A CORRECT SYSTEM OF AXIOMS FOR A SYMMETRIC GENERALIZED UNIFORM SPACE

By

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Let X be a set with power set $P(X)$. In [2] *M. W. Lodato* gives the following axioms for a symmetric generalized proximity $\mathfrak{P}$ on X: (where $\mathfrak{P}$ is a subset of $P(X) \times P(X)$)

L_1: $(A, B) \notin \mathfrak{P}$ implies $(B, A) \notin \mathfrak{P}$;
L_2: $(A, B) \notin \mathfrak{P}$ implies $A \cap B = \phi$;
L_3: $(A, B) \notin \mathfrak{P}$ and $(A, C) \notin \mathfrak{P}$ implies $(A, (B \cup C)) \notin \mathfrak{P}$;
L_4: $(A, B) \in \mathfrak{P}$ implies $A \neq \phi$ and $B \neq \phi$;
L_5: $(A, B) \in \mathfrak{P}$ and $(\{b\}, C) \in \mathfrak{P}$ for all b in B implies that $(A, C) \in \mathfrak{P}$.

He also shows (in [2]) that a topology $\mathfrak{T}(\mathfrak{P})$ (called the proximity topology) on X can be defined: $x \in \bar{A}$ iff $(\{x\}, A) \in \mathfrak{P}$. Also, if $(X, \mathfrak{T})$ is a topological space then $\mathfrak{T}$ is the proximity topology for some proximity $\mathfrak{P}$ on X iff $\mathfrak{T}$ is symmetric (i. e $x \in \bar{y}$ implies $y \in \bar{x}$ for all x, y in X)

Let $P(X \times X)$ denote the power set of $(X \times X)$.

Definition 1. A subset $\mathfrak{U}$ of $P(X \times X)$ is a *generalized uniformity on* X iff for every U in $\mathfrak{U}$ U^{-1} contains a member of $\mathfrak{U}$.

Let $\mathfrak{U}$ be a generalized uniformity on X. Consider the following axioms:

B_1: For every U in $\mathfrak{U}$ $U \supseteq \Delta$;
B_2: For every A in $P(X)$ and U, V in $\mathfrak{U}$ there is a W in $\mathfrak{U}$ such that $W[A] \subseteq U[A] \cap V[A]$;
B_3: For every A, B in $P(X)$ and U in $\mathfrak{U}$ $V[A] \cap B \neq \phi$ for all V in $\mathfrak{U}$ implies there exists x in B and there exists a W in $\mathfrak{U}$ such that $W[x] \subseteq U[A]$.

Thorem 1. *Suppose* $\mathfrak{P}$ *and* $\mathfrak{U}$ *satisfy the relation*: (A, B) *in* $\mathfrak{P}$ *iff for every* U *in* $\mathfrak{U}$ $U[A] \cap B \neq \phi$. *Then* $\mathfrak{P}$ *satisfies* L_1, L_2, L_3, L_4 *and* L_5 *iff* $\mathfrak{U}$ *satisfies* B_1, B_2 *and* B_3.

Proof. We first show that if $\mathfrak{U}$ satisfies B_1, B_2 and B_3, then $\mathfrak{P}$ satisfies L_1, L_2, L_3, L_4 and L_5.

L_1: Suppose $(A, B) \notin \mathfrak{P}$. There exists (by hypothesis) a U in $\mathfrak{U}$ such that $U[A] \cap B = \phi$. Suppose $U^{-1}[B] \cap A \neq \phi$. Let $x_0 \in U^{-1}[B] \cap A$. Then $x_0 \in U^{-1}[B]$; so that there exists $y_0 \in B$ such that $(y_0, x_0) \in U^{-1}$ implies (by definition) that $(x_0, y_0) \in U$; so that $y_0 \in U[A] \cap B$ which is a contradiction. Hence $U^{-1}[B] \cap A = \phi$. But $U^{-1} \supseteq V$ where $V \in \mathfrak{U}$, and $V[B] \cap A = \phi$. Hence $(B, A) \notin \mathfrak{P}$.

L_2: Suppose $A \cap B \neq \phi$; (by B_1) for all U in $\mathfrak{U}$ $U[A] \cap B \neq \phi$; so that $(A, B) \in \mathfrak{P}$.

L_3: Suppose $(A, B) \notin \mathfrak{P}$ and $(A, C) \notin \mathfrak{P}$. Then there exists U, V in $\mathfrak{U}$ such that $U[A] \cap B = \phi$, and $V[A] \cap C = \phi$. There exists (by B_2) a W in $\mathfrak{U}$ such that $W[A] \subseteq U[A] \cap V[A]$; so that $W[A] \cap (B \cup C) = \phi$; so that $(A, (B \cup C)) \notin \mathfrak{P}$.

L_4: Immediate from the definition of $\mathfrak{P}$ and the fact that the members of $\mathfrak{U}$ are are nonempty (by B_1).

L_5: To prove this it is sufficient to show that $(A, -B) \notin \mathfrak{P}$ and $(A, C) \in \mathfrak{P}$ implies there exists x in C such that $(\{x\}, -B) \notin \mathfrak{P}$. $(A, -B) \notin \mathfrak{P}$ implies that there exists U in $\mathfrak{U}$ such that $U[A] \cap (-B) = \phi$; so that $U[A] \subseteq B$. Since $(A, C) \in \mathfrak{P}$ we have that $V[A] \cap C \neq \phi$ for all V in $\mathfrak{U}$; so that (by B_3) there exists $X \in C$ and there exists W in $\mathfrak{U}$ such that $W[x] \subseteq U[A] \subseteq B$; so that $W[x] \cap (-B) = \phi$; so that $(\{x\}, -B) \notin \mathfrak{P}$.

To prove the converse we now show that if $\mathfrak{P}$ satisfies L_1, L_2, L_3, L_4 and L_5 then $\mathfrak{U}$ satisfies B_1, B_2, and B_3.

B_1: Let $x \in X$. Let $U \in \mathfrak{U}$. $\{x\} \cap \{x\} \neq \phi$ implies (by L_2) that $(\{x\}, \{x\}) \in \mathfrak{P}$; so that $U[\{x\}] \cap \{x\} \neq \phi$; so that $(x, x) \in U$. Hence $U \supseteq \Delta$.

B_2: Suppose not true. Then there exists A in $P(X)$ and U and V in $\mathfrak{U}$ such that for every W in $\mathfrak{U}$ there exists $x \in W[A]$ such that $x \notin U[A] \cap V[A]$. For each W we define $M_W = \{x \mid x \in W[A]$ and $x \notin U[A] \cap V[A]\}$. Let M be the union of all M_W. Suppose there exists U_1 such that $U_1[A] \cap M = \phi$. Then $U_1[A] \subseteq U[A] \cap V[A]$; but by assumption this is not possible. Hence (by definition) $(A, M) \in \mathfrak{P}$. Put $M_{W_1} = \cup \{x \mid x \in W[A]$ and $x \notin U[A]\}$, $M_{W_2} = \cup \{x \mid x \in W[A]$ and $x \notin V[A]\}$, $M_1 = \cup M_{W_1}$ and $M_2 = \cup M_{W_2}$. Then $M = M_1 \cup M_2$ (possible that $M_1 \cap M_2 \neq \phi$) and $U[A] \cap M_1 = \phi$ and $V[A] \cap M_2 = \phi$; so that $(A, M) \notin \mathfrak{P}$ which is a contradiction.

B_3: L_5 is equivalent to the statement: $(A, -B) \notin \mathfrak{P}$ and $(A, C) \in \mathfrak{P}$ implies there exists x in C such that $(x, -B) \notin \mathfrak{P}$. $U[A] \cap (-U[A]) = \phi$ implies $(A, -U[A]) \notin \mathfrak{P}$. But $V[A] \cap B \neq \phi$ for all V in $\mathfrak{U}$ implies that $(A, B) \in \mathfrak{P}$; so that (by L_5) there exists $x \in B$ such that $(x, \sim U[A]) \notin \mathfrak{P}$ implies there exists $W \in \mathfrak{U}$ such that $W[x] \cap (\sim U[A]) = \phi$; so that $W[x] \subseteq U[A]$.

Definition 2. A non-void subset $\mathfrak{U}$ o $P(X\times X)$ is a *symmetric generalized uniformity on* X iff the following axioms are satisfied;

M_1: For every U in $\mathfrak{U}$ $U \supseteq \Delta$;

M_2: If U in $\mathfrak{U}$, then $U=U^{-1}$;

M_3: For every A in $P(X)$ and U, V in $\mathfrak{U}$ there exists a W in $\mathfrak{U}$ such that $W[A] \subseteq U[A] \cap V[A]$;

M_4: For every A, B in $P(X)$ and U in $\mathfrak{U}$: $V[A] \cap B \neq \phi$ for all V in $\mathfrak{U}$ implies there exists an x in B and there exists a W in $\mathfrak{U}$ such that $W[x] \subseteq U[A]$;

M_5: If U in $\mathfrak{U}$ and $U \subseteq V$ (symmetric) $\subseteq (X\times X)$ then V is in $\mathfrak{U}$.

Clearly (by theorem 1) corresponding to any symmetric generalized uniform space $(X, \mathfrak{U})$ there exists a symmetric generalized proximity space $(X, \mathfrak{P}(\mathfrak{U}))$ where $(A, B) \in \mathfrak{P}(\mathfrak{U})$ iff $U[A] \cap B \neq \phi$ for all U in $\mathfrak{U}$.

Theorem 2. *Let* $(X, \mathfrak{U})$ *be a symmetric generalized uniform space. The function* $g: P(X)$ *into* $P(X)$ *defined by* $x \in g(A)$ *iff* $U[x] \cap A \neq \phi$ *for all* U *in* $\mathfrak{U}$ *is a Kuratowski closure function.*

The proof is straightforward.

Definition 3. The topology induced on X by the Kuratowski closure function g in theorem 2 is called the uniform topology on X induced by $\mathfrak{U}$ (notation: $\mathfrak{T}(\mathfrak{U})$).

Theorem. 3. *Let* $(X, \mathfrak{U})$ *be a symmetric generalized uniform space. Then* $\mathfrak{T}(\mathfrak{U}) = \mathfrak{T}(\mathfrak{P}(\mathfrak{U}))$.

Proof. $U[x] \cap A \neq \phi$ for every U in $\mathfrak{U}$ iff $(\{x\}, A) \in \mathfrak{P}(\mathfrak{U})$ (by definition)

Theorem 4. *Let* $(X, \mathfrak{U})$ *be a symmetric generalized uniform space. Then*

1. A *is in* $\mathfrak{T}(\mathfrak{U})$ *iff for every* x *in* A *there exists* U *in* $\mathfrak{U}$ *such that* $U[x] \subseteq A$;
2. *For cvery* A *in* $P(X)$ *we have that* $A^\circ = \{x \mid U\{x\} \subseteq A$ *for some* U *in* $\mathfrak{U}\}$;
3. *For every* A *in* $P(X)$ *we have that* $\bar{A} = \cap \{U[A] \mid U \in \mathfrak{U}\}$;
4. $\mathfrak{T}(\mathfrak{U})$ *is* T_0 *iff* $\cap \{U \mid U \in \mathfrak{U}\} = \Delta$;
5. $(X, \mathfrak{U})$ *has a closed base implies* $\mathfrak{T}(\mathfrak{U})$ *is regular.*

The proof is straightforward.

Definition. 4. $\mathfrak{B}$ is a base for some symmetric generalized uniformity on X iff

1. V in $\mathfrak{B}$ implies $V=V^{-1}$
2. $\mathfrak{U} = \{U \mid U=U^{-1}$ and $U \supseteq V$ for some V in $\mathfrak{P}\}$ is a symmetric generalized uniformity on X.

Theorem 5. $\mathfrak{B}$ *is a base for some symmetric generalized uniformity on* X *iff* $\mathfrak{B}$ *satisfies* M_1, M_2, M_3 *and* M_4.

The proof is straightforward.

Theorem 6. *A topology $\mathfrak{T}$ on X is the uniform topology for some symmetric generaliezed uniformity on X iff $\mathfrak{T}$ is symmetric (i. e. $x\in\bar{y}$ implies $y\in\bar{x}$ for all x, y in X).*

Proof. Suppose $\mathfrak{T}=\mathfrak{T}(\mathfrak{U})$ where $\mathfrak{U}$ is some symmetric generalized uniformity on X. Let $x\in\bar{y}$. Then $U[x]\cap\{y\}\neq\phi$ for all U in $\mathfrak{U}$. But since U is symmetric, we have that $U[y]\cap\{x\}\neq\phi$ for all U in $\mathfrak{U}$; so that $y\in\bar{x}$.

By theorem 2.3 in [1] page 418 and by theorem 3 to prove the converse it is sufficient to prove the following

Lemma *For every symmetric generalized proximity space, $(X, \mathfrak{P})$ there exists a symmetric generalized uniform space, $(X, \mathfrak{U}(\mathfrak{P})_1)$, such that $\mathfrak{P}(\mathfrak{U}(\mathfrak{P})_1)=\mathfrak{P}$.*

Proof. Let X be a set with power set $P(X)$. For every A, B in $P(X)$ let $U_{A,B}$ equal $(X\times X)-((A\times B)\cup(B\times A))$. Let $\mathfrak{B}=\{U_{A,B}\mid(A, B)\notin\mathfrak{P}\}$. Clearly, $\mathfrak{B}$ satisfies M_2. Suppose $A\bar{\mathfrak{P}}B$. Then $U_{A,B}[A]\cap B=\phi$. Conversely suppose there exists C, D such that $C\bar{\mathfrak{P}}D$ and $U_{C,D}[A]\cap B=\phi$. Then it is easily shown that $(A\subseteq C$ and $B\subseteq D)$ or $(A\subseteq D$ and $B\subseteq C)$. Hence $A\bar{\mathfrak{P}}B$. So that by theorem 1 we have that $\mathfrak{B}$ satisfies M_1, M_3, and M_4. Let $\mathfrak{U}(\mathfrak{P})_1=\{U\mid U=U^{-1}$ and $U\supseteq V$ for some V in $\mathfrak{B}\}$. $\mathfrak{U}(\mathfrak{P})_1$ (by theorem 5) is a symmetric generalized uniformity on X. It is easy to show that $\mathfrak{P}(\mathfrak{U}(\mathfrak{P})_1)=\mathfrak{P}$. (c. f. [5] page 194).

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