ON SELF-RECIPROCAL FUNCTIONS RELATING TO GENERALISED HANKEL TRANSFORM $\chi_{v,k,m}$

By

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1. **Introduction**: A generalisation of the Hankel Transform, namely,

$$g(x)=\int_0^\infty \sqrt{xy}\, J_\nu(xy)\, f(y)\, dy \tag{1.1}$$

has been introduced by *Roop Narain* (1) in the form

$$g(x)=\left(\frac{1}{2}\right)^{\nu}\int_0^\infty (\chi y)^{\nu+\frac{1}{2}}\chi_{\nu,k,\mu}\left(\frac{x^2 y^2}{4}\right)f(y)\, dy \tag{1.2}$$

where

$$\chi_{\nu,k,\mu}(x)=\frac{\Gamma(2\mu)\,\Gamma(\frac{3}{2}+\nu+\mu-k)}{\Gamma(\frac{1}{2}-k+\mu)\,\Gamma(1+\nu+\mu\pm\mu)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}+k-\mu,\ \frac{3}{2}+\nu+\mu-k\,; \\ 1-2\mu,\ 1+\nu+\mu\pm\mu\,;\end{matrix}\ -x\right]$$

$$+\frac{\Gamma(-2\mu)\,\Gamma(\frac{3}{2}+\nu+3\mu)}{\Gamma(\frac{1}{2}-k-\mu)\,\Gamma(1+\nu+3\mu\pm\mu)}\ {}_2F_3\left[\begin{matrix}\frac{1}{2}+k-\mu,\ \frac{3}{2}+\nu+3\mu-k\,; \\ 1+2\mu,\ 1+\nu+3\mu\pm\mu\,;\end{matrix}\ -x\right]$$

$$=x^{-\nu}\, G^{21}_{24}\left(x\left|\begin{matrix}k-\mu-\frac{1}{2},\ \frac{1}{2}+\nu-k+\mu \\ \nu,\quad \nu+2\mu,\quad -2\mu,\quad 0\end{matrix}\right.\right) \tag{1.3}$$

provided $R(\nu+4\mu+1)>0$, and $2m$ is not zero or an integer and the integral (1.2) converges absolutely, In particular, when $k+m=\frac{1}{2}$; (1.2) yields the well-known Hankel Transform (1.1). The reciprocal relation of (1.2) is of the form

$$f(x)=\left(\frac{1}{2}\right)^{\nu}\int_0^\infty (xy)^{\nu+\frac{1}{2}}\chi_{\nu,k,m}\left(\frac{x^2 y^2}{4}\right)g(y)\, dy \tag{1.4}$$

provided the generalised Hankel Transform of $|f(x)|$ and $|g(x)|$ exist; $R(\nu+1+\mu\pm\mu)>0$ and $2m$ is not an integer or zero.

The function $g(x)$ given by (1.2) is called the $\chi_{\nu,k,m}$ transform of $f(x)$.

In particular, if $f(x)=g(x)$, so that $f(x)$ is its own $\chi_{\nu,k,\mu}$ transform, then $f(x)$ is said to be self-reciprocal function in $\chi_{\nu,k,\mu}$ transform and is denoted by $R_{\nu\,(k,\mu)}$ while function selfreciprocal in Hankel transform (1.1) is denoted by R_ν.

The object of the present paper is to investigate some theorems connected with the above generalisation and to use them to evaluate certain integrals and to establish certain relations.

2. Thorem 1: *Let*

(i) $\phi(p) \doteqdot f(x)$

(ii) $t^{\nu-\lambda-\frac{1}{2}} f(t)$ be $R_{\nu\,(k,\mu)}$,

then

$$x^{2\nu-\lambda} f(x) \doteqdot \frac{2^{\nu-\lambda-1}\Gamma\left(\nu+\frac{3}{2}\right)\Gamma(\mu-k)\,\Gamma\left(\nu+2\mu+\frac{3}{2}\right)p^2}{\Gamma\left(\lambda+\frac{3}{2}\right)\Gamma\left(\frac{\lambda}{2}+2\right)\Gamma(\nu+\mu-k+2\,\Gamma\left(2\mu-\frac{1}{2}\right)}$$

$$x\int_0^\infty t^{\lambda+1}\phi(t)\ {}_3F_4\left(\begin{matrix}\nu+\frac{3}{2},\nu+2\mu+\frac{3}{2},\frac{3}{2}-2\mu\,; \\ \frac{\lambda+3}{2},\frac{\lambda}{2}+2,\nu+\mu-k+2,\mu-k-1\,;\end{matrix}\ \frac{p^2t^2}{4}\right)dt \tag{2.1}$$

provided $f(x), x^{\nu-\lambda-\frac{1}{2}}$ *and* $x^{2\nu-\lambda} f(x)$ *are continuous aud absolutely integrable in* $(0,\infty)$, *and* $R(\nu)\geq-\frac{1}{2}$, $R(\lambda+2)>0$, *and* 2μ *is not an integer or zero.*

Proof: Let $p^{m+\nu+\frac{1}{2}}\chi_{\nu,k,\mu}\left(\frac{p^2}{4}\right)\doteqdot F(x)$,

then

$$\chi^\lambda F\left(\frac{1}{x}\right)\doteqdot p^{\frac{1-\lambda}{2}}\int_0^\infty x^{\frac{\lambda-1}{2}+m} J_{\lambda+1}(2\sqrt{xp})\,x^{\nu+\frac{1}{2}}\chi_{\nu,k,\mu}\left(\frac{x^2}{4}\right)dx \tag{2.2}$$

$$R(\lambda+2)>0,\ R(\lambda+m+\nu)>-\frac{3}{2},\ R(\lambda+m+\mu\pm\mu)>-\frac{3}{2}.$$

or, $$x^\lambda F\left(\frac{1}{x}\right)\doteqdot\frac{2^\nu p^{1-m-\lambda}}{2\pi i}\int_{c+i\infty}^{c+i\infty}\frac{\Gamma(\lambda+m+s)}{\Gamma(-m-s+2}\,2^{\frac{1}{2}-s}\,\frac{\Gamma\left(\frac{\nu-s}{2}+\frac{3}{4}\right)\Gamma\left(\frac{\nu-s}{2}+2\mu+\frac{3}{4}\right)}{\Gamma\left(\frac{\nu+s}{2}+\frac{1}{4}\right)\Gamma\left(\frac{\nu-s}{2}+\mu-k+\frac{5}{4}\right)}$$

$$\times\frac{\Gamma\left(\frac{\nu+s}{2}+\mu-k+\frac{3}{4}\right)}{\Gamma\left(\frac{\nu+s}{2}+2\mu+\frac{1}{4}\right)}p^{-s}\,ds \tag{2.3}$$

or

$$x^{\lambda} F\left(\frac{1}{x}\right) \doteqdot \frac{2^{\lambda+2n+\nu-\frac{3}{2}}}{2\pi i\, p^{\lambda+m-1}} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\left(\frac{\lambda+m+s}{2}\right)\Gamma\left(\frac{\lambda+m+s+1}{2}\right)\Gamma\left(\frac{\nu-s}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{2-m-s}{2}\right)\Gamma\left(\frac{3-m-s}{2}\right)\Gamma\left(\frac{\nu+s}{2}+\frac{1}{4}\right)}$$

$$\times \frac{\Gamma\left(\frac{\nu-s}{2}+2\mu+\frac{3}{4}\right)\Gamma\left(\frac{\nu+s}{2}+\mu-k+\frac{3}{4}\right)}{\Gamma\left(\frac{\nu-s}{2}+\mu-k+\frac{5}{4}\right)\Gamma\left(\frac{\nu+s}{2}+2\mu+\frac{1}{4}\right)} 2^{s} p^{-s}\, ds \tag{2.4}$$

By putting $m=\nu-\lambda+\frac{1}{2}$ and then evaluating the integral, we get (poles are given by $s=-2n-\nu-\frac{3}{2}$, n is$+ve$ integer)

or,

$$x^{\lambda} F\left(\frac{1}{x}\right) \doteqdot p^{2}\, 2^{2\nu-\lambda-1} \sum_{n=0}^{\infty} \frac{(-1)^{n}}{n\,!} \frac{\Gamma\left(n+\nu+\frac{3}{2}\right)\Gamma\left(n+\nu+2\mu+\frac{3}{2}\right)}{\Gamma\left(n+\frac{\lambda}{2}+\frac{3}{2}\right)\Gamma\left(\frac{\lambda}{2}+n+2\right)}$$

$$\times \frac{\Gamma(\mu-k-n)}{\Gamma(n+\nu+\mu-k+2)\,\Gamma\left(2\mu-n-\frac{1}{2}\right)} \times \frac{p^{2n}}{2^{2n}} \tag{2.5}$$

or

$$\left(\frac{x}{t}\right)^{\lambda} F\left(\frac{t}{x}\right) \doteqdot 2^{2\nu-\lambda-1} \frac{\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\nu+2\mu+\frac{3}{2}\right)\Gamma(\mu-k)\, p^{2} t^{2}}{\Gamma\left(\frac{\lambda+3}{2}\right)\Gamma\left(\frac{\lambda}{2}+2\right)\Gamma(\nu+\mu-k+2)\,\Gamma\left(2\mu-\frac{1}{2}\right)}$$

$$\times\, {}_{3}F_{4}\left(\begin{matrix}\nu+\frac{3}{2},\ \nu+2\mu+\frac{3}{2},\ \frac{3}{2}-2\mu \\ \frac{\lambda+3}{2},\ \frac{\lambda}{2}+2,\ \nu+\mu-k+2,\ \mu-k-1\ ;\end{matrix}\ \frac{p^{2} t^{2}}{4}\right) \tag{A}$$

Let (i) $p^{2\nu-\lambda+1} \chi_{\nu,k,\mu}\left(\frac{p^{2}}{4}\right) \doteqdot F(x)$ (B)

(ii) $\psi(p) \doteqdot f(x)$ (C)

Also from (B) $(ap)^{2\nu-\lambda+1} \chi_{\nu,k,\nu}\left(\frac{p^{2} a^{2}}{4}\right) \doteqdot F\left(\frac{x}{a}\right)$ (D)

Applying *Goldstein's* theorem to (C) and (D), we get

$$\int_{0}^{\infty} f(t)\, (at)^{2\nu-\lambda+1} \chi_{\nu,k,\mu}\left(\frac{a^{2} t^{2}}{4}\right) \frac{dt}{t} = \int_{0}^{\infty} \psi(t)\, F\left(\frac{t}{a}\right) \frac{dt}{t}$$

On writing x for a and multiplying both sides by x^{λ}, it follows that

$$x^{\lambda}\int_0^{\infty} f(t)\,(xt)^{2\nu-\lambda+1}\,\chi_{\nu,k,\mu}\left(\frac{x^2t^2}{4}\right)\frac{dt}{t}=\int_0^{\infty}\phi(t)\,F\left(\frac{t}{x}\right)\left(\frac{x}{t}\right)^{\lambda} t^{\lambda-1}\,dt$$

Interpreting with the help of (A), we get

$$x^{2\nu+1}\int_0^{\infty} f(t)\,t^{2\nu-\lambda}\,\chi_{\nu,k,\mu}\left(\frac{x^2t^2}{4}\right)dt \doteqdot \frac{2^{2\nu-\lambda-1}\,\Gamma\left(\nu+\frac{3}{2}\right)\Gamma(\mu-k)\,\Gamma\left(\nu+2\mu+\frac{3}{2}\right)}{\Gamma\left(\lambda+\frac{3}{2}\right)\Gamma\left(\frac{\lambda}{2}+2\right)\Gamma(\nu+\mu-k+2)}$$

$$\times\frac{p^2}{\Gamma\left(2\mu-\frac{1}{2}\right)}\int_0^{\infty} t^{\lambda+1}\,\phi(t)\;{}_3F_4\left(\begin{matrix}\nu+\frac{3}{2},\ \nu+2\mu+\frac{3}{2},\ \frac{3}{2}-2\mu\,;\\ \frac{\lambda+3}{2},\ \frac{\lambda}{2}+2,\ \nu+\mu-k+2,\ \mu-k-1\,;\end{matrix}\ -\frac{p^2t^2}{4}\right)dt \quad (2.6)$$

If $t^{\nu-\frac{1}{2}-\lambda}f(t)$ is $R_{\nu}(k,\mu)$, we obtain the required result.

3. In particular, if we take $k=\frac{1}{2}-\mu$, we get the following corollary.

Cor: Let (i) $\phi(p)\doteqdot f(x)$

(ii) $t^{\nu-\lambda-1}f(t)$ *be* R_{ν},

then

$$x^{2\nu-\lambda}f(x)\doteqdot\frac{2^{\nu-\lambda-1}\,\Gamma\left(\nu+\frac{3}{2}\right)p^2}{\Gamma\left(\lambda+\frac{3}{2}\right)\Gamma\left(\frac{\lambda}{2}+2\right)}\int_0^{\infty} t^{\lambda+1}\,\phi(t)\;{}_1F_2\left(\begin{matrix}\nu+\frac{3}{2}\,;\\ \frac{\lambda+3}{2},\ \frac{\lambda}{2}+2\,;\end{matrix}\ -\frac{p^2t^2}{4}\right)dt \quad (3.1)$$

provided $f(x), x^{\nu-\frac{1}{2}}f(x)$ are continuous and absolutely integrable in $(0,\infty)$; $R(\nu)\geq-\frac{1}{2}$, $R(\lambda)>-\frac{3}{2}$.

4. Theorem 2: *Let* (i) $f(x)\doteqdot g(p)$

(ii) $x^{\nu-\frac{1}{2}}g(x)$ *be* $R_{\nu}(k,\mu)$

then,

$$p^{2\nu-\lambda+1}g(p)\doteqdot 2^{\nu-\lambda}\,\frac{\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\nu+2\mu+\frac{3}{2}\right)\Gamma(\mu-k)\,t^{\lambda+1}}{\Gamma\left(\frac{\lambda}{2}+1\right)\Gamma\left(\frac{\lambda+3}{2}\right)\Gamma(\nu+\mu-k+2)\,\Gamma\left(2\mu-\frac{1}{2}\right)}$$

$$\times\int_0^{\infty} xf(x)\;{}_3F_4\left(\begin{matrix}\nu+\frac{3}{2},\ \nu+2\mu+\frac{3}{2},\ \frac{3}{2}-2\mu\,;\\ \frac{\lambda}{2}+1,\ \frac{\lambda+3}{2},\ \nu+\mu-k+2,\ \mu-k-1\,;\end{matrix}\ \frac{x^2t^2}{4}\right)dx \quad (4.1)$$

Provided $f(x)$ *and* $x^{\nu-\frac{1}{2}}g(x)$ *are continuous and absolutely integrable in* $(0,\infty)$; $R(\nu)\geq-\frac{1}{2}$ *and* 2μ *is not an integer or zero*; $R(\lambda)\geqslant 0$.

Proof: Let $x^{m+\nu+\frac{1}{2}}\chi_{\nu,k,\mu}\left(\frac{x^2}{4}\right)\doteqdot\Phi(p)$;

$R\left(\nu+\frac{1}{2}\right)>0$, $R(\nu+1+\mu\pm2\mu)>0$, $R\left(m+\nu+\frac{3}{2}\right)>0$, and then

$$p^{1-\lambda}\Phi\left(\frac{1}{p}\right)\doteqdot\frac{t^{\lambda-m}2^{\nu}}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s+m)}{\Gamma(\lambda-s-m+1)}\times2^{\frac{1}{2}-s}\frac{\Gamma\left(\frac{\nu}{2}-\frac{s}{2}+\frac{3}{4}\right)\Gamma\left(\frac{\nu-s}{2}+2\mu+\frac{3}{4}\right)}{\Gamma\left(\frac{\nu+s}{2}+\frac{1}{4}\right)\Gamma\left(\frac{\nu-s}{2}+\mu-k+\frac{5}{4}\right)}$$

$$\times\frac{\Gamma\left(\frac{\nu}{2}+\frac{s}{2}+\mu-k+\frac{3}{4}\right)}{\Gamma\left(\frac{\nu}{2}+\frac{s}{2}+2\mu+\frac{1}{4}\right)}t^{-s}\,ds \tag{4.2}$$

or,

$$p^{1-\lambda}\Phi\left(\frac{1}{p}\right)\doteqdot\frac{2^{2m-\lambda+\nu+\frac{1}{2}}t^{\lambda-m}}{2i\pi}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma\left(\frac{s+m}{2}\right)\Gamma\left(\frac{s+m+1}{2}\right)\Gamma\left(\frac{\nu-s}{2}+\frac{3}{4}\right)}{\Gamma\left(\frac{\lambda-s-m+1}{2}\right)\Gamma\left(\frac{\lambda-s-m+2}{2}\right)\Gamma\left(\frac{\nu+s}{2}+\frac{1}{4}\right)}$$

$$\times\frac{\Gamma\left(\frac{\nu-s}{2}+2\mu+\frac{3}{4}\right)\Gamma\left(\frac{\nu+s}{2}+\mu-k+\frac{3}{4}\right)}{\Gamma\left(\frac{\nu+s}{2}+2\mu+\frac{1}{4}\right)\Gamma\left(\frac{\nu-s}{2}+\mu-k+\frac{5}{4}\right)}2^{s}t^{-s}\,ds \tag{4.3}$$

Now on putting $m=\nu+\frac{1}{2}$ and then evaluating the integral, we get

$$p^{1-\lambda}\Phi\left(\frac{1}{p}\right)\doteqdot\frac{2^{2\nu-\lambda}t^{\lambda+1}}{2\pi i}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{\Gamma\left(\nu+n+\frac{3}{2}\right)\Gamma\left(\nu+n+2\mu+\frac{3}{2}\right)\Gamma(\mu-n-k)}{\Gamma\left(\frac{\lambda}{2}+n+1\right)\Gamma\left(\frac{\lambda}{2}+n+\frac{3}{2}\right)\Gamma(\nu+n+\mu-k+2)}$$

$$\times\frac{\left(\frac{t}{2}\right)^{2n}}{\Gamma\left(2\mu-n-\frac{1}{2}\right)} \tag{4.4}$$

or

$$p^{1-\lambda}\Phi\left(\frac{1}{p}\right)\doteqdot2^{2\nu-\lambda}\,t^{\lambda+1}\frac{\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\nu+2\mu+\frac{3}{2}\right)\Gamma(\mu-k)}{\Gamma\left(\frac{\lambda}{2}+1\right)\Gamma\left(\frac{\lambda+3}{2}\right)\Gamma(\nu+\mu-k+2)\Gamma\left(2\mu-\frac{1}{2}\right)}$$

$$\times{}_3F_4\left(\begin{matrix}\nu+\frac{3}{2},\nu+2\mu+\frac{3}{2},\frac{3}{2}-2\mu\,;\\ \frac{\lambda}{2}+1,\frac{\lambda+3}{2},\nu+\mu-k+2,\mu-k-1\,;\end{matrix}\ \frac{t^2}{4}\right) \tag{A}$$

Let (i) $f(x)\doteqdot g(p)$ (B)

and (ii) $(ax)^{2\nu+1}\,\chi_{\nu,k,\mu}\left(\frac{x^2a^2}{4}\right) \doteqdot \Phi\left(\frac{p}{a}\right)$ (C)

Applying *Goldstein's* theorem to (C) and (B), we get

$$\int_0^\infty f(x)\,\Phi\left(\frac{x}{a}\right)\frac{dx}{x}=\int_0^\infty g(x)\,(ax)^{2\nu+1}\,\chi_{\nu,k,\mu}\left(\frac{a^2x^2}{4}\right)\frac{dx}{x}$$

On writing p for a and then multiplying both sides by $p^{1-\lambda}$, we get

$$\int_0^\infty f(x)\,\Phi\left(\frac{x}{p}\right)\left(\frac{p}{x}\right)^{1-\lambda}\frac{dx}{x^\lambda}=p^{\nu-\lambda+\frac{3}{2}}\int_0^\infty x^{\nu-\frac{1}{2}}\,g(x)\,(px)^{\nu+\frac{1}{2}}\,\chi_{\nu,k,\mu}\left(\frac{p^2x^2}{4}\right)dx$$

Interpreting with the help of (A), we obtain

$$p^{\nu-\lambda+\frac{3}{2}}\int_0^\infty x^{\nu-\frac{1}{2}}\,g(x)\,(px)^{\nu+\frac{1}{2}}\,2^{-\nu}\,\chi_{\nu,k,\mu}\left(\frac{x^2p^2}{4}\right)dx$$

$$\doteqdot 2^{\nu-\lambda}\frac{\Gamma\left(\nu+\frac{3}{2}\right)\Gamma\left(\nu+2\mu+\frac{3}{2}\right)\Gamma(\mu-k)\,t^{\lambda+1}}{\Gamma\left(\frac{\lambda}{2}+1\right)\Gamma\left(\frac{\lambda+3}{2}\right)\Gamma(\nu+\mu-k+2)\,\Gamma\left(2\mu-\frac{1}{2}\right)}$$

$$\int_0^\infty x f(x)\,{}_3F_4\left(\begin{matrix}\nu+\frac{3}{2},\,\nu+2\mu+\frac{3}{2},\,\frac{3}{2}-2\mu\,;\\ \frac{\lambda}{2}+1,\,\frac{\lambda+3}{2},\,\nu+\mu-k+2,\,\mu-k-1\,;\end{matrix}\;\frac{x^2t^2}{4}\right)dx \tag{4.5}$$

If $x^{\nu-\frac{1}{2}}g(x)$ is $R_\nu(k,\mu)$, we obtain the required result.

5. Corollary: *If we put* $k+\mu=\frac{1}{2}$ *in the above relation, then we obtain the following:*

Let (i) $f(x)\doteqdot g(p)$

(ii) $x^{\nu-\frac{1}{2}}g(x)$ *be* R_ν;

then

$$p^{2\nu-\lambda+1}g(p)\doteqdot\frac{2^{\nu-\lambda}\,\Gamma\left(\nu+\frac{3}{2}\right)t^{\lambda+1}}{\Gamma\left(\frac{\lambda}{2}+1\right)\Gamma\left(\frac{\lambda+3}{3}\right)}\int_0^\infty x f(x)\,{}_1F_2\left(\begin{matrix}\nu+\frac{3}{2}\,;\\ \frac{\lambda}{2}+1,\,\frac{\lambda+3}{2}\,;\end{matrix}\;-\frac{x^2t^2}{4}\right)dx \tag{5.1}$$

Provided $f(x)$ and $x^{\nu-\frac{1}{2}}g(x)$ are *continuous and absolutely integrable in* $(0,\infty)$, $R(\lambda)\geq 0$.

6. Applications:

(a) Let $x^{\nu+\frac{1}{2}}\,e^{\frac{x^2}{4}}D_{-2\nu-3}(x)$ be R_ν

$\therefore\;\; g(x)=x\,e^{\frac{x^2}{4}}D_{-2\nu-3}(x)$

and $g(p)=p\, e^{\frac{p^2}{4}} D_{-2\nu-3}(p) \doteqdot \dfrac{1}{\Gamma(2\nu+3)}\, e^{-\frac{t^2}{2}} t^{2\nu+2} \equiv f(t)$ Hence from (5.1), we get

$$p^{2\nu-\lambda+2} e^{\frac{p^2}{4}} D_{-2\nu-3}(p) \doteqdot \frac{2^{2\nu-\lambda+1}\Gamma(\nu+2)\Gamma\left(\nu+\frac{3}{2}\right)}{\Gamma(2\nu+3)\Gamma\left(\frac{\lambda}{2}+1\right)\Gamma\left(\frac{\lambda+3}{2}\right)}$$

$$\times {}_2F_2\left(\begin{matrix}\nu+\frac{3}{2}, \nu+2\,; \\ \frac{\lambda}{2}+1, \frac{\lambda+3}{2}\,;\end{matrix}\; -\frac{t^2}{2}\right),$$

$R(\lambda)>-2.$

(b) Let $x^{\nu-2m-\frac{1}{2}} e^{\frac{x^2}{4}} W_{3m-\nu-\frac{1}{2}, m}\left(\frac{x^2}{2}\right)$ *be* R_ν

$$\therefore \quad g(p)=p^{-2m} e^{\frac{p^2}{4}} W_{3m-\nu-\frac{1}{2}, m}\left(\frac{p^2}{2}\right)$$

$$\doteqdot \frac{2^{\nu-3m+\frac{1}{2}} t^{2\nu-4m+1}}{\Gamma(2\nu-4m+2)}\, {}_1F_1\left(\nu-4m+1\,; \nu-2m+\frac{3}{2}\,; -\frac{t^2}{2}\right) \equiv f(t)$$

Hence from (5.1), we get

$$\int_0^\infty x^{2\nu-4m+2}\, {}_1F_1\left(\nu-4m+1\,; \nu-2m+\frac{3}{2}\,; -\frac{x^2}{2}\right) {}_1F_2\left(\begin{matrix}\nu+\frac{3}{2}\,; \\ \frac{\lambda}{2}+1, \frac{\lambda+3}{2}\,;\end{matrix}\; -\frac{x^2 t^2}{4}\right) dx$$

$$= \frac{2^{\lambda-\nu}\Gamma\left(\frac{\lambda}{2}+1\right)\Gamma\left(\frac{\lambda+3}{2}\right)\Gamma(2\nu-4m+2)}{\Gamma\left(\nu+\frac{3}{2}\right)\Gamma(\lambda-4m+1)\, t^{4m+1}}\, {}_2F_2\left(\begin{matrix}\nu-2m+1, \nu-4m+1\,; \\ \frac{\lambda}{2}-2m+\frac{1}{2}, \frac{\lambda}{2}-2m+1\,;\end{matrix}\; -\frac{t^2}{2}\right);$$

$R(\nu-2m)>-1, R(\lambda-4m-\nu) > \frac{1}{2}.$

(c) Let $x^{\nu+\frac{1}{2}}(x^2+1)^{-\frac{\nu+1}{4}} K_{\frac{\nu+1}{2}}(\sqrt{x^2+1})$ *be* R_ν

$$\therefore \quad g(p)=\frac{P}{(p^2+1)^{\frac{\nu+1}{4}}} K_{\frac{\nu+1}{2}}(\sqrt{p^2+1}) \doteqdot \begin{cases} 0, \text{ if } 0<t<1 \\ \sqrt{\frac{\pi}{2}}\,(t^2-1)^{\frac{\nu}{4}} J_{\frac{\nu}{2}}(\sqrt{t^2-1}), \text{ if } t>1. \end{cases}$$

Hence from (5.1), we get

$$\frac{p^{2\nu-\lambda+2}}{(p^2+1)^{\frac{\nu+1}{4}}} K_{\frac{\nu+1}{2}}(\sqrt{p^2+1}) \doteqdot \frac{2^{\nu-\lambda-\frac{1}{2}}\sqrt{\pi}\,\Gamma\left(\nu+\frac{3}{2}\right) t^{\lambda+1}}{\Gamma\left(\frac{\lambda}{2}+1\right)\Gamma\left(\frac{\lambda+3}{2}\right)}$$

$$\times \int_0^\infty x(x^2-1)^{\frac{\nu}{4}} J_{\frac{\nu}{2}}(\sqrt{x^2-1})\, {}_1F_2\begin{pmatrix} \nu+\frac{3}{2}; & \\ & -\frac{x^2 t^2}{4} \\ \frac{\lambda}{2}+1;\ \frac{\lambda+3}{2}; & \end{pmatrix} dx$$

$$R(\nu)>-1,\ R(\lambda)>-2,\ R\left(\lambda-\frac{3\nu}{2}\right)>1.$$

In particular, if $\lambda=2\nu+1$, we get

$$\int_1^\infty \frac{(x^2-1)^{\frac{\nu}{4}}}{x^{\nu+\frac{1}{2}}} \sqrt{xt}\ J_{\nu+1}(xt)\, J_{\frac{\nu}{2}}(\sqrt{x_2-1})\, dx$$

$$=\frac{(t^2-1)^{\frac{\nu}{4}}}{t^{\nu+\frac{1}{2}}} J_{\frac{\nu}{2}}(\sqrt{t^2-1}),\ t>1,\ R(\nu)>-2.$$

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