_{\lambda_1+1}(x)+J_{\lambda_2}(x-s)\,F_{\lambda_2-1}(x)+J_{\lambda_3+1}(x-s)\,F_{\lambda_3}(x)+J_{\lambda_4-1}(x-s)\,F_{\lambda_4}(x)]\,dx, \tag{2.4}$$

provided that $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are all positive.

Proof of (2.2):

Let the R. H. S. of (2.2)$=j_1+j_2+j_3+j_4$.

Then, assuming first that

$$f(s)=\int_a^b \phi(c)\cos\mu\,(s-c)\,dc,$$

we find that the absolute convergence of

$$\int_0^\infty \frac{J_\lambda(u)}{u}\,du$$

in conjunction with the restrictions imposed on $\phi(c)$ enables us to change the order of integration in the integral

$$\int_0^\infty \lambda\frac{J_\lambda(u)}{u}\,du\int_a^b \phi(c)\cos\mu\,(u-c)\,dc.$$

Hence

$$j_1=\int_s^\infty J_{\lambda_1}(x-s)\,dx\int_{-\infty}^x (\lambda_1+1)\frac{J_{\lambda_1+1}(x-t)}{x-t}\,dt\int_a^b \phi(c)\cos\mu\,(t-c)\,dc$$

$$=\int_s^\infty J_{\lambda_1}(x-s)\,dx\int_a^b \phi(c)\,dc\int_0^\infty (\lambda_1+1)\frac{J_{\lambda_1+1}(u)}{u}\cos\mu\,(x-u-c)\,du,$$

in which, after changing the order of integration, t is replaced by $(x-u)$.

Now, on using Watson's formula

$$\lambda\int_0^\infty \frac{J_\lambda(\alpha t)}{t}\,{\sin \atop \cos}(\beta t)\,dt={\sin \atop \cos}(\lambda\sin^{-1}\beta/\alpha), \tag{2.5}$$

where $\beta<\alpha$ and $Re(\lambda)>0$, we obtain

$$j_1=\int_s^\infty J_{\lambda_1}(x-s)\,dx\int_a^b \phi(c)\cos(\mu\,\overline{x-c}-\overline{\lambda_1+1}\sin^{-1}\mu)\,dc. \tag{2.6}$$

From the asymptotic expansion of the Bessel function it is obvious that the integral

$$\int_s^\infty J_{\lambda_1}(x-s)\cos(\mu\,\overline{x-c}-\overline{\lambda_1+1}\sin^{-1}\mu)\,dx$$

is uniformly convergent with respect to c if c is such that $-1<a\leq c\leq b<1$. Hence, given an ϵ, we can find a λ_0 such that

$$\left| \int_a^b \phi(c)\,dc \int_\lambda^\infty J_{\lambda_1}(x-s) \cos\{\mu(x-c)-(\lambda_1+1)\sin^{-1}\mu\}\,dx \right|$$

$$\leq \int_a^b |\phi(c)|\,dc \left| \int_\lambda^\infty J_{\lambda_1}(x-s) \cos\{\mu(x-c)-(\lambda_1+1)\sin^{-1}\mu\}\,dx \right| <\epsilon$$

whenever $\lambda>\lambda_0$. It follows that if λ is sufficiently large, then

$$\left| \int_s^\lambda J_{\lambda_1}(x-s)\,dx \int_a^b \phi(c) \cos\{\mu(x-c)-(\lambda_1+1)\sin^{-1}\mu\}\,dc \right.$$

$$\left. -\int_a^b \phi(c)\,dc \int_s^\infty J_{\lambda_1}(x-s) \cos\{\mu(x-c)-(\lambda_1+1)\sin^{-1}\mu\}\,dx \right| <\epsilon.$$

Now using (2.6), replacing $(x-s)$ in the second integral inside the modulus sign by u and making $\lambda\to\infty$, we obtain

$$\int_s^\infty J_{\lambda_1}(x-s)\,dx \int_a^b \phi(c) \cos\{\mu(x-c)-(\lambda_1+1)\sin^{-1}\mu\}\,dc$$

$$=\int_a^b \phi(c)\,dc \int_0^\infty J_{\lambda_1}(u) \cos\{\mu(s+u-c)-(\lambda_1+1)\sin^{-1}\mu\}\,du.$$

But we know that

$$\int_0^\infty J_\lambda(\alpha t)\cdot \begin{matrix}\sin\\ \cos\end{matrix}(\beta t)\,dt=\frac{1}{\sqrt{(\alpha^2-\beta^2)}} \begin{matrix}\sin\\ \cos\end{matrix}(\lambda \sin^{-1}\beta/\alpha) \tag{2.7}$$

where $\beta<\alpha$ and $Re(\lambda)>0$.

Hence, we get

$$j_1=\int_a^b \frac{\phi(c)}{\sqrt{(1-\mu^2)}} \cos\{\mu(s-c)-\sin^{-1}\mu\}\,dc.$$

Similarly, we obtain

$$j_2=\int_a^b \frac{\phi(c)}{\sqrt{(1-\mu^2)}} \cos\{\mu(s-c)+\sin^{-1}\mu\}\,dc$$

$$j_3=\int_a^b \frac{\phi(c)}{\sqrt{(1-\mu^2)}} \cos\{\mu(s-c)+\sin^{-1}\mu\}\,dc$$

and

$$j_4=\int_a^b \frac{\phi(c)}{\sqrt{(1-\mu^2)}} \cos\{\mu(s-c)-\sin^{-1}\mu\}\,dc.$$

Adding these we arrive at :

$$j_1+j_2+j_3+j_4=2\int_a^b \frac{\phi(c)}{\sqrt{(1-\mu^2)}}\,[\cos\{\mu(s-c)+\sin^{-1}\mu\}+\cos\{\mu(s-c)-\sin^{-1}\mu\}]\,dc$$

$$=4\int_a^b \phi(c).\cos\{\mu(s-c)\}\,dc=4f(s),$$

which proves (2.2).

Proceeding on parallel lines the theorem could have been proved for the other three formulae. Furthermore, the theorem may also be proved for all integral values of $\lambda' s$. The truth of (2.2), however, for negative integral values of $\lambda' s$ (zero and one excluded) follows from the usual definition of the Bessel function, namely,

$$J_{-n}(x)=(-1)^n J_n(x), \tag{2.8}$$

where n is a positive integer. The case when $\lambda' s$ take the value 0 or 1 are somewhat difficult. When all the $\lambda' s$ are zero, the two terms on the R. H. S. become the products of zero and a divergent integral. The other term, however, is the same as the integral on the left of Bateman's result (1.1). In view of the reason given above, the theorem also fails to hold good for $\lambda_2=1=\lambda_4$. And so we may say, in a sense, that (2.2) is a generalisation of (1.1) for all values of $\lambda' s$ other than zero and unity.

Theorem IA :

In the above theorem if the function $\phi(\mu)$ *is such that* $(1-\mu)^n\phi(\mu)$ *is monotonic near* $\mu=1$, *where* $0\leq n\leq\frac{1}{2}$, *then the theorem still holds good provided that it holds for* $-1<a<b<1$.

This may easily be shown to be true on proceeding exactly as in the corresponding result given by Fox [3].

Particular Cases of (2.2) :

(i) Putting $\lambda_1+1=\lambda_2=\lambda_3+1=\lambda_4=\nu$ we obtain

$$2f(s)=\int_s^\infty dx\int_{-\infty}^x\Big[J_{\nu-1}(x-s)\,\nu\,\frac{J_\nu(x-t)}{x-t}+J_\nu(x-s)(\nu-1)\,\frac{J_{\nu-1}(x-t)}{x-t}\Big]f(t)\,dt \tag{2.9}$$

which is a result due to Fox [3]. The inversion formula which it leads to, as stated by Fox, is

$$2f(s)=\int_s^\infty \Big[J_\nu(x-s)\,F_{\nu-1}(x)+J_{\nu-1}(x-s)\,F_\nu(x)\Big]\,dx,$$

where
$$F_\nu(x)=\int_{-\infty}^x \frac{J_\nu(x-t)}{x-t}\,f(t)\,dt. \tag{2.10}$$

(ii) On taking $\lambda_1+1=\lambda_2-1=\lambda_3=\lambda_4=\nu$, we get

$$f(s)=\int_s^\infty \nu\,\frac{J_\nu(x-s)}{x-s}\,dx\int_{-\infty}^x \nu\,\frac{J_\nu(x-t)}{x-t}\,f(t)\,dt, \tag{2.11}$$

a result given by Brij Mohan [4]. It leads to the inversion formula:

$$f(s)=\nu\int_s^\infty \frac{J_\nu(x-s)}{x-s}\,F_\nu(x)\,dx,$$

where $F_\nu(x)$ is given by (2.10).

(iii) Another known result due to Brij Mohan [4], namely

$$2f(s)=\int_s^\infty dx\int_{-\infty}^x J_\nu(x-s)\Big[(\nu+1)\,\frac{J_{\nu+1}(x-t)}{x-t}+(\nu-1)\frac{J_{\nu-1}(x-t)}{x-t}\Big]f(t)\,dt, \tag{2.12}$$

is arrived at when the substitutions,

$\lambda_1=\lambda_2=\lambda_3+1=\lambda_4-1=\nu$ are made in (2.2).

This leads to the inversion formula:

if
$$F_\nu(x)=\int_{-\infty}^x \nu\,\frac{J_\nu(x-t)}{x-t}\,f(t)\,dt,$$

then

$$f(s)=\tfrac{1}{2}\int_s^\infty J_\nu(x-s)\,[F_{\nu-1}(x)+F_{\nu+1}(x)]\,dx. \tag{2.13}$$

(iv) Making the substitutions $\lambda_1+2=\lambda_2=\nu=\lambda_3=\lambda_4$ in (2.2) we obtain

$$4f(s)=\int_s^\infty dx\int_{-\infty}^x\Big[2(\nu-1)^2\,\frac{J_{\nu-1}(x-s)\,J_{\nu-1}(x-t)}{(x-s)\quad(x-t)}+2\nu^2\,\frac{J_\nu(x-s)\,J_\nu(x-t)}{(x-s)\quad(x-t)}\Big]f(t)\,dt, \tag{2.14}$$

which leads to the inversion formula

$$2f(s)=\int_s^\infty\Big[(\nu-1)\,\frac{J_{\nu-1}(x-s)}{x-s}\,F_{\nu-1}(x)+\nu\,\frac{J_\nu(x-s)}{x-s}\,F_\nu(x)\Big]\,dx,$$

where

$$F_\nu(x)=\int_{-\infty}^x \nu\,\frac{J_\nu(x-t)}{x-t}\,f(t)\,dt. \tag{2.15}$$

Rewriting (2.14) in the form

$$4f(s)=\int_s^\infty dx\int_{-\infty}^x\Big[(\nu-1)\frac{J_{\nu-1}(x-s)}{x-s}\{J_\nu(x-t)+J_{\nu-2}(x-t)\}+\nu\frac{J_\nu(x-s)}{x-s}$$

$$\{J_{\nu+1}(x-t)+J_{\nu-1}(x-t)\}\Big]f(t)\,dt \tag{2.16}$$

and subtracting from it the following result

$$2f(s)=\int_s^\infty dx\int_{-\infty}^x\Big[\nu\frac{J_\nu(x-s)}{x-s}J_{\nu-1}(x-t)+(\nu-1)\frac{J_{\nu-1}(x-s)}{x-s}J_\nu(x-t)\Big]f(t)\,dt$$

due to Brij Mohan [4, p. (iii)] , we obtain

$$2f(s)=\int_s^\infty dx\int_{-\infty}^x\Big[\nu\frac{J_\nu(x-s)}{x-s}J_{\nu-1}(x-t)+(\nu-1)\frac{J_{\nu-1}(x-s)}{x-s}J_\nu(x-t)\Big]f(t)\,dt. \tag{2.17}$$

It leads to the inversion formula

$$f(s)=\tfrac{1}{2}\int_s^\infty\Big[(\nu-1)\frac{J_{\nu-1}(x-s)}{x-s}F_{\nu-2}(x)+\nu\frac{J_\nu(x-s)}{x-s}F_{\nu+1}(x)\Big]dx,$$

where

$$F_\nu(x)=\int_{-\infty}^x J_\nu(x-t)f(t)\,dt.$$

The rigorous proof of (2.17) follows exactly on the lines of the proof of the main theorem.

(v) Substituting ν for each of λ_1+2, λ_2-2, λ_3-1 and λ_4+1, we obtain

$$2f(s)=\int_s^\infty dx\int_{-\infty}^x\Big[(\nu-1)J_{\nu-2}(x-s)\frac{J_{\nu-1}(x-t)}{x-t}+(\nu+1)\frac{J_{\nu+2}(x-s)J_{\nu+1}(x-t)}{x-t}\Big]$$

$$f(t)\,dt. \tag{2.18}$$

The inversion formula which it leads to is

$$f(s)=\tfrac{1}{2}\int_s^\infty\Big[J_{\nu-2}(x-s)\,G_{\nu-1}(x)+J_{\nu+2}(x-s)\,G_{\nu+1}(x)\Big]dx, \tag{2.19}$$

where $$G_\nu(x)=\int_{-\infty}^x\nu\frac{J_\nu(x-t)}{x-t}f(t)\,dt. \tag{2.20}$$

(vi) Putting ν for each of the numbers λ_1+2, λ_2, λ_3-1 and λ_4-1 in (2.2), we get

$$2f(s)=\int_s^\infty dx\int_{-\infty}^x\Big[(\nu-1)^2\frac{J_{\nu-1}(x-s)J_{\nu-1}(x-t)}{(x-s)(x-t)}+(\nu+1)^2\frac{J_{\nu+1}(x-s)}{(x-s)}$$

$$\frac{J_{\nu+1}(x-t)}{(x-t)}\Big]f(t)\,dt \tag{2.21}$$

which leads to the inversion formula

$$f(s)=\frac{1}{2}\int_{s}^{\infty}\left[(\nu-1)\frac{J_{\nu-1}(x-s)}{x-s}H_{\nu-1}(x)+(\nu+1)\frac{J_{\nu+1}(x-s)}{x-s}H_{\nu+1}(x)\right]dx, \tag{2.22}$$

where

$$H_{\nu}(x)=\int_{-\infty}^{x}\frac{J_{\nu}(x-t)}{x-t}f(t)\,dt. \tag{2.23}$$

Writing $J_{\nu}(x-t)+J_{\nu-2}(x-t)$ for $2(\nu-1)\dfrac{J_{\nu-1}(x-t)}{x-t}$ in (2.22)

and thereafter subtracting from it another result due to Brij Mohan [4], namely

$$2f(s)=\int_{s}^{\infty}dx\int_{-\infty}^{x}\left[(\nu-1)\frac{J_{\nu-1}(x-t)}{x-t}J_{\nu}(x-s)+(\nu+1)\frac{J_{\nu+1}(x-s)}{x-s}J_{\nu}(x-t)\right]f(t)\,dt \tag{2.24}$$

we arrive at

$$2f(s)=\int_{s}^{\infty}dx\int_{-\infty}^{x}\left[(\nu-1)\frac{J_{\nu-1}(x-s)}{x-s}J_{\nu-2}(x-t)+(\nu+1)\frac{J_{\nu+1}(x-s)}{x-s}J_{\nu+2}(x-t)\right]f(t)\,dt, \tag{2.25}$$

which leads to the inversion formula

$$2f(s)=\int_{s}^{\infty}\left[(\nu-1)\frac{J_{\nu-1}(x-s)}{x-s}G_{\nu-2}(x)+(\nu+1)\frac{J_{\nu+1}(x-s)}{x-s}G_{\nu+2}(x)\right]dx,$$

where

$$G_{\nu}(x)=\int_{-\infty}^{x}J_{\nu}(x-t)f(t)\,dt.$$

Similar cases may be arrived at on particularizing the parameters in the above manner.

3. Theorem 2 :

If $f(s)$ be defined by either of the formulae given in theorem 1, then

$$4f(s)=\int_{s}^{\infty}dx\int_{-\infty}^{x}\left[(\lambda_1+1)\frac{J_{\lambda_1+1}(x-s)}{x-s}J_{\lambda_1}(x-t)+(\lambda_2-1)\frac{J_{\lambda_2-1}(x-s)}{x-s}J_{\lambda_2}(x-t)\right.$$
$$\left.+\lambda_3\frac{J_{\lambda_3}(x-s)}{x-s}J_{\lambda_3+1}(x-t)+\lambda_4\frac{J_{\lambda_4}(x-s)}{x-s}J_{\lambda_4-1}(x-t)\right]f(t)\,dt, \tag{3.1}$$

provided that

$$\int_a^b |\phi(\mu)|\, d\mu$$

exists for $-1<a<b<1$ *and* $\lambda_1, \lambda_2, \lambda_3$ *and* λ_4 *are all positive.*

In addition, if the function $\phi(\mu)$ *is such that* $(1-\mu)^\alpha \phi(\mu)$ *is monotonic near* $\mu=1$, *where* $0<\alpha<\frac{1}{2}$, *then the theorem still holds provided that it holds for* $-1<a<b<1$.

Obviously (3.1) leads to the inversion formula

$$4f(s)=\int_s^\infty \Big[(\lambda_1+1)\frac{J_{\lambda_1+1}(x-s)}{x-s}G_{\lambda_1}(x)+(\lambda_2-1)\frac{J_{\lambda_2-1}(x-s)}{x-s}G_{\lambda_2}(x)$$

$$+\lambda_3\frac{J_{\lambda_3}(x-s)}{x-s}G_{\lambda_3+1}(x)+\lambda_4\frac{J_{\lambda_4}(x-s)}{x-s}G_{\lambda_4-1}(x)\Big]f(t)\,dt,$$

where

$$G_\lambda(x)=\int_{-\infty}^x J_\lambda(x-t)f(t)\,dt.$$

The proof of this theorem may be given in a manner similar to that of theorem 1.

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