GENERATING FUNCTIONS FOR A CLASS OF POLYNOMIALS*

By

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1. Introduction.

Put

$$E[z]=\sum_{n=0}^{\infty}\frac{z^n}{n!},\quad G[z]=\sum_{n=0}^{\infty}g_n z^n\,(g_n\neq 0), \tag{1.1}$$

and let the coefficients $\{\Psi_n^{(\lambda)}(x)\,|\,n=0,1,2,\cdots\}$ be generated by

$$E[M(x)t^r]\,G[Q(x)t^m]=\sum_{n=0}^{\infty}\frac{t^n}{(\lambda+1)_n}\Psi_n^{(\lambda)}(x), \tag{1.2}$$

where $(\nu)_n$ is the usual Pochhammer symbol defined by

$$(\nu)_n=\nu(\nu+1)(\nu+2)\cdots(\nu+n-1),\ n\geqq 1,\ (\nu)_0=1,$$

$M(x)\neq 0, Q(x)\neq 0$ are real functions, and m, r are positive integers.

In an earlier paper [8] the present author has proved, amongst other results, the following

Theorem. *If the coefficients* $\{\Psi_n^{(\lambda)}(x)\,|\,n=0,1,2,\cdots\}$ *be generated by* (1.2), *then for arbitrary parameter* ν,

$$\begin{aligned}&[1-M(x)t^r]^{-\nu}\,H\left(Q(x)\left[\frac{t^r}{1-M(x)t^r}\right]^{m/r}\right)\\ &\qquad=\sum_{n=0}^{\infty}\frac{\Gamma\left(\nu+\frac{n}{r}\right)}{(\lambda+1)_n}\Psi_n^{(\lambda)}(x)\,t^n,\end{aligned} \tag{1.3}$$

provided

$$H(z)=\sum_{n=0}^{\infty}\Gamma\left(\nu+\frac{mn}{r}\right)g_n z^n. \tag{1.4}$$

It may be of interest to recall that the fundamental importance of this theorem in the theory of generating functions lies in the following fact. Suppose that $G[z]$ is a specified hypergeometric function. Then, since ν is an arbirtary parameter, the theorem

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readily gives for $\Psi_n^{(\lambda)}(x)$ a class of generating functions involving a hypergeometric function of superior order. For instance, if we set

$$G[z] = {}_pF_q\left[\begin{matrix}\alpha_1, \cdots, \alpha_p ; \\ \beta_1, \cdots, \beta_q ;\end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!}, \tag{1.5}$$

and for the sake of simplicity, let $r=1$ so that (1.2) assumes the form

$$\begin{aligned} &E[M(x)t]\; {}_pF_q\left[\begin{matrix}\alpha_1, \cdots, \alpha_p ; \\ \beta_1, \cdots, \beta_q ;\end{matrix}\; Q(x)t^m\right] \\ &= \sum_{n=0}^{\infty} \frac{[M(x)t]^n}{n!}\; {}_{m+p}F_q\left[\begin{matrix}\Delta(m; -n), \alpha_1, \cdots, \alpha_p ; \\ \beta_1, \cdots, \beta_q ;\end{matrix}\; \left(-\frac{m}{M(x)}\right)^m Q(x)\right], \end{aligned} \tag{1.6}$$

then our theorem yields

$$\begin{aligned} &[1-M(x)t]^{-\nu}\; {}_{m+p}F_q\left[\begin{matrix}\Delta(m; \nu), \alpha_1, \cdots, \alpha_p ; \\ \beta_1, \cdots, \beta_q ;\end{matrix}\; \left(\frac{mt}{1-M(x)t}\right)^m Q(x)\right] \\ &= \sum_{n=0}^{\infty} \frac{(\nu)_n}{n!} [M(x)t]^n\; {}_{m+p}F_q\left[\begin{matrix}\Delta(m; -n), \alpha_1, \cdots, \alpha_p ; \\ \beta_1, \cdots, \beta_q ;\end{matrix}\; \left(-\frac{m}{M(x)}\right)^m Q(x)\right], \end{aligned} \tag{1.7}$$

where ν is an arbitrary parameter, $m=1, 2, 3, \cdots$, and for convenience, $\Delta(m; \lambda)$ is taken to abbreviate the sequence of m paramaters

$$\frac{\lambda}{m}, \frac{\lambda+1}{m}, \cdots, \frac{\lambda+m-1}{m}, \quad m \geqq 1.$$

Various special cases of the formulas (1.6) and (1.7) appear in the literature. To quote but a few such instances, we note that the formula (1.7) reduces to the generating function (25), p. 62 of Chaundy [5] when $M(x)=1, Q(x)=-x$ and $m=1$. For $M(x)=1$ and $Q(x)=(-m)^{-m}. x$, (1.6) corresponds to the formula (28), p. 947 of Brafman [2], while (1.7) gives us the relatively recent formula ([3], p. 187 (55)) which, in turn, yields his earlier result (24), p. 947 in [2] when $m=2$. For an alternative derivation of the aforementioned result of Chaundy, using certain operational techniques, see formula (4.15), p. 24 in [7]. Note also that several generalizations of the results of Brafman, Chaundy, and others, have appeared in our earlier papers [9] and [10].

Now we return to the generating relation (1.3). For $r=1$, it evidently has the elegant form

$$(1.8)\qquad [1-M(x)t]^{-\nu}\, J\left(Q(x)\left[\frac{t}{1-M(x)t}\right]^{m}\right)=\sum_{n=0}^{\infty}\frac{(\nu)_n}{(\lambda+1)_n}R_n^{(\lambda)}(x)\,t^n,$$

where

$$(1.9)\qquad J(z)=\sum_{n=0}^{\infty}(\nu)_{mn}\,g_n\,z^n\;(g_n\neq 0)$$

and

$$(1.10)\qquad R_n^{(\lambda)}(x)=\sum_{k=0}^{[n/m]}\frac{(\lambda+1)_n}{(n-mk)!}\,g_k\,[Q(x)]^k\,[M(x)]^{n-mk},$$

for positive integral values of m, and $n=0,1,2,\cdots\cdots$.

Starting from (1.10), our main object in the present paper is to prove a generalization of (1.8) in the following form:-

$$(1.11)\qquad \sum_{n=0}^{\infty}\frac{([a_\rho])_n}{([b_\sigma])_n}R_n^{(\lambda)}(x)\frac{t^n}{(\lambda+1)_n}$$
$$=\sum_{n=0}^{\infty}\frac{([a_\rho])_{mn}}{([b_\sigma])_{mn}}\,g_n\,[Q(x)\,t^m]^n\;{}_\rho F_\sigma\left[\begin{matrix}[a_\rho]+mn\,; & \\ & M(x)\,t\\ [b_\sigma]+mn\,; & \end{matrix}\right],$$

where, for the sake of brevity, $[a_\rho]$ denotes the sequence of ρ parameters

$$a_1,a_2,\cdots,a_\rho,$$

$([a_\rho])_n$ has the interpretation

$$\prod_{j=1}^{\rho}(a_j)_n,$$

with $(a_j)_n$ defined above, and so on.

2. Proof of the Generating Relation (1.11).

On substituting for the coefficients $\{R_n^{(\lambda)}(x)\,|\,n=0,1,2,\cdots\}$ from (1.10), we notice that

$$\sum_{n=0}^{\infty}\frac{([a_\rho])_n}{([b_\sigma])_n}R_n^{(\lambda)}(x)\frac{t^n}{(\lambda+1)_n}$$
$$=\sum_{n=0}^{\infty}\frac{([a_\rho])_n}{([b_\sigma])_n}\,t^n\sum_{k=0}^{[n/m]}g_k\,[Q(x)]^k\,\frac{[M(x)]^{n-mk}}{(n-mk)!}$$

$$= \sum_{k=0}^{\infty} g_k \frac{[Q(x)]^k}{[M(x)]^{mk}} \sum_{n=mk}^{\infty} \frac{([a_\rho])_n}{([b_\sigma])_n} \frac{[M(x)]^n}{(n-mk)!}$$

$$= \sum_{k=0}^{\infty} g_k [Q(x) t^m]^k \sum_{n=0}^{\infty} \frac{([a_\rho])_{n+mk}}{([b_\sigma])_{n+mk}} \frac{[M(x)t]^n}{n!}$$

$$= \sum_{k=0}^{\infty} \frac{([a_\rho])_{mk}}{([b_\sigma])_{mk}} g_k [Q(x) t^m]^k \, {}_\rho F_\sigma \left[\begin{matrix} [a_\rho]+mk; \\ \\ [b_\sigma]+mk; \end{matrix} \; M(x)t \right],$$

since

$$(\nu)_{n+mk} = (\nu)_{mk} (\nu+mk)_n,$$

and the formula (1.11) follows immediately.

3. Particular Cases.

When $\rho-1=\sigma=0$ and $a_1=\nu$, the hypergeometric function on the right-hand side of (1.11) reduces to

$${}_1F_0 \left[\begin{matrix} \nu+mn; \\ \\ \text{———}; \end{matrix} \; M(x)t \right],$$

which of course is the binomial

$$[1-M(x)t]^{-\nu-mn},$$

and the formula (1.11) leads us at once to the generating relation (1.8).

For $\rho=\sigma$ and $a_j=b_j$, $j=1, 2, 3, \cdots, \rho$ (or σ), (1.11) would readily yield the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{(\lambda+1)_n} R_n^{(\lambda)}(x) = E[M(x)t] \, G[Q(x)t^m], \tag{3.1}$$

which can naturally be recovered from (1.2) on setting $r=1$.

Next we consider, for the sake of simplicity, the special case of (1.11) when $G[z]$ takes the hypergeometric form, i. e., when

$$g_n = \frac{\prod_{j=1}^{p} (\alpha_j)_n}{n! \prod_{j=1}^{q} (\beta_j)_n}, \quad n=0, 1, 2, \cdots\cdots \tag{3.2}$$

From (1.10) and (1.11) we shall then have

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\rho} (a_j)_n}{\prod_{j=1}^{\sigma} (b_j)_n} \frac{[M(x)t]^n}{n!} {}_{m+p}F_q \left[\begin{matrix} \Delta(m;-n), \alpha_1, \cdots, \alpha_p; \\ \beta_1, \cdots, \beta_q; \end{matrix} \left(-\frac{m}{M(x)}\right)^m Q(x) \right] \tag{3.3}$$

$$= \frac{\prod_{j=1}^{\sigma} \Gamma(b_j) \prod_{j=1}^{q} \Gamma(\beta_j)}{\prod_{j=1}^{\rho} \Gamma(a_j) \prod_{j=1}^{p} \Gamma(\alpha_j)} S^{\rho:\,p;\,0}_{\sigma:\,q;\,0} \begin{pmatrix} Q(x)t^m \\ M(x)t \end{pmatrix},$$

where the S-function on the right-hand side is the generalized Kampé de Fériet function in two variables defined by (see [11], p. 199, eqn. (2.1))

$$S^{\rho:\,p;\,0}_{\sigma:\,q;\,0} \begin{pmatrix} X \\ Y \end{pmatrix} = S^{\rho:\,p;\,0}_{\sigma:\,q;\,0} \left(\begin{matrix} \{[a_\rho]:m,1\}:\{[\alpha_p]:1\};-; \\ \{[b_\sigma]:m,1\}:\{[\beta_q]:1\};-; \end{matrix} X, Y \right) \tag{3.4}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{\rho} \Gamma(a_j+mn+k) \prod_{j=1}^{p} \Gamma(\alpha_j+n)}{\prod_{j=1}^{\sigma} \Gamma(b_j+mn+k) \prod_{j=1}^{q} \Gamma(\beta_j+n)} \frac{X^n}{n!} \frac{Y^k}{k!},$$

it being understood, in order to be in complete agreement with our earlier notation in [11], that the m_j are all equal to m.

In particular, when $m=1$ the double hypergeometric function can be expressed as an ordinary Kampé de Fériet's function. Following the notation of Burchnall and Chaundy [4, p. 112] in preference, for the sake of generality and elegance, to the earlier one introduced by Kampé de Fériet himself (see [1], p. 150), we thus have

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\rho} (a_j)_n}{\prod_{j=1}^{\sigma} (b_j)_n} {}_{p+1}F_q \left[\begin{matrix} -n, \alpha_1, \cdots, \alpha_p; \\ \beta_1, \cdots, \beta_q; \end{matrix} \frac{Q(x)}{M(x)} \right] \frac{[M(x)t]^n}{n!} \tag{3.5}$$

$$= F \left[\begin{matrix} a_1, \cdots, a_\rho : \alpha_1, \cdots, \alpha_p; -; \\ b_1, \cdots, b_\sigma : \beta_1, \cdots, \beta_q; -; \end{matrix} Q(x)t, M(x)t \right].$$

The special case $\rho-1=\sigma=1$ of the last formula is worthy of note. On expressing the hypergeometric function on the right-hand side as an infinite series of Gauss's ${}_2F_1$, if we make use of Euler's transformation [6, p. 64]

$$
{}_2F_1\left[\begin{matrix} a, b\,; \\ \\ c\,; \end{matrix}\; z\right] = (1-z)^{-a}\; {}_2F_1\left[\begin{matrix} a, c-b\,; \\ \\ c\,; \end{matrix}\; \frac{z}{z-1}\right], \tag{3.6}
$$

we at once get

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda)_n (\mu)_n}{(\nu)_n}\; {}_{p+1}F_q\left[\begin{matrix} -n, \alpha_1, \cdots, \alpha_p\,; \\ \\ \beta_1, \cdots, \beta_q\,; \end{matrix}\; -\frac{Q(x)}{M(x)}\right] \frac{[M(x)t]^n}{n!} \\
&= [1-M(x)t]^{-\lambda}\; F\left[\begin{matrix} \lambda : \mu, \alpha_1, \cdots, \alpha_p\,; \nu-\mu\,; \\ \\ \nu : \quad \beta_1, \cdots, \beta_q\,; \text{——}\,; \end{matrix}\; \frac{Q(x)t}{1-M(x)t}, -\frac{M(x)t}{1-M(x)t}\right].
\end{aligned} \tag{3.7}
$$

For $M(x)=1$ and $Q(x)=-x$, the formula (3.7) leads us to our earlier generating function (18) in [12] which, in turn, has several interesting special forms involving Jacobi and Laguerre polynomials (see [12], §§ 2 and 3).

Finally, we observe that when $\mu \to \nu$, the double series in (3.7) reduces to a single one of the generalized hypergeometric ${}_{p+1}F_q$ function, and we thus have

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\; {}_{p+1}F_q\left[\begin{matrix} -n, \alpha_1, \cdots, \alpha_p\,; \\ \\ \beta_1, \cdots, \beta_q\,; \end{matrix}\; -\frac{Q(x)}{M(x)}\right] [M(x)t]^n \\
&= [1-M(x)t]^{-\lambda}\; {}_{p+1}F_q\left[\begin{matrix} \lambda, \alpha_1, \cdots, \alpha_p\,; \\ \\ \beta_1, \cdots, \beta_q\,; \end{matrix}\; \frac{Q(x)t}{1-M(x)t}\right],
\end{aligned} \tag{3.8}
$$

which follows immediately from our earlier generating function (1.7) if we set $m=1$. Notice also that by letting $M(x)=1$ and $Q(x)=-x$ in (3.8) we shall again arrive at the well-known formula (25), p. 62 of Chaundy [5].

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