ON THE TYPE OF ENTIRE FUNCTIONS

By

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Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an entire function of regular growth of order ρ.

It is known [1],

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{n}{e\rho} \left| a_n \right|^{\rho/n} = \begin{matrix}T\\ t\end{matrix},$$

where T and t are upper and lower types of $f(z)$, defined by

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{r^\rho} = \begin{matrix}T\\ t\end{matrix},$$

where $M(r)=\max_{|z|=r} |f(z)|$.

The object of this paper is to investigate the relationship between the types of two or more entire functions of regular growth.

Now, we define *RATIO TYPE* M of $f(z)$ as,

$$\frac{T}{t}=M, \quad (1\leqslant M\leqslant\infty).$$

Theorem

Let $f_i(z)=\sum_{n=0}^{\infty} a_n^{(i)} z^n, (i=1,2,3,\cdots,k)$ *be* k *entire functions of regular growth of orders* $\rho_1, \rho_2, \cdots, \rho_k$ *and types* $T_1, T_2, \cdots, T_k$, *lower types* $t_1, t_2, \cdots, t_k$. $(0<t_i\leqslant T_i<\infty)$ *and* $|a_n^{(i)}/a_{n+1}^{(i)}|$ *are non decreasing functions of* n,

then

$$f(z)=\sum_{n=0}^{\infty} a_n z^n,$$

where

$$\log|a_n| \sim \log\{|a_n^{(1)}|^{m_1} |a_n^{(2)}|^{m_2} \cdots\cdots |a_n^{(k)}|^{m_k}\} \quad\cdots\cdots\cdots\cdots(\mathrm{I})$$

(m_i'*s are constants*)

is an entire function of order ρ, *type* T *and lower type* t, *given by*

$$\prod_{i=1}^{k-1} (\rho_i t_i)^{m_i/\rho_i} \leqslant \left\{ \frac{(\rho t)^{1/\rho}}{(\rho_k t_k)^{m_k/\rho_k}} \cdot \frac{(\rho T)^{1/\rho}}{(\rho_k T_k)^{m_k/\rho_k}} \right\} \leqslant \prod_{i=1}^{k-1} (\rho_i T_i)^{m_i/\rho_i} \quad\cdots\cdots\cdots(\mathrm{II})$$

Corollary 1 *If any* k *functions out of* $k+1$ *functions* $f(z), f_1(z), f_2(z), \cdots, f_k(z)$

are of perfectly regular growth, then all $k+1$ *functions are also of perfectly regular growth, and*

$$(\rho T)^{1/\rho}=\prod_{i=1}^{k}(\rho_i T_i)^{m_i/\rho_i}$$

Remark: It improves the result of *Pavankumar* [2] and *R, S. L. Srivastav* [3].

Corollary 2:

$$M^{1/\rho}\leqslant\prod_{i=1}^{k}(M_i)^{m_i/\rho_i}$$

where $M, M_1, M_2, \cdots M_k$ *are ratio types of* $f(z), f_1(z), f_2(z), \cdots f_k(z)$ *respectively.*

Proof of Theorem We know [1],

$$\lim_{n\to\infty}\ \begin{matrix}\sup\\ \inf\end{matrix}\ n\left|a_n^{(i)}\right|^{\rho_i/n}=\begin{matrix}e.\rho_i.T_i\\ e.\rho_i.t_i\end{matrix},$$

$$n\left|a_n^{(i)}\right|^{\rho_i/n}<(e.\rho_i.T_i+\epsilon)\ \text{for}\ n\geq n_0.$$

$$n\left|a_n^{(i)}\right|^{\rho_i/n}<(e.\rho_i.t_i+\epsilon)\ \text{for a sequence of values of}\ n\to\infty$$

so,

$$\left|a_n^{(i)}\right|^{m_i}<\left(\frac{e.\rho_i.T_i+\epsilon}{n}\right)^{nm_i/\rho_i}\quad\text{for}\quad n\geq n_0\cdots\cdots\cdots\cdots\text{(III)}$$

$$\left|a_n^{(k)}\right|^{m_k}<\left(\frac{e.\rho_k.t_k+\epsilon}{n}\right)^{nm_k/\rho_k}\quad\text{for a sequence of values of}\ n\to\infty\quad\cdots\cdots\text{(IV)}$$

Taking $i=1,2,\cdots,k-1$ in (III) and multiplying all together with (IV), we get

$$\log\prod_{i=0}^{k}\left|a_n^{(i)}\right|^{m_i}<\log\left\{\prod_{i=1}^{k-1}\left\{\frac{e.\rho_i.T_i+\epsilon}{n}\right\}^{nm_i/\rho_i}\cdot\left\{\frac{e.\rho_k.t_k+\epsilon}{n}\right\}^{nm_k/\rho_k}\right\}$$

Hence, using (I) we get,

$$\log|a_n|<\log\left\{\prod_{i=1}^{k-1}\left\{\frac{e.\rho_i.T_i+\epsilon}{n}\right\}^{nm_i/\rho_i}\cdot\left\{\frac{e.\rho_k.t_k+\epsilon}{n}\right\}^{nm_k/\rho_k}\right\}$$

$$\log|a_n|<\sum_{i=1}^{k-1}\frac{nm_i}{\rho_i}\log(e.\rho_i.T_i+\epsilon)+\frac{nm_k}{\rho_k}\log(e.\rho_k.t_k+\epsilon)$$
$$-n\left(\frac{m_1}{\rho_1}+\frac{m_2}{\rho_2}+\cdots+\frac{m_k}{\rho_r}\right)\cdot\log n$$

But by [4]

$$\frac{1}{\rho}=\frac{m_1}{\rho_1}+\frac{m_2}{\rho_2}+\cdots+\frac{m_k}{\rho_k} \quad \text{(which can be easily proved)}$$

Hence, we get

$$\frac{n}{\rho} \log |a_n|^{\rho/n}+\frac{n}{\rho} \log n<\sum_{i=1}^{k-1} \frac{nm_i}{\rho_1} \log (e.\ \rho_i.\ T_i+\epsilon)+\frac{nm_k}{\rho_k} \log (e.\ \rho_k.\ t_k+\epsilon)$$

i. e.,

$$\log \{n|a_n|^{\rho/n}\}^{1/\rho}<\sum_{i=1}^{k-1} \frac{m_i}{\rho_1} \log (e.\ \rho_i.\ T_i+\epsilon)+\frac{m_k}{\rho_k} \log (e.\ \rho_k.\ t_k+\epsilon)$$

so,

$$\liminf_{n\to\infty} \{n|a_n|^{\rho/n}\}^{1/\rho} \leq \prod_{i=1}^{k-1} (e.\ \rho_i.\ T_i)^{m_i/\rho_i}.\ (e.\ \rho_k,\ t_k)^{m_k/\rho_k}$$

i, e.,

$$(e.\ \rho.\ t)^{1/\rho} \leq \prod_{i=1}^{k-1} (e.\ \rho_i.\ T_i)^{m_i/\rho_i}.\ (e.\ \rho_k.\ t_k)^{m_k/\rho_k}$$

i. e.,

$$\frac{(\rho t)^{1/\rho}}{(\rho_k.\ t_k)^{m_k/\rho_k}} \leq \prod_{i=1}^{k-1} (\rho_i.\ T_i)^{m_i/\rho_i}$$

which proves a part of right hand side of (II).

Taking $i=1, 2, \cdots, k$ in (III) and proceeding similarly, we get

$$\frac{(\rho T)^{1/\rho}}{(\rho_k.\ T_k)^{m_k/\rho_k}} \leq \prod_{i=1}^{k-1} (\rho_i.\ T_i)^{m_i/\rho_i}$$

which proves the other part of right hand side of (II).

Similarly left hand side follows.

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