ON A GENERATING FUNCTION OF HERMITE POLYNOMIALS

By

BIBHUTI BHUSAN SAHA

(Received July 14, 1969)

1. In a recent paper [1] Chatterjea has proved the following generating function for the Hermite Polynomials from the view-point of continuous transformation groups:

$$\exp(2xt-t^2)=\sum_{n=0}^{\infty}\frac{[(\mu-\sqrt{\mu^2-1})\,t]^n}{n!}H_n((\mu+\sqrt{\mu^2-1})x-\sqrt{\mu^2-1}\cdot t), \tag{1.1}$$

where $H_n(x)$ is defined by the Rodrigues' formula

$$H_n(x)=(-1)^n e^{x^2}D^n(e^{-x^2})$$

and μ is quite arbitrary.

In particular, when $\mu=1$, one obtains the usual generating function

$$\exp(2xt-t^2)=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x). \tag{1.2}$$

The object of this note is to point out that one can make elementary verification of Chatterjea's result (1.1) by means of any one of the following formulas of Hermite Polynomials:

Explicit representation [2]

$$H_n(x)=\sum_{m=0}^{[n/2]}\frac{(-1)^m\,n!\,(2x)^{n-2m}}{m!\,(n-2m)!}. \tag{1.3}$$

Addition theorem [2, p. 255, 254]

$$H_n(x+y)=\sum_{m=0}^{n}\binom{n}{m}H_m(x)(2y)^{n-m}, \tag{1.4}$$

$$H_n(x+y)=2^{-n/2}\sum_{m=0}^{n}\binom{n}{m}H_m(x\sqrt{2})\,H_{n-m}(y\sqrt{2}). \tag{1.5}$$

Multiplication theorem [3]

$$H_n(\mu x)=\sum_{m=0}^{[n/2]}\mu^n\frac{n!}{m!}\left(1-\frac{1}{\mu^2}\right)^m\frac{H_{n-2m}(x)}{(n-2m)!}. \tag{1.6}$$

In other words, we wish to show that Chatterjea's generating function (1.1) contains many properties of Hermite Polynomials, viz. the explicit representation, the addition formulas, the multiplication formula, in addition to the usual generating function.

2. For our purpose, we write (1.1) in the form

$$\exp(2xt-t^2)=\sum_{n=0}^{\infty}\frac{(t/\lambda)^n}{n!}H_n\left(\lambda x+\frac{1-\lambda^2}{2\lambda}t\right). \tag{2·1}$$

First we notice that

$$\exp(2xt-t^2)=\exp(2xt-t^2+t^2/\lambda^2)\cdot\exp(-t^2/\lambda^2)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(t/\lambda)^{n+2k}\frac{(-)^k\left(2\lambda x+\frac{1-\lambda^2}{\lambda}t\right)^n}{k!\,n!} \tag{2.2}$$

$$=\sum_{n=0}^{\infty}\frac{(t/\lambda)^m}{n!}\cdot\sum_{k=0}^{[n/2]}\frac{(-1)^k n!\left(2\lambda x+\frac{1-\lambda^2}{\lambda}t\right)^{n-2k}}{k!\,(n-2k)!}.$$

If follows therefore from (2.1) and (2.2) that

$$H_n\left(\lambda x+\frac{1-\lambda^2}{2\lambda}t\right)=\sum_{k=0}^{[n/2]}\frac{(-1)^k n!\left(2\lambda x+\frac{1-\lambda^2}{\lambda}t\right)^{n-2k}}{k!\,(n-2k)!},$$

which is the explicit representation (1.3) for Hermite Polynomials. Next we have

$$\exp(2xt-t^2)=\exp(t^2(1-\lambda^2)/\lambda^2)\exp\left(2\lambda x\cdot\frac{t}{\lambda}-\frac{t^2}{\lambda^2}\right)$$

$$=\sum_{n=0}^{\infty}\frac{\left[\frac{t^2(1-\lambda^2)}{\lambda^2}\right]^n}{n!}\sum_{m=0}^{\infty}\frac{(t/\lambda)^m}{m!}H_m(\lambda x) \tag{2.3}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(t/\lambda)^{n+m}}{n!\,m!}H_m(\lambda x)\left(\frac{1-\lambda^2}{\lambda}t\right)^n$$

$$=\sum_{n=0}^{\infty}\frac{(t/\lambda)^n}{n!}\sum_{m=0}^{n}\binom{n}{m}H_m(\lambda x)\left(\frac{1-\lambda^2}{\lambda}t\right)^{n-m}.$$

It follows therefore from (2.1) and (2.3) that

$$H_n\left(\lambda x+\frac{1-\lambda^2}{2\lambda}t\right)=\sum_{m=0}^{n}\binom{n}{m}H_m(\lambda x)\left(\frac{1-\lambda^2}{\lambda}t\right)^{n-m},$$

which is the addition formula (1.4) for the Hermite Polynomials. Again we have

$$\exp(2xt-t^2)$$

$$=\exp\left(2\cdot\frac{1-\lambda^2}{\lambda\sqrt{2}}t\cdot\frac{t}{\lambda\sqrt{2}}-\frac{t^2}{2\lambda^2}\right)\exp\left(2\lambda x\sqrt{2}\cdot\frac{t}{\lambda\sqrt{2}}-\frac{t^2}{2\lambda^2}\right)$$

(2.4) $$=\sum_{n=0}^{\infty}\frac{(t/\lambda\sqrt{2})^n}{n!}H_n\left(\frac{1-\lambda^2}{\lambda\sqrt{2}}t\right)\sum_{m=0}^{\infty}\frac{(t/\lambda\sqrt{2})^m}{m!}H_m(\lambda x\sqrt{2})$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(t/\lambda\sqrt{2})}{n!\,m!}H_m(\lambda x\sqrt{2})H_n\left(\frac{1-\lambda^2}{\lambda\sqrt{2}}t\right)$$

$$=\sum_{n=0}^{\infty}\frac{(t/\lambda)^n}{n!}\sum_{m=0}^{n}2^{-\frac{n}{2}}\binom{n}{m}H_m(\lambda x\sqrt{2})H_{n-m}\left(\frac{1-\lambda^2}{\lambda\sqrt{2}}t\right).$$

It follows therefore from (2.1) and (2.4) that

$$H_n\left(\lambda x+\frac{1-\lambda^2}{2\lambda}t\right)=2^{-\frac{n}{2}}\sum_{m=0}^{n}\binom{n}{m}H_m(\lambda x\sqrt{2})H_{n-m}\left(\frac{1-\lambda^2}{2\lambda}t\sqrt{2}\right),$$

which is the addition formula (1.5) for the Hermite Polynomials.

Lastly we note that

$$\exp(2xt-t^2)$$

$$=\exp(t^2(1-\lambda^2)/\lambda^2).\quad\exp(2xt-t^2).\quad\exp(t^2(\lambda^2-1)/\lambda^2)$$

$$=\sum_{n=0}^{\infty}\frac{\left[\frac{t^2(1-\lambda^2)}{\lambda^2}\right]^n}{n!}\cdot\sum_{m=0}^{\infty}\frac{t^m}{m!}H_m(x)\sum_{k=0}^{\infty}\frac{\left[\frac{t^2(\lambda^2-1)}{\lambda^2}\right]^k}{k!}$$

(2.5) $$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{\left[\frac{t^2(1-\lambda^2)}{\lambda^2}\right]^n}{n!}(t)^{n+2k}\left(\frac{\lambda^2-1}{\lambda^2}\right)^k\frac{H_m(x)}{k!\,m!}$$

$$=\sum_{n=0}^{\infty}\frac{\left[\frac{t^2(1-\lambda^2)}{\lambda^2}\right]^n}{n!}\sum_{m=0}^{\infty}\frac{(t/\lambda)^m}{m!}\sum_{k=0}^{[m/2]}\lambda^m\frac{m!}{k!}\left(1-\frac{1}{\lambda^2}\right)^k\frac{H_{m-2k}(x)}{(m-2k)!}.$$

Now in deriving (2.3) we have noticed

(2.6) $$\exp(2xt-t^2)$$

$$=\sum_{n=0}^{\infty}\frac{\left[\frac{t^2(1-\lambda^2)}{\lambda^2}\right]^n}{n!}\sum_{m=0}^{\infty}\frac{(t/\lambda)^m}{m!}H_m(\lambda x).$$

Thus it follows from (2.5) and (2.6) that

$$H_m(\lambda x)=\sum_{k=0}^{[m/2]}\lambda^m\frac{m!}{k!}\left(1-\frac{1}{\lambda^2}\right)^k\frac{H_{m-2k}(x)}{(m-2k)!},$$

which is the multiplication formula (1.6) for Hermite Polynomials.

I am indebted to Dr. S. K. Chatterjea for his kind help in preparing this note.

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Bangabasi College,
Calcutta 9, West Bengal, India.