SOME RESULTS ON FIXED POINT THEOREMS *

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Let (X, d) be a metric space. A mapping $T: X \longrightarrow X$ is called a contraction mapping if there is a real number k, $0 < k < 1$, such that

$$d(Tx, Ty) \leq k\, d(x, y) \text{ for all } x, y \in X.$$

The well-known Banach contraction principle states that a contration mapping of a complete metric space X into itself has a unique fixed point. This theorem has been extensively used in proving existence and uniqueness of solutions to various functional equations, particularly differential and integral equations. Because of its widespread applicability there has been a search for generalizations of the Banach contraction principle. The works of Chu and Diaz [2] and Edelstein [3], [4], are worth mentioning.

Recently Kannan [5] proved the following result.

Theorem A: If T is a map of the complete metric space X into itself such that

$$d(Tx, Ty) \leq \alpha \{d(x, Tx) + d(y, Ty)\},$$

for x, y in X and $0 < \alpha < \frac{1}{2}$, then T has a unique fixed point in X.

The aim of this paper is to give some more general results. The result given by Kannan [5] may be taken as a corollary to our result.

Theorem 1: *If T is a map of the complete metric space X into itself and if T^n (n is a positive integer) satisfies the condition*

$$d(T^n x, T^n y) \leq \alpha \{d(x, T^n x) + d(y, T^n y)\}$$

for x, y in X and $0 < \alpha < \frac{1}{2}$, then T has a unique fixed point in X.

Proof: Since T^n satisfies the condition given in Theorem A, therefore T^n has a unique fixed point. Let x_0 be a unique fixed point of T^n. Then $T^n x_0 = x_0$. We know that

$$T(T^n x_0) = T^n (T x_0).$$

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* This work was done while the author was a fellow of the Summer Research Institute, McGill University, Montreal, 1969.

Thus Tx_0 is a fixed point of T^n. But T^n has a unique fixed point x_0 therefore $Tx_0=x_0$. Hence x_0 is a unique fixed point of T. Corollary : In case $n=1$, we get a theorem given by Kannan [5].

In order to illustrate the theorem the following examples are worth mentioning.

Example 1 : Let $X=[0, 1]$, and let

$T: X \longrightarrow X$ be defined by

$Tx=\frac{x}{3}$ for all $x \in [0, 1]$.

Then T does not satisfy the condition

$$d(Tx, Ty) \leq \alpha \{d(x, Tx)+d(y, Ty)\},$$

for $x, y \in [0, 1]$ and $0 < \alpha < \frac{1}{2}$; as one can easily see by taking $x=\frac{1}{3}$ and $y=0$. But T^2 satisfies the condition and therefore T^2 has a unique fixed point. It then follows that T has a unique fixed point.

Example 2 : Let $X=[0, 1]$ and let

$T: X \longrightarrow X$ be defined by

$Tx=\frac{9}{10}x$ for all $x \in [0, 1]$.

Then $T, T^2, T^3, \cdots, T^9$ and T^{10} do not satisfy the condition, but T^{11} does satisfy and therefoıe T^{11} has a unique fixed point. Hence T has a unique fixed point.

Remark : If X is simply a metric space not necessarily a complete metric space and $T: X \longrightarrow X$ is a map such that T^n has a unique fixed point, then T has a unique fixed point.

Theorem 2 : *Let X be a complete metric space and let T be any map of X onto itself. If there exists a mapping K of X into itself which has a right inverse (i. e. $KK^{-1}=I$, identity mapping) and which makes $K^{-1}TK$ to satisfy the condition*

$$d(K^{-1}TKx, K^{-1}TKy) \leq \alpha \{d(x, K^{-1}TKx) + d(y, K^{-1}TKy)\} \quad \textit{for} \quad x, y \in X$$

and $0 < \alpha < \frac{1}{2}$, then T has a unique fixed point.

Proof : Since $K^{-1}TK$ satisfies condition of Theorem A and X is a complete metric space therefore $K^{-1}TK$ has a unique fixed point. Let us assume that x_0 be a unique fixed point of $K^{-1}TK$.

Then $K^{-1}TKx_0=x_0$,

or $KK^{-1}TKx_0=Kx_0$

or $TKx_0=Kx_0$.

Therefore T has a unique fixed point.

In the end we prove a theorem on sequence of mappings. If a sequence of mappings T_n with fixed points U_n converges to T then does the sequence of corresponding fixed points converge to the fixed point of T? Some work on this line has been done by Bonsall [1] and Nadler Jr. [6]. Nadler Jr. has shown that if the sequence of contraction mappings with different Lipschitz constants converges pointwise to a conraction mapping T then sequence of their fixed points does not converge to the fixed point of T. He also proved the following theorem:

If $T_n : X \longrightarrow X$ is a map for $n=1, 2, \cdots$, with fixed point $U_n\,(n=1, 2, \cdots,)$ and if T_n converges to T uniformly, where T is a contraction map with fixed point U, then U_n converges to U.

We prove the following theorem. Since the contraction mapping and the mapping given in Theorem A are independent therefore this theorem is different from that given by Nadler Jr.

Theorem 3: *Let*

(1) *$T_n : X \longrightarrow X$ be a map with fixed point U_n for $n=1, 2, \cdots$,* and

(2) *T_n converges uniformly to T where $T: X \longrightarrow X$ is a map such that*

$$d(Tx, Ty) \leq \alpha \{d(x, Tx)+d(y, Ty)\}$$

for x, y in X and $0<\alpha<\frac{1}{2}$, with fixed point U. Then U_n converges to U.

Proof: By uniform convergence we get that for given $\epsilon > 0$ there exists a positive integer N such that $n \geq N$ implies

$$d(T_n x, Tx) < \frac{\epsilon}{1+\alpha} \quad \text{for all } x \in X.$$

Hence for $n \geq N$,

$$\begin{aligned} d(U_n, U) &= d(T_n U_n, TU) \\ &\leq d(T_n U_n, TU_n)+d(TU_n, TU) \\ &\leq d(T_n U_n, TU_n)+\alpha\{d(U_n, TU_n)+d(U, TU)\} \\ &\leq d(T_n U_n, TU_n)+\alpha\, d(U_n, T_n U_n)+\alpha\, d(T_n U_n, TU_n) \end{aligned}$$

$[d(U, TU) = 0$ since U is a fixed point of $T.]$

$$\begin{aligned} &= (1+\alpha)\, d(T_n U_n, TU_n)+0 \\ &< \epsilon, \end{aligned}$$

so that U_n converges to U.

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