

AN INDUCTION PRINCIPLE IN SET THEORY I.

By

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In this paper we shall present an induction principle in *Bernays-Gödel* set theory (BG) by analogy with the weak recursion theorem in recursion theory. This will enable one to define a spacial class or an operation (in the sense of [4]) whose implicit definition is expressed by a certain kind of formula. Since Gödel's existence theorems M1~M4 (in [4]) have been proved for normal formulas but not in general for formulas which is not normal, the induction principle we shall consider has of course a certain restriction.

Proofs that all the elements of an inductively defined class have certain property, are often carried out inductively along the definition of it. Theorem 1 justifies such proofs for classes defined by the induction principle we shall introduce. We shall use the notations in [4] without any reference.

First we shall prove the following theorem :

Theorem 1. *Let $\varphi(x, X)$ be a normal formula (i.e. a formula of set theory without class quantifiers). Assume that*

(i) $\forall x \forall X \forall Y [X \subseteq Y \wedge \varphi(x, X) \supset \varphi(x, Y)]$ and

(ii) $\forall x \forall X [\varphi(x, X) \supset \exists y [y \subseteq X \wedge \varphi(x, y)]]$.

Then,

(iii) $\exists ! A [\forall x [x \in A \equiv \varphi(x, A)] \wedge \forall X [\forall x [\varphi(x, X) \supset x \in X] \supset A \subseteq X]]$.

(More precisely, (i) \wedge (ii) \supset (iii) is provable under Bernays-Gödel axiom system **A**, **B** and **C**.)

Proof. Assume (i) and (ii). The uniqueness of such A in (iii) is trivial. To prove the existence of such an A , let

$$A = \{a \mid \exists y \exists \alpha \exists f [a \in f' \alpha \wedge \forall \beta \leq \alpha \forall x [x \in f' \beta \equiv x \in y \wedge \varphi(x, \mathfrak{S}(f'' \beta))]]\}.$$

(The class A exists since the formula in the above abstraction term is normal.) We shall show that the A is the desired class.

First we claim that

$$(*) \quad \varphi(a, A) \supset a \in A.$$

Suppose that $\varphi(a, A)$. By (ii), there exists a set u such that $u \subseteq A$ and $\varphi(a, u)$.

Hence we have that

$$(1) \quad \forall t \in u \exists y \exists \alpha \exists f \psi(t, y, \alpha, f),$$

where

$$\psi(t, y, \alpha, f) \equiv t \in f' \alpha \wedge \forall \beta \leq \alpha \forall x [x \in f' \beta \equiv x \in y \wedge \varphi(x, \mathfrak{S}(f'' \beta))].$$

By the axiom of replacement, (1) implies that there exists a set z such that

$$(2) \quad \forall t \in u \exists y \in z \exists \alpha \in z \exists f \in z \psi(t, y, \alpha, f).$$

Now let

$$y_0 = \mathfrak{S}(z) \cup \{a\} \text{ and } \alpha_0 = \mathfrak{S}(z \cap 0n) + 1.$$

As in the proof of 7.5 in [4], there exists an f_0 such that

$$(3) \quad f_0 \text{Fn } \alpha_0 + 1 \wedge \forall \beta \leq \alpha_0 \forall x [x \in f_0' \beta \equiv x \in y_0 \wedge \varphi(x, \mathfrak{S}(f_0'' \beta))].$$

Let b be an arbitrary element of u . Then, by (2), there exist $y_1, \alpha_1, f_1 \in z$ such that $\psi(b, y_1, \alpha_1, f_1)$. In view of the definitions of y_0 and α_0 , we have that $y_1 \subseteq y_0$ and $\alpha_1 < \alpha_0$. Moreover we have that $b \in f_1' \alpha_1$ and that

$$(4) \quad \forall x \forall \beta \leq \alpha_1 [x \in f_1' \beta \equiv x \in y_1 \wedge \varphi(x, \mathfrak{S}(f_1'' \beta))].$$

Now we shall prove

$$(5) \quad \forall \beta \leq \alpha_1 [f_1' \beta \subseteq f_0' \beta]$$

by the induction on β . Assume that $\beta \leq \alpha_1$ and $\forall \gamma < \beta [f_1' \gamma \subseteq f_0' \gamma]$. Then $\mathfrak{S}(f_1'' \beta) \subseteq \mathfrak{S}(f_0'' \beta)$. Let $x \in f_1' \beta$. Then, by (4), $x \in y_1$ and $\varphi(x, \mathfrak{S}(f_1'' \beta))$. So, by (i), $x \in y_0 \wedge \varphi(x, \mathfrak{S}(f_0'' \beta))$. Hence $x \in f_0' \beta$ by (3). We have proved $f_1' \beta \subseteq f_0' \beta$. Now the induction is complete and we have (5). In particular, $b \in f_1' \alpha_1 \subseteq f_0' \alpha_1 \subseteq \mathfrak{S}(f_0'' \alpha_0)$ since $\alpha_1 < \alpha_0$. Hence we have shown that $u \subseteq \mathfrak{S}(f_0'' \alpha_0)$. Therefore $\varphi(a, \mathfrak{S}(f_0'' \alpha_0))$, since $\varphi(a, u)$. Hence by (3), $a \in f_0' \alpha_0$. Hence $\psi(a, y_0, \alpha_0, f_0)$, which implies $a \in A$. Hence (*) is proved. Next we claim that

$$(**) \quad \forall X [\forall x [\varphi(x, X) \supset x \in X] \supset A \subseteq X].$$

Assume that $\forall x [\varphi(x, X) \supset x \in X]$ and $a \in A$. We have to show that $a \in X$. By definition of A , there exist y, α and f such that $a \in f' \alpha$ and such that

$$\forall \beta \leq \alpha \forall x [x \in f' \beta \equiv x \in y \wedge \varphi(x, \mathfrak{S}(f'' \beta))].$$

Now we propose to show that $f' \beta \subseteq X$ for every $\beta \leq \alpha$. We prove this by the induction on β . Suppose that $\forall \gamma < \beta [f' \gamma \subseteq X]$. Then $\mathfrak{S}(f'' \beta) \subseteq X$. Let $t \in f' \beta$. Then $t \in y$ and $\varphi(t, \mathfrak{S}(f'' \beta))$. From this and (i) it follows that $\varphi(t, X)$, from which follows that $t \in X$ using the assumption. Hence $f' \beta \subseteq X$ and the induction is complete. In particular we have $a \in f' \alpha \subseteq X$, as was to be shown. Hence we have (**). Finally we claim that

$$(***) \quad a \in A \equiv \varphi(a, A).$$

Let $B = \{x \mid \varphi(x, A)\}$. By (*), $B \subseteq A$. Hence, by (i), $\forall x [\varphi(x, B) \supset \varphi(x, A)]$, that is, $\forall x [\varphi(x, B) \supset x \in B]$. From this and (**) it follows that $A \subseteq B$. Therefore $A = B$, which means (***). This completes the proof of theorem. *q. e. d.*

We refer to the A in the theorem as the class inductively defined by the formula φ and denote it as

$$a \in A \equiv \varphi(a, A)^{ind}$$

or

$$A = \{a \mid \varphi(a, A)\}^{ind}.$$

Now let $\chi(a)$ be a normal formula. We think of it as a property of a set a . Let $G = \{a \mid \chi(a)\}$. Then, in order to prove

$$(6) \quad \forall a [a \in A \supset \chi(a)],$$

it suffices to prove

$$(7) \quad \forall a [\varphi(a, G) \supset a \in G],$$

in view of theorem 1. A proof of (7) may be considered as a proof of (6) along the inductive definition of A . Actually, instead of (7),

$$(8) \quad \forall a [\varphi(a, G \cap A) \supset a \in G]$$

suffices to conclude (6). For, (8) implies

$$\forall a [\varphi(a, G \cap A) \supset a \in G \cap A],$$

which, in turn, implies $A \subseteq G \cap A \subseteq G$. Some examples of such proofs will be given in the subsequent paper.

Next we shall examine what formulas satisfy the conditions (i) and (ii). Theorem 2 will give a sufficient condition for it.

Definition. Let φ be a formula in set theory (class variables and special classes may occur in it).

1. φ is called a bounded formula¹⁾ iff no class quantifiers ($\forall X, \exists X$) occur in it and each set quantifier in it is of the type $\forall x [x \in y \supset \dots]$ or $\exists x [x \in y \wedge \dots]$ (which we abbreviate as $\forall x \in y [\dots]$ or $\exists x \in y [\dots]$ respectively).

2. φ is called a quasi-bounded formula iff no class quantifiers occur in it and each set quantifier in it is of the type $\forall x [x \in y \supset \dots]$, $\exists x [x \in y \wedge \dots]$, $\forall x [x \subseteq y \supset \dots]$ or $\exists x [x \subseteq y \wedge \dots]$. (The last two types of quantifiers are abbreviated by $\forall x \subseteq y [\dots]$ or $\exists x \subseteq y [\dots]$.)

1) This definition is essentially due to A. Lévy [1].

3. $\Sigma_1 = \{\exists x [\phi] \mid \phi \text{ is a bounded formula}\}.$
4. $\tilde{\Sigma}_1 = \{\exists x [\phi] \mid \phi \text{ is a quasi-bounded formula}\}.$
5. Similarly, $\Pi_1, \Sigma_2, \Pi_2, \dots, \tilde{\Pi}_1, \tilde{\Sigma}_2, \tilde{\Pi}_2, \dots,$ are defined.
6. $\Sigma_1^{\text{BG}} = \{\varphi \mid \text{BG} \vdash \varphi \equiv \phi \text{ and } \phi \in \Sigma_1\},$
 $\tilde{\Sigma}_1^{\text{BG}} = \{\varphi \mid \text{BG} \vdash \varphi \equiv \phi \text{ and } \phi \in \tilde{\Sigma}_1\},$

etc.

Lemma. Suppose that $\varphi(X)$ is a quasi-bounded formula and that it has no occurrences of the form $X \in Z$ or $X \in a$. Then,

(1) if each occurrence of the form $u \in X$ (or $Y \in X$) in $\varphi(X)$ lies in a positive part (in this case, briefly, we say $\varphi(X)$ is positive), then

$$(1.1) \quad \text{BG} \vdash \varphi(X) \supset \exists y [y \subseteq X \wedge \varphi(y)] \text{ and}$$

(2) if each occurrence of the form $u \in X$ (or $Y \in X$) in $\varphi(X)$ lies in a negative part (in this case, briefly, we say $\varphi(X)$ is negative), then

$$(2.1) \quad \text{BG} \vdash \forall y [y \subseteq X \supset \varphi(y)] \supset \varphi(X).$$

Proof. We shall prove (1) and (2) simultaneously by the induction on the number of logical symbols in φ .

Case 1. $\varphi(X) \equiv a \in X$ or $Y \in X$.

If it is the latter case, then Y is a set. So we only prove the lemma for the former case. $\varphi(X)$ is positive but not negative.

Now suppose that $\varphi(X)$ holds. Let $y = \{a\}$. Then, $y \subseteq X$ and $\varphi(y)$. Hence we have (1.1).

Case 2. $\varphi(X)$ is an atomic formula which is not of the form in the case 1. Then $\varphi(X)$ does not contain X . So, $\varphi(X)$ is positive and at the same time negative. (1.1) and (2.1) trivially hold.

Case 3. $\varphi(X) \equiv \neg \psi(X)$.

Suppose that $\varphi(X)$ is positive. Then $\psi(X)$ is clearly negative. So, by the induction hypothesis, we have

$$\forall y [y \subseteq X \supset \psi(y)] \supset \psi(X).$$

By contraposing it,

$$\psi(X) \supset \exists y [y \subseteq X \wedge \psi(y)],$$

which is (1.1). Similarly if $\varphi(X)$ is negative, we have (2.1).

Case 4. $\varphi(X) \equiv \phi(X) \wedge \chi(X)$.

Suppose first that $\varphi(X)$ is positive. Then, both $\phi(X)$ and $\chi(X)$ are positive. By the induction hypothesis,

$$\phi(X) \supset \exists y [y \subseteq X \wedge \phi(y)]$$

and

$$\chi(X) \supset \exists y [y \subseteq X \wedge \chi(y)].$$

Now assume that $\varphi(X)$ is negative. Then $\phi(X)$ and $\chi(X)$ are negative. Hence by the above, there exist $y_1, y_2 \subseteq X$ such that $\phi(y_1)$ and $\chi(y_2)$. Let $y = y_1 \cup y_2$. Since $\phi(X)$ and $\chi(X)$ are negative, we easily have that $\phi(y)$ and $\chi(y)$. Hence

$$\exists y [y \subseteq X \wedge \phi(y)].$$

This proves (1.1). Next suppose that $\varphi(X)$ is negative. Then both $\phi(X)$ and $\chi(X)$ are negative. By the induction hypothesis,

$$\forall y [y \subseteq X \supset \phi(y)] \supset \phi(X)$$

and

$$\forall y [y \subseteq X \supset \chi(y)] \supset \chi(X).$$

Now assume that $\forall y [y \subseteq X \supset \phi(y)]$. Then,

$$\forall y [y \subseteq X \supset \phi(y)] \wedge \forall y [y \subseteq X \supset \chi(y)].$$

From the above it follows that $\phi(X) \wedge \chi(X)$. This proves (2.1).

Case 5. $\varphi(X) \equiv \forall x \in a [\phi(X, x)]$.

Suppose that $\varphi(X)$ is positive. Then $\phi(X, x)$ is positive. Hence, by the induction hypothesis, we have

$$\phi(X, x) \supset \exists y [y \subseteq X \wedge \phi(y, x)].$$

Assume that $\varphi(X)$ is negative. Then $\phi(X, x)$ is negative for every $x \in a$.

Hence

$$\forall x \in a \exists y [y \subseteq X \wedge \phi(y, x)].$$

Using the axiom of replacement, we have

$$\exists z \forall x \in a \exists y \in z [y \subseteq X \wedge \phi(y, x)].$$

Let z_0 be such that $\forall x \in a \exists y \in z_0 [y \subseteq X \wedge \phi(y, x)]$. Let $y = \bigcup (z_0 \cap X)$. Let $x \in a$. Then there exists $y_x \in z_0$ such that $y_x \subseteq X \wedge \phi(y_x, x)$. Clearly $y_x \subseteq y$. Hence $\phi(y, x)$. y is independent of x . Therefore we have $\exists y [y \subseteq X \wedge \forall x \in a [\phi(y, x)]]$, that is, $\exists y [y \subseteq X \wedge \varphi(y)]$. This proves (1.1). Next suppose that $\varphi(X)$ is negative. Then $\phi(X, x)$ is also negative. By the induction hypothesis,

$$\forall y [y \subseteq X \supset \phi(y, x)] \supset \phi(X, x).$$

Assume that $\forall y [y \subseteq X \supset \phi(y)]$. Then

$$\forall x \in a \forall y [y \subseteq X \supset \phi(y, x)].$$

By the above, we have $\forall x \in a \phi(X, x)$, that is, $\phi(X)$. This proves (2.1).

Case 6. $\phi(X) \equiv \forall x \subseteq a \phi(X, x)$.

Similar to the case 5. *q. e. d.*

Theorem 2. *If $\phi(x, X)$ is in $\tilde{\Sigma}_1^{\text{BG}}$, it has no occurrences of the form $X \in Z$ or $X \in a$ and each occurrence of the form $y \in X$ (or $Y \in X$) in it lies in a positive part in ϕ , then $\phi(x, X)$ satisfies (i) and (ii) (in theorem 1).*

Proof. (i) easily follows from the last two assumptions. To prove (ii), suppose that

$$\text{BG} \vdash \phi(x, X) \equiv \exists u \phi(u, x, X),$$

where $\phi(u, x, X)$ is a quasi-bounded formula. Assume that $\phi(x, X)$. Then there exists a set u such that $\phi(u, x, X)$. $\phi(u, x, X)$ does not contain a subformula $X \in Z$ or $X \in a$ but is positive. Hence by the lemma, there exists a $y \subseteq X$ such that $\phi(u, x, y)$. Hence $\exists u \phi(u, x, y)$, that is, $\phi(x, y)$. This proves (ii). *q. e. d.*

Now suppose that $\phi(x, X)$ satisfies (i) and (ii).

Let

$$G = \{ \langle x, y \rangle \mid \phi(x, y) \}.$$

Since $\phi(x, X)$ is normal, G exists. We can readily prove that

$$\phi(x, X) \equiv \exists y [y \subseteq X \wedge \langle x, y \rangle \in G].$$

Since $y \subseteq X \equiv \forall t \in y [t \in X]$, the right-hand side formula of the equivalence is in Σ_1 and positive. Moreover it has no occurrences of the form $X \in Z$ or $X \in a$. Hence the condition of theorem 2 is also necessary in the sense of equivalence.

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