EXTENSION THEOREM AND ITS APPLICATION TO THE FRENET FORMULAS OF CURVES

By

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1. Introdution.

Tietze's extension theorem asserts that a continuous real-valued function defined on a closed subset of a normal space admits a continuous extension over the whole space. (See [1] or [2, p. 60]) In his paper [3], [4] *H. Whitney* has given a sufficient condition for a real-valued function defined on a closed subset of the euclidIan n-space to have an analytic extension over the whole space. The proof of *Whitney*'s extension theorem is rather complicated.

In this paper, we shall present two extension theorems of differentiable functions for particular cases with relatively simple proofs, and an application to *Frenet* Formulas of curves in E^3. Since a curve is defined on a closed interval $[a, b]$, we therefore consider only extension of a differentiable real-valued function defined on a subset of $[a, b]$,

In order that the differentiability of a real-valued function makes sense, it is natural to consider only those subsets which contain no isolated points. Let f be a real-valued function defined on a subset A of $[a, b]$ containing no isolated points. Then f is said to be *k-normal* on A if the *i-th* derivatives $f^{(i)}$ $(0\leqslant i\leqslant k)$ of f exist and

$$f^{(i)}(x)-P_i^k(x;f,t)=o((x-t)^{k-i})$$

uniformly on A, where

$$P_i^k(x;f,t)=f^{(i)}(t)+f^{(i+1)}(t)(x-t)+\cdots+\frac{f^{(k)}(t)}{(k-i)!}(x-t)^{k-i}.$$

We shall prove in § 2 that a k-normal function is a C^k-function but not conversely. Further propertes of k-normal functions are also discussed in § 2. The main result of this paper stated in § 5 is that f has a C^k-extension over $[a, b]$ i.e., there is $g\in C^k[a, b]$ such that

$$g(x)=f(x) \qquad \text{for} \quad x\in A,$$

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iff f is k-normal on A. In order to prove the main theorem we shall introduce the so-called relative polynomials in § 3 and present some propositions on real analysis in § 4.

An application to the Frenet Formulas of curve in E^3 is given in § 6.

2. k-normal Functions.

Let $[a, b]$ be a closed interval with the usual topology, and A a subset of $[a, b]$ containing no isolated points and f a real-valued function on A.

For any $x_0 \epsilon A$, if the limit

$$f'(x_0) = \lim_{\substack{x \to x_0 \\ x \epsilon A \setminus \{x_0\}}} \frac{f(x) - f(x_0)}{x - x_0}$$

exists then we call it the first derivative of f at x_0 relative to A. If at each $x \epsilon A, f'(x)$ exists then there derives a real-valued function denoted by f' and said to be the first derivative of f on A. We now define inductively the i-th derivative of f on A by the formula

$$f^{(i)} = (f^{(i-1)})'$$

for $i = 1, 2, 3, \cdots$, where $f^{(0)}$ means f itself.

Clearly, if $f^{(i)}$ exists on A then $f, f', \cdots, f^{(i-1)}$ are all continuous on A. Furthermore, if $A = [a, b]$ then $f^{(i)}$ is the usual i-th derivative of f.

In what follows we always assume that sets under consideration are subsets of $[a, b]$ and contain no isolated points.

Definition 2.1. Let f be a real-valued function on A and $t \epsilon A$. The polynomial

$$P_i^k(x; f, t) = f^{(i)}(t) + f^{(i+1)}(t)(x-t) + \cdots + \frac{f^{(k)}(t)}{(k-i)!}(x-t)^{k-i}$$

for $i = 0, 1, 2, \cdots, k$ is called a *T_i^k-polynomial* of f at the point t.

It is clear that $P_i^k(x; f, t)$ is a polynomial in x defined on $[a, b]$, provided that $f^{(k)}$ exists.

Definition 2.2. A real-valued function f on a set A is said to be a *k-normal function* on A, and denoted by $f \epsilon N^k(A)$ if $f^{(k)}$ exists on A and for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, t \epsilon A \text{ and } 0 < |x - t| < \delta \Rightarrow \left| \frac{f^{(i)}(x) - P_i^k(x; f, t)}{(x-t)^{k-i}} \right| < \varepsilon$$

for $i = 0, 1, \cdots, k$.

The following two propositions are trival.

Proposition 2.1. *f is 0-normal on A iff f is uniformly continuous on A.*

Proposition 2.2. *If $f \in N^k(A)$ and $B \subset A$ then the restriction of f on B is a k-normal function on B.*

To show that $f \in N^k(A)$ is strictly stronger than that $f \in C^k(A)$, we state the following proposition and counter-example.

Proposition 2.3. *If $f \in N(A)$ then $f, f', \cdots, f^{(k)}$ are all uniformly continuous on A.*

Proof. Since for every $t \in A$

$$P_k^k(x; f, t) = f^{(k)}(t),$$

by Definition 2.2, $f^{(k)}$ is uniformly continuous on A. Therefore, it suffices to show that for $0 \leqslant i < k$, if $f^{(i+1)}, \cdots, f^{(k)}$ are all uniformly continuous on A then so is $f^{(i)}$. Since uniform continuity on A implies boundedness we may assume that $|f^{(s)}(x)| \leqslant M$ for $s = i+1, \cdots, k$ and all $x \in A$ where M is a constant. From the definition of $P_i^k(x; f, t)$ we get the following inequality:

$$|P_i^k(x; f, t) - f^{(i)}(t)| \leqslant M\left\{|x-t| + \frac{|x-t|^2}{2!} + \cdots + \frac{|x-t|^{k-i}}{(k-i)!}\right\}$$

for $x, t \in A$. Therefore, for any given $\varepsilon > 0$ there exists $\delta_1 > 0$ such that

$$x, t \in A \text{ and } |x-t| < \delta_1 \Rightarrow |P_i^k(x; f, t) - f^{(i)}(t)| < \frac{\varepsilon}{2}.$$

Since $f \in N^k(A)$ there also exists $\delta_2 > 0$ such that

$$x, t \in A \text{ and } |x-t| < \delta_2 \Rightarrow |P_i^k(x; f, t) - f^{(i)}(x)| < \frac{\varepsilon}{2}.$$

The uniform continuity of $f^{(i)}$ on A follows from the above two implications.

Example. Let

$$A = \cup \left\{\left[\frac{1}{2n}, \frac{1}{2n-1}\right] : n = 1, 2, \cdots\right\} \cup \{0\}$$

and

$$f(x) = \begin{cases} 0 & x = 0 \\ \dfrac{1}{(2n)^2} & x \in \left[\dfrac{1}{2n}, \dfrac{1}{2n-1}\right] \text{for } n = 1, 2, \cdots \end{cases}$$

It is clear that $A \subset [0, 1]$ and contains no isolated points. Moreover, A is a bounded closed subset in R hence is compact.

It is easy to verify that $f^{(i)}(x) = 0$ for all $i > 0$ and $x \in A$ and hence $f \in C^\infty(A)$.

To show that $f \notin N^2(A)$, we consider T_0^2–polynomial of f at 0,

$$P_0^2(x;f,0)=f(0)+f'(0)(x-0)+\frac{f''(0)}{2!}(x-0)^2=0.$$

Thus

$$\left|\frac{P_0^2(x;f,0)-f(x)}{(x-0)^2}\right|=\frac{\frac{1}{(2n)^2}}{x^2} \qquad \text{for} \quad x \in \left[\frac{1}{2n}, \frac{1}{2n-1}\right],$$

hence

$$\frac{\frac{1}{(2n)^2}}{\frac{1}{(2n-1)^2}} \leqslant \left|\frac{P_0^2(x;f,0)-f(x)}{(x-0)^2}\right| \leqslant 1,$$

for $x \in \left[\frac{1}{2n}, \frac{1}{2n-1}\right]$. This implies that

$$\lim_{\substack{x\to 0\\ x\in A\setminus\{0\}}} \left|\frac{P_0^2(x;f,0)-f(x)}{(x-0)^2}\right| =1.$$

and by definition f is not 2–normal.

However, in case $A=[a,b]$, the difference between k–normality and being of C^k vanishes. Namely, we have the following proposition

Proposition 2.4. *$f \in N^k([a,b])$ iff $f \in C^k([a,b])$*

Proof. In proposition 2.3, we have proved that k–normal functions are all of C^k. We now prove that any C^k–function on $[a,b]$ is k–normal on $[a,b]$. By Taylor's expansion, for each t, $x \in [a,b]$

$$f^{(i)}(x)=f^{(i)}(t)+f^{(i+1)}(t)(x-t)+\cdots+\frac{f^{(i)}(\xi)}{(k-i)!}(x-t)^{k-i}$$

where $|t-\xi|<|t-x|$. Therefore

$$f^{(i)}(x)-P_i^k(x;f,t)=(f^{(k)}(\xi)-f^{(k)}(t))\frac{(x-t)^{k-i}}{(k-i)!}.$$

Since $f^{(k)}$ is uniformly continuous on $[a,b]$ the above equality implies that $f \in N^k([a,b])$.

In order to prove the main theorems in §5, we need a few more propositions as the following.

Proposition 2.5. *If $f \in N^k(A)$, then $f \in N^h(A)$* for $0 \leqslant h \leqslant k$.

Proof. It is clear from the definftion of $P_i^h(x;f,t)$ that

$$P_i^h(x;f,t)=P_i^k(x;f,t)-\left\{\frac{f^{(h+1)}(t)}{(h-i+1)!}(x-t)^{h-i+1}+\cdots\right.$$

$$\left.+\frac{f^{(k)}(t)}{(k-i)!}(x-t)^{k-i}\right\}$$

for $i=0,1,\cdots,h$. Hence we have

$$\frac{|P_i^h(x;f,t)-f^{(i)}(x)|}{|x-t|^{h-i}}\leqslant\frac{|P_i^k(x;f,t)-f^{(i)}(x)|}{|x-t|^{k-i}}|x-t|^{k-h}$$

$$+\frac{|f^{(h+1)}(t)|}{(h-i+1)!}|x-t|+\cdots+\frac{|f^{(k)}(t)|}{(k-i)|}|x-t|^{k-h}.$$

Since $f\in N^k(A)$ and by Proposition 2.3, $f^{(i)}$ are all uniformly continuous and hence bounded on A, it follows from the above inequality that $f\in N^h(A)$.

Proposition 2.6. *If $f\in N^k(A)$, then for given $\varepsilon>0$, there exists $\delta>0$ such that for any pair of points*

$$x,\ t\in A \quad \text{such that} \quad 0<|x-t|<\delta,$$

we have

$$\left|\frac{f^{(i)}(x)-f^{(i)}(t)}{x-t}-f^{(i+1)}(t)\right|<\varepsilon$$

for all $i=0,1,\cdots,k-1$.

Proof. Since

$$P_i^k(x;f,t)=f^{(i)}(t)+f^{(i+1)}(t)(x-t)+\cdots+\frac{f^{(k)}(t)}{(k-i)!}(x-t)^{k-i}$$

we get

$$f^{(i)}(x)-P_i^k(x;f,t)=f^{(i)}(x)-\{f^{(i)}(t)+f^{(i+1)}(t)(x-t)+\cdots+\frac{f^k(t)}{(k-i)!}(x-t)^{k-i}\}$$

and hence

$$\left|\frac{f^{(i)}(x)-f^{(i)}(t)}{x-t}-f^{(i+1)}(t)\right|\leqslant\frac{|f^{(i)}(x)-P_i^k(x;f,t)|}{|(x-t)^{k-i}|}|x-t|^{k-i-1}+$$

$$\frac{|f^{(i+2)}(t)|}{2!}|x-t|+\cdots+\frac{|f^k(t)|}{(k-i)!}|x-t|^{k-i-1},$$

holds for $i=0,1,\cdots,k-1$, and $x,\ t\in A$. For $f\in N^k(A)$ and all $f^{(i)}$ bounded on A, the proposition follows from the above inequality.

Proposition 2.7. *If $f\in N^k(A)$, then $R_i^k(x,t)=f^{(i)}(x)-P_i^k(x,f,t)$ is continuous with respect to x and t.*

Proof. This follows from the fact that $P_i^k(x;f,t)$ is a polynomial in x and all $f^{(i)}$ are continuous.

3. Relative Polynomials.

Let $f \in N^k(A)$ and s, t be two distinct points of A.

Consider the following system of linear equations with unknowns $y_0, y_1, \cdots, y_k$.

$$f(s) - P_0^k(s; f, t) = (s-t)^{k+1} y_0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f^{(i)}(s) - P_i^k(s; f, t) = \sum_{j=0}^{i} C_j^i \frac{(k+1)!}{(k-j+1)!}(s-t)^{k-j+1} y_{i-j} \tag{3.1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f^{(k)}(s) - P_k^k(s; f, t) = \sum_{j=0}^{k} C_j^k \frac{(k+1)!}{(k-j+1)!}(s-t)^{k-j+1} y_{k-j}$$

It is clear that for given $f \in N^k(A)$ and distinct points s, t in A, $\{y_0, y_1, \cdots, y_k\}$ is uniquely determined by the above equations. Therefore, for given $f \in N^k(A)$, y_i are functions of s and t.

Lemma 3.1. *For given $\varepsilon > 0$, there is $\delta > 0$ such that*

$$s, t \in A, \quad 0 < |s-t| < \delta \Rightarrow |y_i (s-t)^{i+1}| < \varepsilon$$

for $i = 0, 1, \cdots, k$.

Proof. For obvious reason, we can prove the above lemma for each i individually. From the first equation of the system we get

$$y_0(s-t) = \frac{f(s) - P_0^k(s; f, t)}{(s-t)^k}.$$

Since $f \in N^k(A)$, the lemma is then true for $i = 0$.

Suppose that the lemma is true for $i = 0, 1, \cdots, r$ where $r < k$. We shall prove it is true for $i = r+1$.

From the $(r+1)$-th equation we get

$$y_{r+1}(s-t)^{r+2} = \frac{f^{(r+1)}(s) - P_{r+1}^k(s; f, t)}{(s-t)^{k-(r+1)}} - \sum_{j=1}^{r+1} C_j^{r+1} \frac{(k+1)!}{(k-j+1)!}(s-t)^{r-j+2} y_{r-j+1}.$$

By induction assumption, $f \in N^k(A)$ and the above equation the lemma is true for $i = r+1$.

We now consider the polynomial

$$R(x) = y_0 + y_1(x-s) + \frac{y_2}{2!}(x-s)^2 + \cdots + \frac{y_k}{k!}(x-s)^k \tag{3.2}$$

where $y_0, y_1, \cdots, y_k$ are determined by (3.1). We have the following lemma.

Lemma 3.2. *For given* $\varepsilon>0$, *there is* $\delta>0$ *such that*

$$s,\ t\in A,\qquad 0<|s-t|<\delta \Rightarrow |(s-t)^{r+1}R^{(r)}(x)|<\varepsilon$$

for $r=0, 1, \cdots, k$ *and* $x\in[s, t]$.

Proof. Since

$$R^{(r)}(x)=y_r+y_{r+1}(x-s)+\cdots+\frac{y_k}{(k-r)!}(x-s)^{k-r}$$

we get

$$|R^{(r)}(x)|\leqslant|y_r|+|y_{r+1}||s-t|+\cdots+\frac{|y_k|}{(k-r)!}|s-t|^{k-r}$$

for $r=0, 1, \cdots, k$.

Therefore

$$|(s-t)^{r+1}R^{(r)}(x)|\leqslant\sum_{j=0}^{k-r}\frac{|y_{r+j}|}{j!}|s-t|^{r+j+1}$$

for $r=0, 1, \cdots, k$.

It is clear that the lemma follows from the above inequality and Lemma 3.1.

Definition 3.3. Let $f\in N^k(A)$ and s, t be two distinct points in A. The polynomial

$$P(x;f, s, t)=P_0^k(x;f, t+(x-t)^{J+1}R(x)$$

is said to be the *relative polynomial* of f at s and t, where $R(x)$ is defined as in (3.2).

Proposition 3.4. *The i-th derivative of* $P(x;f, s, t)$ *at* s *and* t *are* $f^{(i)}(s)$ *and* $f^{(i)}(t)$ *respectively.*

Proof. It can be verified directly from the construction of the relative polynomial.

Proposition 3.5. *For given* $\varepsilon>0$ *there is* $\delta>0$ *such that*

$s, t\in A$, $\quad 0<|s-t|<\delta\Rightarrow|P^{(i)}(x; f, s, t)-f^{(i)}(t)|<\varepsilon$ *for* $i=0, 1, \cdots, k$ *and* $x\in[s, t]$.

Proof. Since

$$P^{(i)}(x;f, s, t)=P_i^k(x;f, t)+\sum_{j=0}^{k}C_j^i\frac{(k+1)!}{(k-j+1)!}(x-t)^{k-j+1}R^{(i-j)}(x)$$

we have

$$P^{(i)}(x;f, s, t)-f^{(i)}(t)=P_i^k(x;f, t)-f^{(i)}(t)+\sum_{j=0}^{i}C_j^i\frac{(k+1)!}{(k-j+1)!}(x-t)^{k-j+1}R^{(i-j)}(x)$$

for $i=0, 1, \cdots, k$.

Therefore, the proposition follows from that $f \in N^k(A)$ and Lemma 3.2.

4. Some Propositions on Real Analysis.

Let $\Delta(A)$ denote the set $\{(x, x) : x \in A\}$. A real-valued function $F(x, t)$ defined on $(A \times A) \setminus \Delta(A)$ is said to be $o((x-t)^0)$ uniformly on A if for any given $\varepsilon > 0$ there is $\delta > 0$ such that

$$x, t \in A \text{ and } 0 < |x-t| < \delta \Rightarrow |F(x, t)| < \varepsilon.$$

Proposition 4.1. *Let $F(x, t)$ be a real-valued function defined and continuous on $(A^- \times A^-) \setminus \Delta(A^-)$. If F is $o((x-t)^0)$ uniformly on A then F is also $o((x-t)^0)$ uniformly on A^-.*

Proof. For given $\varepsilon > 0$, we take $\delta_0 > 0$ so that

$$x, t \in A \text{ and } 0 < |x-t| < \delta_0 \Rightarrow |F, (x, t)| < \frac{\varepsilon}{2}.$$

Let $\delta = \frac{\delta_0}{3}$. For $x', t' \in A^-$ and $0 < |x'-t'| < \delta$, it is seen that there are $\{x_n\}, \{t_n\} \subset A$ such that

$$0 < |x_n - t_n| < \delta_0$$

and $x_n \to x', t_n \to t'$ respectively. So we have

$$|F(x', t')| = \lim_{n\to\infty} |F(x_n, t_n)| \leqslant \frac{\varepsilon}{2} < \varepsilon.$$

Proposition 4.2. (*A generalization of the mean value theorem*) *Let $\phi(x)$ and $\psi(x)$ be two continuous functions on $[a, b]$ and A a closed subset of $[a, b]$.*

If

$$\phi'_A(x) = \psi_A(x) \qquad \text{on } A$$

and

$$\phi'_B(x) = \psi_B(x) \qquad \text{on } B = [a, b] \setminus A,$$

where ϕ_E, ψ_E denote the restriction of ϕ, ψ on E respectively, then for each pair of points $x_1, x_2 \in [a, b]$ $(x_1 < x_2)$ there exists $c \in (x_1, x_2)$ such that

$$\phi(x_2) - \phi(x_1) = \psi(c)(x_2 - x_1). \tag{4.1}$$

Proof. Let

$$\bar{\phi}(x) = \phi(x) - \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1}(x - x_1)$$

and

$$\bar{\psi}(x) = \psi(x) - \frac{\phi(x_2) - \phi(x_1)}{x_2 - x_1}.$$

It is clear that $\bar{\phi}(x)$ and $\bar{\psi}(x)$ are both continuous on $[a, b]$ and

(i) $\bar{\phi}'_B=\bar{\psi}_B$,

(ii) $\bar{\phi}'_A=\bar{\psi}_A$,

(iii) $\bar{\phi}(x_1)=\bar{\phi}(x_2)$.

Moreover, for proving (4.2) it is sufficient to prove that there exists $c\in(x_1, x_2)$ such that

$$\bar{\psi}(c)=0.$$

Since $\bar{\phi}(x)$ is continuous on $[x_1, x_2]$ it has either a maximal or a minimal value at some point $c\in(x_1, x_2)$. Suppose that $\bar{\phi}(c)$ is a maximal value, we have then

$$\frac{\bar{\phi}(x)-\bar{\phi}(c)}{x-c}\leqslant 0 \qquad \text{for } x\in(c, x_2)$$

and

$$\frac{\bar{\phi}(x)-\bar{\phi}(c)}{x-c}\geqslant 0 \qquad \text{for } x\in(x_1, c).$$

If there exists $t_n\in(x, c)\subset A$ such that $t_n\to c$, then $c\in A$ and since $\bar{\phi}'_A=\bar{\psi}_A$ we have

$$\bar{\psi}(c)=\lim_{n\to\infty}\frac{\bar{\phi}(t_n)-\bar{\phi}(c)}{t_n-c}\geqslant 0.$$

Otherwise we have some point x' such that

$$(x', c)\subset(x_1, c)\cap B.$$

Taking $x'_n\in(x' c)$ such that $x'_n\to c$ and using the Largrange mean value theorem on $[x_n, c]$ there exist $\xi_n\in(x'_n, c)$ such that

$$\bar{\psi}(\xi_n)=\frac{\bar{\phi}(x_n)-\bar{\phi}\ c)}{x_n-c}\geqslant 0.$$

By the continuity of $\bar{\psi}$ at c we have

$$\bar{\psi}(c)=\lim_{n\to\infty}\bar{\psi}(\xi_n)\geqslant 0.$$

Similarly, we can show that

$$\bar{\psi}(c)\leqslant 0$$

Thus the proposition is proved.

Proposition 4.3. *Let ϕ and ψ be two continuous functions on $[a, b]$ and A a closed subset of $[a, b]$. If*

$$\phi'_A(x)=\psi_A(x) \qquad \textit{on } A$$

and

$$\phi'_B(x)=\psi_B(x) \qquad on\ B=[a,b]\setminus A,$$

then

$$\phi'(x)=\psi(x) \qquad on\ [a,b].$$

Proof. For each $x\in[a,b]$, we consider

$$\frac{\phi(t)-\phi(x)}{t-x}.$$

By the above proposition there is c between t and x such that

$$\psi(c)=\frac{\phi(t)-\phi(x)}{t-x}.$$

By the continuity of ψ, for any given $\varepsilon>0$ there is $\delta>0$ such that

$$0<|s-x|<\delta\Rightarrow|\psi(s)-\psi(x)|<\varepsilon$$

Therefore

$$\begin{aligned}0<|t-x|<\delta&\Rightarrow 0<|c-x|<\delta\\&\Rightarrow|\psi(c)-\psi(x)|<\varepsilon\\&\Rightarrow\left|\frac{\phi(t)-\phi(x)}{t-x}-\psi(x)\right|<\varepsilon\end{aligned}$$

This proves the proposition.

5. Extension Theorems.

In this section we shall prove our main results stated in Theorems 5.1 and 5,4, In order to prove Theorem 5.1, we need the following known theorem.

Theorem A ([5], p. 55) *If A is a dense subset of a metric space E and f a uniformly continuous function of A into a complete metric space E', then f has a unique uniformly continuous extension over E.*

Theorem 5.1. *Let A be a subset of $[a,b]$ containing no isolated points. If $f\in N^k(A)$ then f has an N^k-extension over A^-.* i.e., *a function $g\in N^k(A^-)$ such that $g_A=f$.*

Proof. Since $f\in N^k(A)$, by Proposition 2.3 for each $i=0,1,\cdots,k$. $f^{(i)}$ is continuous on A. By Theorem A. for each $i=0,1,\cdots,k$ there exists a uniformly continuous g_i on A^- which extends $f^{(i)}$. we shall prove that

(*i*) $g'_i=g_{i+1}$ for $i=0,1,\cdots,k-1$ on A^-, and

(*ii*) $g_0\in N^k(A^-)$.

For (*i*), we consider for each $i=0,1.\cdots,k-1$,

$$F_i(x,t)=\frac{g_i(x)-g_i(t)}{x-t}-g_{i+1}(x).$$

It is seen that $F_i(x,t)$ continuous on $(A^-\times A^-)\setminus\Delta(A^-)$. Furthermore, for $x, t\in A$, we have that

$$F_i(x,t)=\frac{f^{(i)}(x)-f^{(i)}(x)}{x-t}-f^{(i+1)}(x)$$

which by Proposition 2.6 is $o((x-t)^0)$ uniformly on A, Therefore, by Proposition 4.1, $F_i(x,t)$ is $o((x-t)^0)$ uniformly on A^- and it means that $g_i'=g_{i+1}$ on A^-.

For (ii), we consider for each $i=0,1,\cdots,k$,

$$G_i(x,t)=\frac{g_0^{(i)}(x)-P_i^k(x;g_0,t)}{(x-t)^{k-i}}.$$

It is also seen that $G_i(x,t)$ is continuous on $(A^-\times A^-)\setminus\Delta(A^-)$. Noticing that for $x, t\in A$

$$G_i(x,t)=\frac{f^{(i)}(x)-P_i^k(x;f,t)}{(x-t)^{k-i}}$$

and hence by k-normality of f on A, $G_i(x,t)$ is $o((x-t)^0)$ uniformly on A. By Proposition 4.1, $G_i(x,t)$ is $o((x-t)^0)$ uniformly on A^-. Thus $g_0\in N^k(A^-)$.

Corollary 5.2. *If* $A^-=[a,b]$ *and* $f\in N^k(A)$ *then there exists* $g\in C^k[a,b]$ *which extends* f.

Corollary 5.3. *If* $F\in N^\infty(A)$ (i.e. $f\in N^k(A)$ *for* $k=0,1,\cdots$) *then* f *has an* N^∞*-extension over* A^-.

Thorem 5.4. *Let* A *be a subset of* $[a,b]$ *containing no isolated points. A function* f *defined on* A *has a* C^k*-extension over* $[a,b]$ *iff* $f\in N^k(A)$.

Proof. *Necessity*: Let g be a C^k-function on $[a,b]$ which extends f. By Proposition 2.4, $g\in N^k([a,b])$ and by Proposition 2.2, $g\in N^k(A)$. Since $f(x)=g(x)$ for $x\in A$, $f\in N^k(A)$.

Sufficiency: It is clear that by Theorem 5.1, we may assume that A is closed. Furthermore, we shall show that without loss of generality we may assume that $a, b\in A$. In fact, if not so we let $[a_1,b_1]$ be the smallest closed interval containing A. If the theorem is true for $[a_1,b_1]$ and $\tilde{f}$ denotes the obtained differentiable extension, then we define the polynomials

$$P_1(x)=\tilde{f}(a_1)+\tilde{f}'(a_1)(x-a_1)+\cdots+\frac{\tilde{f}^{(k)}(a_1)}{k!}(x-a_1)^k$$

on $[a,a_1]$ and

$$P_2(x)=\tilde{f}(b_1)+\tilde{f}'(b_1)(x-b_1)+\cdots+\frac{\tilde{f}^{(k)}(b_1)}{k!}(x-b_1)^k$$

on $[b_1, b]$. It will be easily seen that the function

$$g(x)=\begin{cases} P_1(x) & x\in[a,a_1],\\ \tilde{f}(x) & x\in[a_1,b_1],\\ P_2(x) & x\in[b_1,b]\end{cases}$$

is the differentiable extension of f over $[a, b]$.

Now, let $a, b\in A$, $B=[a,b]\setminus A$ is then an open subset of the real line. It is well-known in real analysis that an open set can be written as the union of a countable disjoint family of open intervals. Let $\{(t_{2n-1}, t_{2n})\}_{n=1,2,\cdots}$ be the disjoint family of open intervals such that

$$B=\cup\ \{(t_{2n-1}, t_{2n}): n=1, 2, \cdots\}.$$

Let $F=\cup\ \{[t_{2n-1}, t_{2n}]: n=1, 2, \cdots\}$ and a function $g(x)$ defined on F as follows

$$g(x)=P(x; f, t_{2n-1}, t_{2n}) \text{ for } x\in[t_{2n-1}, t_{2n}],$$

where $P(x; f t_{2n-1}, t_{2n})$ is the relative polynomial of f at t_{2n-1}, t_{2n}.

We shall prove that $g_B\in C^k(B)$. It is seen that $g_B^{(i)}$ exists on B for $i=0, 1, \cdots, k$. Let

$$F_1(\delta)=\{[t_{2n-1}, t_{2n}]: |t_{2n}-t_{2n-1}|<\delta\}$$

and

$$F_2(\delta)=\{[t_{2n-1}, t_{2n}]: |t_{2n}-t_{2n-1}|\geqslant\delta\}.$$

It is clear that (i) $F_1(\delta)$ and $F_2(\delta)$ are disjoint, (ii) $F_1(\delta)\cup F_2(\delta)=F$ and (iii) $F_2(\delta)$ is the union of a finite number of closed intervals. Since on each $[t_{2n-1}, t_{2n}]$, g is a polynomial, $g^{(i)}$ is uniformly continuous on $F_2(\delta)$ for each $i=0, 1, \cdots, k$.

For given $\varepsilon>0$, by the uniform continuity of $f^{(i)}$ on A, there is $\delta_1>0$ such that

$$s, t\in A,\ 0<|x-t|<\delta_1 \Rightarrow |f^{(i)}(s)-f^{(i)}(t)|<\frac{\varepsilon}{3} \tag{5.1}$$

By definition of g and proposition 3.5, there is $\delta_2>0$ such that

$$|t_{2n}-t_{2n-1}|<\delta_2 \Rightarrow |g^{(i)}(x)-f^{(i)}(t_{2n})|+|g^{(i)}(x)-f^{(i)}(t_{2n-1})|<\frac{\varepsilon}{3}$$

$$\text{for } x\in[t_{2n-1}, t_{2n}] \text{ and } i=0,1,\cdots,k. \tag{5.2}$$

By the uniform continuity of $g^{(i)}$ on $F_2(\delta_2)$, there is $\delta_3>0$ such that

$$x_1, x_2\in F_2(\delta_2),\ |x_1-x_2|<\delta_3 \Rightarrow |g^{(i)}(x_1)-g^{(i)}(x_2)|<\frac{\varepsilon}{3} \text{ for } i=0, 1, \cdots, k. \tag{5.3}$$

Let $\delta=\text{Min}\ \{\delta_1, \delta_2, \delta_3\}$, we shall prove that

$$x', x'' \in B, |x'-x''| < \delta \Rightarrow |g^{(i)}(x') - g^{(i)}(x'')| < \varepsilon \tag{5.4}$$

We consider the following cases:

Case (*a*) If $x', x'' \in F_2(\delta_2)$, then (5.4) follows from (5.3).

Case (*b*) If $x', x'' \in F_1(\delta_2)$ and $x', x'' \in (t_{2n-1}, t_{2n})$ for some n, then (5.4) follows from (5.2).

Case (*c*) If $x', x'' \in F_1(\delta_2)$ and there are

$$t_{2n-1} < t_{2n} < t_{2m-1} < t_{2m}$$

such that

$$x' \in (t_{2n-1}, t_{2n}) \text{ and } x'' \in (t_{2m-1}, t_{2m})$$

then (5.4) follows from (5.1) and (5.2).

Case (*d*) If $x' \in F_1(\delta_2)$ and $x'' \in F_2(\delta_2)$ then (5.4) follows from (5.1), (5.2) and (5.3).

From the above discussion, we have proved that $g_B \in C^k(B)$.

Let

$$\tilde{f}_i(x) = \begin{cases} f^{(i)}(x) & x \in A, \\ g^{(i)}(x) & x \in B. \end{cases}$$

Then $\tilde{f}_0$ is defined on $[a, b]$ and extends f. We shall prove that $\tilde{f} \in C^k([a, b])$.

We show first that $\tilde{f}_{(i)}$ are all continuous. To see this we need only to prove that for any sequence $\{s_n\}$ in B such that $s_n \to s \in A$

$$\lim_{n \to \infty} \tilde{f}_i(s_n) = \tilde{f}_i(s).$$

It is obvious that we may assume $s_n < s$ for all n. If s is a right end point of some interval (t_{2n-1}, t_{2n}) then the assertion follows from the definition of g. If s is not a right end point of any intervals (t_{2n-1}, t_{2n}) then the assertion follows from (5.1) and (5.2). Therefore $\tilde{f}_i$ are all continuous on $[a, b]$.

Now, in order to show that $f_0 \in C^k([a, b])$ it suffices to prove that

$$\tilde{f}'_{i-1} = \tilde{f}_i$$

for $i = 1, 2, \cdots, k$.

From the definition of f_i, we see that the derivative of the restriction of $\tilde{f}_i$ on A (resp. B) is the restriction of $\tilde{f}_{i+1}$ on A (resp. B).

By Proposition 4.3, we have that

$$\tilde{f}'_{i-1} = \tilde{f}_i$$

for $i = 1, \cdots, k$. The theorem is thus proved.

6. Application to the Frenet Formulas.

The fundamental theorem of curves in the Euclidean 3-space E^3 is the following

Theorem 6.1. (*The Fundamental Theorem*) *For any given integer* $k \geqslant 1$, *if* $\kappa(s)$ *and* $\tau(s)$ *are two functions of class* C^{k-1} *on a closed interval* $[0, L]$ *then there exists a* C^{k+1}*-curve* $X(s)$ *in* E^3 *and a* C^k*-family of orthonormal frames* $X e_1 e_2 e_3(s)$ *along* $X(s)$ *satisfying the equations*

$$\begin{aligned} dX(s)/ds &= e_1(s) \\ de_1(s)/ds &= \kappa(s)e_2(s) \\ de_2(s)/ds &= -\kappa e_1(s) + \tau(s)e_3(s) \\ de_3(s)/ds &= -\tau(s)e_2(s). \end{aligned} \tag{6.1}$$

Moreover, $X(s)$ *is unique up to a motion.*

For a given curve $X(s)$, *a* C^k-family of orthonormal frames $Xe_1 e_2 e_3(s)$ along $X(s)$ satisfying (6.1) with suitably chosen functions $\kappa(s)$ and $\tau(s)$ is called a C^k-Frenet frame of $X(s)$. It is known that even a C^∞-curve may not have a C^0-Frenet frame. For instance, in his paper [7, p. 111], *K. Nomizu* has constructed such an example.

What is then a necessary and sufficient condition for a given C^{k+1}-curve $X(s)$ to admit a C^k-Frenet frame? (Of course, $\kappa(s)$ and $\tau(s)$ would then be of class C^{k-1}.)

It is well known that if we restrict ourselves to the case where $\kappa(s)$ is always positive, then a necessary and sufficient condition for a given C^{k+1}-curve $X(s)$ to admit a C^k-Frenet frame is that $|X''(s)| > 0$. Some authors have tried to remove this restriction. In the case where $\kappa(s) \geqslant 0$ is assumed, *A. Wintner* stated in [8] a necessary and sufficient condition for a given C^2-curve $X(s)$ to admit a C^1-Frenet frame. (We note that his conclusion holds only for the case where the set $\{s : \kappa(s) \neq 0\}$ is dense in $[0, L]$.) Without requiring that $\kappa(s)$ is non-negative everywhere, *K. Nomizn* stated in [7] a sufficient but not necessary condition for a given C^∞-curve to admit a C^∞-Frenet frame. Under his condition, the number of zeros of $\kappa(s)$ must be finite.

In this section we give a necessary and sufficient condition for a given C^{k+1}-curve to admit a C^k-Frenet frame, in the case where the set $\{s : \kappa(s) \neq 0\}$ is dense in $[0, L]$ but without requiring $\kappa(s) \geqslant 0$.

We call a curve $X(s)$, $0 \leqslant s \leqslant L$, in E^3 a k-Frenet curve if it has a family of C^k-Frenet frame. It is obvious that every k-Frenet curve must be of class C^{k+1}. In the following we always assume that $k \geqslant 1$!

Definition 6.2. If $X(s)$ is a C^{k+1}-curve defined on $[0, L]$ with $|X'(s)| = 1$ and there exists a function $\kappa(s) \in C^{k-1}$ such that

(i) $|\kappa(s)| = |X''(s)|$ $\quad s \in [0, L]$,

(ii) $(X''(s)/\kappa(s))_i \in N^k(A)$ $\quad i=1, 2, 3$,

where $A=\{s : \kappa(s) \neq 0\}$ and $(X''(s)/\kappa(s))_i$ is the i-th component of $X''(s)/\kappa(s)$, then we say that $X(s)$ is a k-normal curve on A.

Remark. In the above definition, by the continuity of $\kappa(s)$, we know that A is an open set in $[0, L]$, hence has no isolated points.

Theorem 6.3. *A C^{+1}-curve $X(s), 0 \leqslant s \leqslant L$, having the property that $A=\{s : |X''(s)| \neq 0\}$ is dense on $[0, L]$ is a k-Frenet curve iff $X(s)$ is a k-normal curve on A.*

Proof. *Necessity*: Let $Xe_1 e_2 e_3(s)$ be a C^k-Frenet frame along $X(s)$. Then each component of $e_2(s)$ is a C^k-function on $[0, L]$, hence k-normal on $A=\{s : \kappa(s) \neq 0\}$. Since $X''(s)/\kappa(s)=e_2(s)$ on A, by Definition 6.2, $X(s)$ is k-normal on A.

Sufficiency: By k-normality of $X(s)$ on A, there is $\kappa(s) \in C^{k-1}([0, L])$ such that

$$\left(\frac{X''(s)}{\kappa(s)}\right)_i \in N^k(A) \qquad i=1, 2, 3$$

where $A=\{s : \kappa(s) \neq 0\}$. Let

$$e_2(s)=(e_{21}(s), e_{22}(s), e_{23}(s))$$

where

$$e_{2i}=\left(\frac{X''(s)}{\kappa(s)}\right)_i \text{ for } i=1, 2, 3, \text{ and } e_1(s)=X'(s).$$

Then we see that

(a) $G_{2i}(s) \in N^k(A)$ $\quad i=1, 2, 3$

(b) $|e_2(s)| = 1$ $\quad$ on A,

(c) $e_1(s) \cdot e_2(s)=0$ $\quad$ on A.

By Theorem 5.1 for each $i, e_{2i}(s)$ has an extension on $A^-=[0, L]$, which is k-normal on $[0, L]$. Denoting the extensions obtained also by $e_{2i}(s)$ and $e_2(s)=(e_{21}(s), e_{22}(s), e_{23}(s))$ we can easily see that

(a′) $e_{2i}(s) \in C^k([0, L])$, $\quad i=1, 2, 3$

(b′) $|e_2(s)| =1$, $\quad$ on $[0, L]$

(c′) $e_1(s) \cdot e_2(s)=0$, $\quad$ on $[0, L]$

by continuity of the inner product.

Let $e_3(s)=e_1(s) \times e_2(s)$. It is obvious that $|e_3(s)| = 1, e_3(s) \cdot e_2(s)=e_3(s) \cdot e_1(s)=0$

and $e_3(s) \in C^k$. By the usual argument there exists a function $\tau(s) \in C^{k-1}$ such that

$$\begin{aligned} e'_2(s) &= -\kappa(s)e_1(s) & &+\tau(s)e_3(s) \\ e'_3(s) &= & -\tau(s)e_2(s) & \end{aligned}$$

By making use of Corollary 5.3, we have similarly the following.

Theorem 6.4. *A C^∞-curve $X(s), 0 \leqslant s \leqslant L$, having the property that $A = \{s : |X''(s)| \neq 0\}$ is dense in $[0, L]$ and with $|X'(s)| = 1$ is a ∞-Frenet curve iff $X(s)$ is a k-normal curve on A for $k = 1, 2, \cdots$.*

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