ON THE PROPERTIES OF AN ENTIRE FUNCTION OF TWO COMPLEX VARIABLES (II)

By

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1. Let

(1.1)
$$f(z_1, z_2) = \sum_{m_1, m_2 \geqq 0} a_{m_1 m_2} z_1^{m_1} z_2^{m_2}$$

be an entire functon of two complex variables z_1 and z_2, holomorphic for $|z_j| \leqslant r_j, j=1, 2$. We know

$$M(r_1, r_2) = \max_{|z_j| \leqslant r_j} |f(z_1, z_2)|, \; j=1,2.$$

Following *Bose* and *Sharma* ([1], pp. 214–215), $\mu(r_1, r_2)$ denotes the maximum term in the double series (1.1) for a given value of r_1 and r_2 and $\nu_1(m_2; r_1, r_2)$ or $\nu_1(r_1, r_2)$, r_2 fixed, $\nu_2(m_1; r_1, r_2)$ or $\nu_2(r_1, r_2)$, r_1 fixed and $\nu(r_1, r_2)$ denote the ranks of the maximum term of the double series (1.1).

In continuation of my paper [2] I have investigated few more results connecting the auxiliary functions $M(r_1, r_2)$ etc.

2. Theorem 1. *Let*

$$\lim_{r_1, r_2 \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log M(r_1, r_2)}{\log \log (r_1 r_2)} = \begin{matrix} T \\ t \end{matrix}, \quad \lim_{r_1, r_2 \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log \log \mu(r_1, r_2)}{\log \log (r_1 r_2)} = \begin{matrix} S \\ \sigma \end{matrix}$$

and

$$\lim_{r_1, r_2 \to \infty} \inf \frac{\log \nu(r_1, r_2)}{\log \log (r_1 r_2)} = \delta,$$

then

(2.1)
$$t = \sigma;$$

(2.2)
$$T = S;$$

(2.3)
$$t = 1 + \delta.$$

Proof: (i) We know ([1], p. 217)

(2.4)
$$M(r_1, r_2) > \mu(r_1, r_2).$$

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Taking limits, we have

$$t \geqslant \sigma \geqslant 1.$$

Now, we prove that $t \leqslant \sigma$, for this we may suppose $\sigma < \infty$ and let us choose a number α such that $\alpha > \sigma + 1$.

Also, we know ([1] pp. 219-220) that

$$\{\nu_2(0\,;r_2)+\nu_1(\nu_2\,;r_1,r_2)\} \log 2 < \log \mu(2r_1, 2r_2)$$

or,

$$\{\nu_1(0\,;r_1)+\nu_2(\nu_1\,;r_1,r_2)\} \log 2 < \log \mu(2r_1\, 2r_2).$$

Hence

$$\text{(2.5)} \qquad \liminf_{r_1, r_2 \to \infty} \frac{\log \nu(r_1, r_2)}{\log \log (r_1 r_2)} \leqslant \liminf_{r_1, r_2 \to \infty} \frac{\log \log \mu(r_1, r_2)}{\log \log (r_1 r_2)} < (\alpha - 1).$$

Let us choose β and η such that $(\alpha-1)<\beta, \dfrac{(\alpha-1)}{\beta}<\eta<1$. Then, we have

$$\text{(2.6)} \qquad \log \nu(r_1, r_2) < \beta\eta \log \log (r_1 r_2),$$

for a sequence of values of $r_1 = x_{1,n}$ and $r_{2,n} = x_{2,n}$ (say) for $x_{1,n}$ and $x_{2,n}$ tending to infinity. Let $X_{1,n} = \exp\{\log(x_{1,n})^{\eta}\}$ $X_{2,n} = \exp\{\log(x_{2,n})^{\eta}\}$ and let $I_{1,n}$ and $I_{2,n}$ denote the intervals $X_{1,n} \leqslant r_1 \leqslant x_{1,n}$ and $X_{2,n} \leqslant r_2 \leqslant x_{2,n}$, respectively. Then for every r_1 in $I_{1,n}$ and r_2 in $I_{2,n}$, we have

$$\text{(2.7)} \qquad \begin{aligned} \log \nu(r_1, r_2) &\leqslant \log \nu(x_{1,n}, x_{2,n}) \\ &< \beta\eta \log \log (x_{1,n}\, x_{2,n}) \\ &= \beta \log \{\log (x_{1,n}\, x_{2,n})\}^{\eta} \\ &\leqslant \beta \log \log (r_1 r_2). \end{aligned}$$

Next, let us take $Y_{1,n} = 1 + X_{1,n}$ and $Y_{2,n} = 1 + X_{2,n}$. Then, for large n, $Y_{1,n}$ and $Y_{1,2n}$ lie inside $I_{1,n}$ and $Y_{2,n}$ and $Y_{2,2n}$ lie inside $I_{2,n}$. Since, we know ([1], p, 218) that

$$\begin{aligned} M(r_1, r_2) &< \mu(r_1, r_2)\left\{3\nu\left(r_1 + \frac{r_1}{\nu_1(r_1, r_2)}, r_2 + \frac{r_2}{\nu_2(r_1, r_2)}\right) + 3\right\} \\ &< \mu(r_1, r_2)\{3\nu(2r_1, 2r_2) + 3\}. \end{aligned}$$

Therefore

$$\text{(2.8)} \qquad \log M(Y_{1,n}, Y_{2,n}) < \log \mu(Y_{1,n}, Y_{2,n}) + \log \nu(2Y_{1,n}, 2Y_{2,n}) + 0(1).$$

Also, we know ([1] pp. 216–217)

$$\log \mu(\nu_1\,; r_1, r_2) = \int_0^{r_1} \nu_1(0\,; x_1)\,\frac{dx_1}{x_1} + \int_0^{r_2} \nu_2(\nu_1\,; x_2)\,\frac{dx_2}{x_2}$$

or

$$\log \mu(\nu_2 ; r_1, r_2)=\int_0^{r_2} \nu_2(0 ; x_2) \frac{dx_2}{x_2}+\int_0^{r_1} \nu_1(\nu_2 ; x_1) \frac{dx_1}{x_1}.$$

Hence,

$$\begin{aligned}\log \mu(\nu_1 ; Y_{1,n}, Y_{2,n}) &< \nu_1(0 ; Y_{1,n}) \log Y_{1,n}+\nu_2(\nu_1 ; Y_{2,n}) \log Y_{2,n} \\ &< \{\nu_1(0 ; Y_{1,n})+\nu_2(\nu_1 ; Y_{2,n})\} \log (Y_{1,n} Y_{2,n})\end{aligned}$$

or

$$\log \mu(\nu_2 ; Y_{1,n}, Y_{2,n})<\{\nu_2(0 ; Y_{2,n})+\nu_1(\nu_2 ; Y_{1,n})\} \log (Y_{1,n} Y_{2,n}).$$

So

$$\log \mu(Y_{1,n}, Y_{2,n})<2 \nu(Y_{1,n}, Y_{2,n}) \log (Y_{1,n} Y_{2,n}). \tag{2.9}$$

Using (2.9) in (2.8), we get

$$\log M(Y_{1,n}, Y_{2,n})<2 \nu(Y_{1,n}, Y_{2,n}) \log (Y_{1,n} Y_{2,n})\{1+o(1)\}. \tag{2.10}$$

Taking logarithim on both the sides and using (2.7), we get

$$\begin{aligned}\log \log M(Y_{1,n}, Y_{2,n}) &<\log \nu(Y_{1,n}, Y_{2,n})+\log \log (Y_{1,n} Y_{2,n})+o(1) \\ &<(\beta+1) \log \log (Y_{1,n} Y_{2,n})+o(1).\end{aligned} \tag{2.11}$$

Taking limits, we get

$$t \leqslant \sigma,$$

which completes the proof of (2.1).

(ii) By an argument similar to above, we can show that $T=S$, and so proof is omitted.

(iii) We first prove that $t \leqslant 1+\delta$, we suppose that $\delta<\infty$. Let us choose numbere α, β and η such that $\delta<\alpha<\beta, \alpha/\beta<\eta<1$.

Then, similar to that as in (i), we have

$$\log \log M(r_1, r_2)<(\beta+1) \log \log (r_1 r_2)+o(1), \tag{2.12}$$

for a sequence of values of r_1 and r_2 tending to infinity.

Hence taking limits, we get

$$t \leqslant 1+\delta. \tag{2.13}$$

Also from (2.5), we get

$$1+\delta \leqslant t, \tag{2.14}$$

since $\sigma=t$, and thus (2.3) follows.

REFERENCES

[1] S. K. Bose & D. Sharma: *Integral functions of two complex variables,* Compositio Math., 15, 1963, pp. 210–226.

[2] A. K. Agarwal: *On the properties of an entire function of two complex variables,* Canadian Jour. Math., 20, 1968, pp. 51–57.

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