PROPOSITIONAL CALCULI AND COMPLETENESS THEOREM [1)]

By

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In my paper [3] I gave a simple and whole proof of the completeness of the first-order functional calculus.

J. Slupecki gave the thought to generalize the proof method for many valued calculi.

In [6], [7], [8] I presented different generalizations of the satisfiability definition and generalizations of completeness theorem with generalizations of Herbrand's theorem but proved by means of the usual completeness theorem and therefore proved on a semantic way.

In this paper I generalize the method given in [3] and I obtain in such way simultaneous generalizations of Gödel's completeness theorem with Skolem-Löwenheim's theorem which include also Herbrand's theorem according to the above and in a syntactic way,

They are proved also completeness of infinite many Boolean important calculus with (finite) truncated introduction of general and existential quantifiers which approximate the first-order functional calculus.

This paper we can divide in two parts: the first part is analogical to [6] and in the second one it is generalized the proof method of [3] with generalizations of the above theorem.

We use notations of [4]-[16] and in particular:

(01) variables: (1′) free: $x_1, \cdots$ (simply x),
(2′) apparent: $a_1, \cdots$ (simply a),

(02) relations signs: $f_1^1, \cdots, f_q^1, \cdots, f_1^t, \cdots, f_q^t$,

(03) logical constants: $'$, $+$, Π ,

(04) $w(E), p(E)$—the number of different free, apparent, variables respectively which occur in the expression E, [2)]

(05) $\{K_m\}$—the sequence $K_1, \cdots, K_m$; $\{K_q^t\}$—the sequence $K_1^1, \cdots, K_q^1, \cdots, K_1^t, \cdots, K_q^t$,

(06) $\{i_{w(E)}\}, \{j_{w(E)}\}$—indices of all free variables occurring in E,

1) The paper is connected with my lectures on J. Slupecki's seminar in 1951-1957 years and on meetings of Polish Mathematical Society at Wroclaw and was written several years ago; results without proofs are published at [12].

2) An expression in which an apparent variable belongs to the scope of two quantifiers Πa is not a formula; if a does not occur in E, then $\Pi a E$ is not a formula.

(07) $i(E)=\max\{i_{w(E)}\}$, $n(E)=\max\{i(E), w(E)+p(E)\}$,

(08) $E(u/z)$—the expression resulting from E by substitution of u for each z in E with known conditions, $E(\{i_j\}/\{t_j\})=E(x_{i_1}/y_{t_1})\cdots(x_{i_j}/y_{t_j})$,

(09) $C\{E\}$—the set of all significant parts of the formula E.

(010) $M, M_1, \cdots$—models; $Q, Q_1, \cdots$ —non-empty sets of models of the same power (for finite models it is used also the word "rank" instead of the word; power): $Q(k)-Q$ is a set of models of the power k.

(011) $A, A_1, \cdots$— sets of indecomposable formulas, i.e. atomic formulas with their negation, in which indices of individual variables are $\leqslant k$, where k is a given numbber; the sets may be infinite; S_A—the set of all free variables which occur in elements of A, therefore S_E—the set of all free variables occurring in the expression E; if all the elements of S_A are all free variables with indices $\leqslant k$ and for each indecomposable formula E, if $S_E \subset S_A$, then $E \epsilon A$ iff $E' \bar{\epsilon} A$, then A is called "set of the power k"; $B, B_1, \cdots$—families of sets A; if elements of $B's$ are only sets of the power k, then B is called: family of the power k; for brevity we shall assume that we only consider $A's$ and $B's$ of a given power,

(012) The pair $<D, \{F_q^t\}>$ denote a model, i.e. that the domain D is an arbitrary non-empty set and $\{F_q^t\}$ is an arbitrary finite sequence of relations such that F_k^m is m-ary relation on $D, k=1, \cdots, q$ and $m=1, \cdots, t$. A model of the power k is such model whose domain has exactly numbers $1, \cdots, k$ (k may be infinite),

(013) $M\{E\}=0$, i.e. E' is true in the model M; $M\{E(\{s_k\})\}=0$, i.e. $\{s_k\}$ are elements of the domain of M, x_j are names of s_j and $\{s_k\}$ do not satisfy E in the model M,

(014) Let $M=<D_k, \{F_q^t\}>, M, A$—have the same power and for each $m_1, \cdots, m_j \leqslant k$ and $j \leqslant t, i \leqslant q: F_i^j(m_1, \cdots, m_j)$ iff $f_i^j(x_{m_1}, \cdots, x_{m_j}) \epsilon A$ and $\sim F_i^j(m_1, \cdots, m_j)$ iff $f_i^j(x_{m_1}, \cdots, x_{m_j}) \epsilon A$—such M is called a description of A,

(015) For each model $M=<D, \{F_q^t\}>$by $M/s_1, \cdots, s_k/$—or briefly: $M/\{s_k\}$—we shall denote a model $<D_k, \phi_q^t>$ of the power k such that for each $r_1, \cdots, r_i \leqslant k$: ϕ_j^t $(r_1, \cdots, r_i)$ iff $F_j^t(s_{r1}, \cdots, s_{ri}), i=1, \cdots, t$ and $j=1, \cdots, q$. So $M/\{s_k\}=<D_k, \{\phi_q^t\}>$; if $\{s_k\}$ is empty, then one holds for all models; $M/\{s_k\}$ is a submodel of M in the meaning of homomorphism,

(016) quantifiers: (K), $(\exists K)$, $(\{K_m\})$, $(\exists\ \{K_m\})$,

(017) $E \epsilon A/s_1, \cdots, \cdots, s_k/$ iff $E(\{s_k\}/\{k\}) \epsilon A$,

(018) $A/\{s_k\}=A/s_1, \cdots, s_k/$; $A/\{s_k\}$ is a coset of A in the meaning of homomorphism. In the following $X, Y, X_1, \cdots$—denote a model M or a set A; $U, U_1, \cdots$—sets Q or B.

If $x_{si} \bar{\epsilon} S_A$ or respective s_i does not belong to the domain of the model, then we assume $X/s_1, \cdots, s_k/=X/s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_{k'}$ Of course:

L. 1. $X/\{s_k\}/\{j_m\}=X/\{s_{j_m}\}$, s. [1]

L. 2. If M_1 is the description of A_1 and M_2 is the description of A_2, and both models have the same power, then: $M_1/\{j_m\}=M_2/\{j_m\}$ iff $A_1/\{j_m\}=A_2/\{j_m\}$

D. 1. $X \epsilon Y[k]$ iff $(\exists \{s_k\})\{X=Y/\{s_k\}\}$
$Y[k]$ is the set of all $Y/\{s_k\}$.

For an arbitrary family Q of models of the same power, for an arbitrary model $M=<D_k, \{F_q^t\}> \epsilon Q$, for an arbitrary formula E and each $\{i_l\} \supset \{i_{w(E)}\}, l+p(E) \leqslant k$ we introduced in [4] the following inductive definition of the functional V:

(1d) $V\{k, Q, M, \{i_l\}, f_j^m(x_{r_1}, \cdots, x_{r_m})\}=1$ iff $F_j^m(r_1, \cdots, r_m)$,

(2d) $V\{k, Q, M, \{i_l\}, F'\}=1$ iff $\sim V\{k, Q, M, \{i_l\}, F\}=1$ iff $\{V k, Q, M, \{i_l\}, F\}=0$,

(3d) $V\{k, Q, M, \{i_l\}, F+G\}=1$ iff $V\{k, Q, M, \{i_l\}, F\}=1 \vee V\{k, Q, M, \{i_l\}, G\}=1$,

(4d) $V\{k, Q, M, i_l, \Pi a F\}=1$ iff $(j)(M_1)\{j \leqslant k) \wedge (M_1/\{i_l\}=M/\{i_l\}) \rightarrow$
$V\{k, Q, M_1, \{i_l\}, j, F(x_j/a)\}=1\}$.

D. 2. $N(k, Q, H)$ iff $(\{i_l\})\{(\{i_l\} \supset \{i_{w(H)}\}) \wedge (l+p(H)<k)$
$\rightarrow (M_1)(i)(V\{k, Q, M_1, \{i_l\}, H\}=1$ iff $V\{k, Q, M_1, \{i_l\}, i, H\}=1\}\}$,

D. 3. $F \epsilon P(k, Q, M, \{i_l\})$ iff $(\exists H)\{(H \epsilon C\{F\}) \wedge (H=\Pi a H_1$ for some $H_1)$ [3)]
$\wedge N(k, Q, H) \rightarrow V\{k, Q, M, \{i_l\}, F\}=1\}$,

D. 4. $F \epsilon P(k, Q, M)$ iff $F \epsilon P(k, Q, M, \{i_{w(F)}\})$,

D. 5. $F \epsilon P\{k\}$ iff $(Q)(M)\{Q(k) \wedge (M \epsilon Q) \rightarrow F \epsilon P(k, Q, M)\}$

D. 6. $E \epsilon P$ iff $(\exists k)\{(k \geqslant n(E)) \wedge (E \epsilon P\{k\})\}$ [4)].

We recall:

$V\{k, Q, M, \{i_l\}, E\}=1$ may be read: the model M satis fies E respectively to Q and $\{i_l\}$.

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3) Instead F we may write here an estabilshed formula E, to consider only parts of this formula and then we shall receive a relative definition of the defined class P as in [4], [6]. The reader may replace E by a set of formulas.

Analogously we may define the satisfiability functional V_1 which depends also on arbitrary sequence $\{z_l\}$ of elements of D_k; for atomic formulas, negations and alternatives the definition of V_1 is usual, s. [3], [17], [20], and analogic to the above, and for quantifiers: (d4) V_1 $\{k, Q, M$ $\{i_l\}, \{z_l\}, \Pi a F\}=1$ iff $(j)(z)(M_1)\{(j=1, 2, \cdots) \wedge (x_j \bar{\epsilon} S_F) \wedge (Z \epsilon D_k) \wedge (M_1/z_{i_l}\}=M/\{z_{i_l}\}) \rightarrow V_1$ $\{k, Q, M_1, \{i_l\} j, \{z_l\}(z/z_j), F(x_j/a)\}=1$.

By means of the functional V_1 we obtain as a special case the usual truth definition and its generalization according to the above.

4) It is easy to see $n(E)$ may be less than used here.

If we assume Q is one elementing, then V is the usual satisfiability functional in the domain of ordinary numbers D_k, D. 2.–4. are then obviously and they create the usual truth definition in M.

If M is a model and $Q=M[k]$, then elements of Q are submodels of M in the meaning of homomorphism, the number j in (4d) is the name of an arbitrary element of the domain of M and D. 3. says that the sequence $\{i_l\}$ has not influence in whole on the introduced truncated satisfiability definition as in one elementing Q; here we need note that the invariant relation $N(k, Q, H)$ holds for connectives of propositional calculus and for quantifiers it is assumed in D. 3.; D. 5.–6. are pictures of the usual truth definition in its generalization introduced here.

In [4] and [6] it is proved that for normal formulas E it suffices to consider only $H=E$ and the implication in the left–hand side of D. 2. instead of the second equivalence.

It is easy to prove suitable:

(d) $(H)\{(H \epsilon C\{E\} \rightarrow N(k, Q, H))\}$ iff $(H)\{(H \epsilon C\{E\}) \wedge (H = \Pi a H_1$ for some $H_1) \rightarrow N(k, Q, H)$.

(3D) $F \epsilon P(k, Q, M, \{i_l\})$ iff $(\exists H)\{(H \epsilon C\{F\}) \wedge (N(k, Q, H) \rightarrow V\{k, Q, M, \{i_l\}, F\}=1)\}$,

L. 3. If $M/\{i_l\}=M^\circ/\{i_l\}$, then :

$V\{k, Q, M, \{i_l\}, E\}=1$ iff $V\{k, Q, M^\circ, \{i_l\}, E\}=1$

(3d′) $V\{k, Q, M, \{i_l\}, F+G\}=0$ iff $V\{k, Q, M, \{i_l\}, F\}=0 \wedge$ $V\{k, Q, M, \{i_l\}, G\}=0$,

(4d′) $V\{k, Q, M, \{i_l\}, \Pi aF\}=0$ iff $(\exists j)(\exists M_1)$ $\{(j \leqslant k) \wedge (M_1/\{i_l\} - M/\{i_l\}) \wedge$ $V\{k, Q, M_1, \{i_l\}, J, F(x_j/a)\}=0\}$,

(5d) $V\{k, Q, M, \{i_l\}, \Sigma aF\}=1$ iff $(\exists j)(\exists M_1)$ $\{(j \leqslant k) \wedge (M_1/\{i_l\} = M/\{i_l\}) \wedge$ $V\{k, Q, M_1, \{i_l\}, j, F(x_j/a)\}=1\}$,

(5d′) $V\{k, Q, M, \{i_l\}, \Sigma aF\}=0$ iff $(j)(M_1)$ $\{(j \leqslant k) \wedge (M_1/\{i_l\} = M/\{i_l\}) \rightarrow$ $V\{k, Q, M_1, \{i_l\} J, F(x_j/a)\}=0\}$.

The proof of (d) and L. 3. are inductivel on the length of the considered formulas; (3D) follows immedately from D. 3. and (d); s. L. 5. in [5] and L. 14. in [13].

In the following the rank of $X, U, \cdots$ we denote by $v(X), v(U), \cdots$ For brevity of considerations we shall assume that the sequence (B) $A_1, A_2, \cdots$
includes all elements of B, i. e. we assume we enumerated all elements of B.

Let $v(B)=k$; then :

For an arbitrary $A \epsilon B$, for an arbitrary formula E such that $n(E) \leqslant k$ we introduce the symbol $A, B \vdash E$ which we read″ E is a thesis of A respectively to B″ :

(11) $A, B \vdash F$, for each $F \epsilon A$,

(12) $A, B \vdash F + F'$, for each F,

(13) If $A, B \vdash F_1 + \cdots\cdots + F_m$ and $k_1, \cdots, k_m$ is an arbitrary permutation of number $\leqslant m$, then $A, B \vdash F_{k1} + \cdots + F_{k_m}$ [5],

(14) If $A, B \vdash F$ and G is a formula, then $A, B \vdash F + G$,

(15) If $A, B \vdash F + G$ and [6] $A, B \vdash F + G'$, $G' \epsilon C\{F\}$ then $A, B \vdash F$,

(16) If $A, B \vdash F + G$, $x_r \bar{\epsilon} S_F$, $x_r \epsilon S_G$ and
If there exists such $\{i_l\} \supset \{i_{w(E+G(a/x_r))}\}$, $l + p(F + G(a/x_r)) \leqslant k$, then :

(1°) for each $j \leqslant k$ we have $A, B \vdash F + G(x_j/x_r)$,

(2°) for each $A^\circ \epsilon B$, if $A^\circ/\{i_l\} = A/\{i_l\}$, then for each $j \bar{\epsilon} \{i_l\}$: $A^\circ, B \vdash F + G(x_j/x_r)$,[7]
then $A, B \vdash F + \Pi a G(a/x_r)$,

(17) If $A, B \vdash F + \Pi a G$, $\Pi a G \epsilon C\{F\}$, then for each $\{i_l\} \supset \{i_{w(F+G)}\}$, $l + p(F + G) \leqslant k$, for each $A^\circ \epsilon B$, if $A^\circ/\{i_l\} = A/\{i_l\}$, then $A^\circ, B \vdash F + G(x_j/a)$ (it suffices to take here only $t = w(F + G)$),

(18) If there exists such $\{i_l\} \supset \{i_{w(F+G)}\}$, $l + p(F + G) \leqslant k$ and there exist such $j \leqslant k$ and $A^\circ \epsilon B$, $A^\circ/\{i_l\} = A/\{i_l\}$ that $A^\circ, B \vdash F + G(x_j/a)$, then $A, B \vdash E + \Sigma a G$.

(19) If $A, B \vdash F + \Sigma a G$, $\Sigma a G E \epsilon C\{F\}$, then for each $\{i_l\} \supset \{i_{w(F+G)}\}$, $l + p(F + G) \leqslant$ there exist such $j \leqslant k$ and $A^\circ \epsilon B$ that $A^\circ/\{i_l\} = A/\{i_l\}$ and $A^\circ, B \vdash F + G(x_j/a)$ [8].

(From the following considerations follows that (18), (19) follows from (11)–(17).)

Of course (12)–(15) are proof rules of the propositional calculus and (12)–(17)—of the first-order functional calculus; the last fact is obviously for B—empty.

The following consideration hold also by replacing $j \leqslant k$ in (16), (1°) by means $j \leqslant k - p(F + G(a/x_r))$ analogic to [6] ; then we modify (4d) according to [6].

Let

$Cl(U)$ iff $(X)(m_1)\cdots(m_k)\{(X \epsilon U) \wedge (m_1, \cdots, m_k$ is a permutation of numbers $\leqslant k) \rightarrow (X/m_1, \cdots, m_k/\epsilon U)\}$.

If $Cl(B)$, then B is not one-elementing and in this condition all following

5) Of course, the rule may be replaced by usual associative and commutative laws.

6) For theses we may assume also $\{i_{w(F+G')}\} = \{i_{w(F)}\}$.

7) If considered sets of formulas are closed under substitutions regarded in (1°) and (2°), then it suffices to assume $j = r$.

8) If we consider a relative definition of the class P respectively to E, s. [4], [6], then in proof rules we must assume all considered formulas are composed of significant parts of E and then assumptions about parts in proof rules are in general less, s. footnote 3).

considerations hold by assuming in (16) only: $j\epsilon\,\{i_l\}$ in (1°), $j=r$; if permutations are also with reiterations, then it suffices only: $j\epsilon\,\{i_{w(G(a/x_r))}\}$ in (1°), and in (2°) as above $j=r$. Then (16) will receive a from of usual quantification rule but we must here use more strong lemma than L. 3., namely:

L. 3'. If $E^\circ=E(\{j_l\}/\{i_l\}, \{i_l\}\supset\{i_{w(E)}\}, \{j_l\}\supset\{j_{w(E^\circ)}\}, M/\{i_l\}=M^\circ/\{i_l\}$ and $Cl(Q)$, then:

$V\{k, Q, M, \{i_l\}, E\}=1$ iff $V\{k, Q, M^\circ, \{j_l\}, E^\circ\}=1$, s. L. 12'. in [9].

If B is non-determined, one-elementing and $v(B)$ infinite, then we also will speak that it is closed under the considered permutations.

Other form of proof rules are regarded in [4], [6], [14] and will be also a topic of my future papers.

We point out that in order to approximate the first-order functional calculus by the above calculi it suffices to consider only B and Q with the above permutation property, s. also [8], [9], [15], [16].

D. 7. The double sequence $E_{i1}, \cdots, E_{in_i}, i=1, 2, \cdots$ is a formal proof of the formula E in A_j respectively to B iff $E=E_{jn_j}$ and for each $i=1, 2, \cdots$ and $t=1, \cdots, n_i$ one of following conditions holds:

1. E_{it} is an element of A_i, s.(B), or $E_{it}=F+F'$, for some F,
2. there exists $d<t$ such that E_{it} results from E_{id} by means of rules (13) or (14),
3. there exist $d, m<t$ such that E_{it} results from E_{id} and E_{im} by means of the rule (15),
4. E_{it} results from the double sequence $E_{d1}, \cdots, E_{dt-1}, d=1, 2, \cdots$ by means of the rule (16),
5. there exist $d<t$ and m such that E_{it} results from E_{md} by means of rules (17) or (18), or (19),

D. 8 The formula E is a thesis of A_j respectively to B—in symbols: $A_j, B\vdash E$—iff there exists a formal proof of E in A_j respectively to B.

$A, B\vdash E$ we read: E is not a thesis of A respectively of B.

D. 9. $B\vdash E$ iff $(A)\,\{(A\epsilon B)\to(A, B\vdash E)\}$ $B\vdash E$ may be read: E is a thesis respectively to B.

D 10. $k\vdash E$ iff $(B)\,\{(v(B)=k)\to(B\vdash E)\}$.

D 11. $\Vdash E$ iff $(\exists k)\,\{k\vdash E)\wedge(k\geqslant n(E))\}$.

$\Vdash E$ may be read: E is a B-thesis.

Of course:

If B is not determined and $B\vdash E$, then k$\vdash E$.

The converse implication follows from generalization of Gödel–Skolem–Lövenheim's theorem, p. 14.

T. 1. If E is a thesis, then $\Vdash E$.

T. 2. If Q is the family of all description of elements of $B, k=v(B)=v(Q), M$ is the description of $A_t, E_{i1}, \cdots, E_{in_i}$ is a formal proof of E in A_t respectively to B, $k \geqslant n(E_{is})$, $s=1, \cdots, n_i$, and $i=1,2,\cdots$, then $E_{ts} \epsilon P(K, Q, M)$.

Proof. Let the assumption of T. 2. hold.

We shall prove T. 2. by induction on $s=n_t$ simultaneously for all $t=1,2,\cdots$

Of couse, if $E_{ts}=F+F'$ or $E_{ts}\epsilon A_t$, then in view of the assumption T. 2. holds; therefore T. 2. holds for $s=1$.

Let T. 2. holds for all $m<s$; we shall prove it for s.

If E_{ts} results from E_{tm}, $m<s$, by means of rules (11)–(14), then T. 2. also holds obviously for E_{ts}; in (14) we use D. 2.

If E_{ts} results from E_{tm} and E_{td}, $d, m<s$, by means of the rule (15), then in view of the assumption and D. 2. we obtain that T. 2. also holds for E_{ts}.

If E_{ts} results from the double sequence $E_{d1}, \cdots, E_{ds-1}, d=1,2,\cdots$, by means of the rule (16), then $E_{ts}=F+\Pi aG(a/x_r), x_r \bar{\epsilon} S_F, x_r \epsilon S_G$, for each $j \leqslant k$ formulas $F+G(x_j/x_r)$ occur in the sequence $E_{t1}, \cdots, E_{ts-1}$, and there exists such $\{i_l\} \supset \{i_{w(F+G(a/x_r))}\}$, $l+p(F+G(a/x_r)) \leqslant k$, if $A_t/\{i_l\}=A_c/\{i_l\}$, then for each $j \bar{\epsilon} (i_l\}$ there exists $m<s$ such that $E_{cm}=F+G(x_j/x_r)$.

Let $E_{ts} \bar{\epsilon} P(k, Q, M)$; therefore in view of D. 3.–4., (d), for each $H \epsilon C\{E_{ts}\}$ we have $N(k, Q, H)$ and $V\{k, Q, M, \{i_{l'}\}, E_{ts}\}=0, l'=w(E_{ts})$, and we may assume by D. 2. $l'=l$ given above; therefore by (3d′) and the above $V\{k, Q, M, \{i_l\}, F\}=0$ and $V\{k, Q, M, \{i_l\}, \Pi aG(a/x_r)\}=0$. Hence by virtue of (4d′) there exist $j \leqslant k$ and $M_1 \epsilon Q$ such that $M_1/\{i_l\}=M/\{i_l\}$ and $V\{k, Q, M_1, \{i_l\}, j, G(x_j/x_r)\}=0$;

We consider two cases:

(1°) $j \epsilon \{i_l\}$

(2°) $j \bar{\epsilon} \{i_l\}$

In the case (1°) in view of L. 3. and the above we also have $V\{k, Q, M, \{i_l\}, G(x_j/x_r)\}=0$ and thus $V\{k, Q, M, \{i_l\}, F+G(x_j/x_r)\}=0$ what according to the above and $C\{E_{ts}\} \supset C\{F+G(x_j/x_r)\}$ gives a contradiction.

In the case (2°) in view of $M_1/\{i_l\}=M/\{i_l\}$ we also have $V\{k, Q, M_1, \{i_l\}, F\}=0$. Because from the assumption $l+p(F)<k$, therefore in view of $N(k, Q, H)$ for each $H \in C\{F+G(x_j/x_r)\}$, the assumption and (3d′) we have $V\{k, Q, M_1, \{i_l\}, j, F\}=0$ and $V\{k, Q, M_1, \{i_l\} j, F+G(x_j/x_r)\}=0$ what as above gives a contradiction.

If B is closed under permutations considered on p. 7 [8)–15)], then Q is also closed under the same permutations ; then we regard simpler rules described on p. 7 [8)–15)] and the same two cases. The case (1°) is as above and in the second case we permute j to r, : afterwards we use L. 3′. instead of L. 3. and we enlarge the sequence $\{i_l\}$, to the sequence $\{i_l\}$, r as above, s. e. g. [13].

If B is closed under permutations with reiterations considered on p. 7 [8)–15)], then we regard two simpler cases :

(1′) $j \in \{i_{w(G(a/x_r))}\}$,

(2′) $j \in \{i_{w(G(a/x_r))}\}$.

The case (1′) is here as (1°) above.

In the case (2′) we permute j to r with reiteration and act as in the case (2°) for B closed only on permutations without reiteration, s. e. g. [9].

The above proves T. 2. in the case of the rule (16).

If E_{ts} results from E_{cm}, $m<s$, by means of the rule (17), then $E_{ts}=F+G(x_j/a)$, $E_{cm}=F+\Pi aG$, $\Pi aG \in C\{F\}$, $m<s$, and $A_c/\{i_l\}=A_t/\{i_l\}$, $\{i_l\} \supset \{i_{w(F+G)}\}$, $l+p(F+G) \leqslant k$.

Let $E_{ts} \bar{\epsilon} P(k, Q, M)$; therefore in view of D. 3.–4., (d), for each $H \in C\{E_{ts}\}$ we have $N(k, Q, H)$ and $V\{k, Q, M, \{i_{l'}\}, E_{ts}\}=0$, $l'=w(E_{ts})$, and by (3d′) and above $V\{k, Q, M, \{i_{l'}\}, F\}=0$ and $V\{k, Q, M, \{i_{l'}\}, G(x_j/a)\}=0$. Hence in view of D. 2. $V\{k, Q, M, \{i_l\}, F\}=0$ and $V\{k, Q, M, \{i_l\}, j, G(x_j/a)\}=0$; therefore by (4d′) also $V\{k, Q, M, \{i_l\}, \Pi aG\}=0$ and by (3d′) $V\{k, Q, M, \{i_l\}, F+\Pi aG\}=0$.

Let M_1 be the description of A_c; then in view of the above and L. 2. we have also $M/\{i_l\}=M_1/\{i_l\}$ and by virtue of L. 3. $V\{k, Q, M_1, \{i_l\}, F+\Pi aG\}=0$ and thus $V\{k, Q, M_1, \{i_l\}, E_{cm}\}=0$.

Because by assumption $C\{E_{cm}\}=C\{E_{ts}\}$, therefore in view of D. 2. and the above we may assume here $l=w(E_{cm})$ what gives a contradiction with the inductive assumption.

Thus T. 2. is also true in the case of the rule (17).

Let E_{ts} results from E_{cm}, $m<s$, by means of the rules (18) or (19), then we need consider two cases according to two rules :

(1°) $E_{ts}=F+\Sigma aG$,
(2°) $E_{ts}=F+G(x_j/a)$, $\Sigma aG \in C\{F\}$.

In the first case $E_{cm}=F+G(x_j/a)$, $m<s$, and for some $\{i_l\}\supset\{i_{w(F+G)}\}$, $l+p(F+G)\leqslant k$, $A_t/\{i_l\}=A_c/\{i_l\}$.

Let M_1 be the description of A_c ; therefore in view of the above and L. 2. we have $M/\{i_l\}=M_1/\{i_l\}$. By the inductive assumption $F+G(x_j/a)\epsilon P(k,Q,M_1)$.

If $F+\Sigma aG\bar{\epsilon}P(k,Q,M)$, then in view of D. 3.–4., (d), for each $H\epsilon C\{F+\Sigma aG\}$ we have $N(k,Q,H)$ and $V\{k,Q,M,\{i_{l_0}\},F+\Sigma aG\}=0$, and by (3d') also $V\{k,Q,M,\{i_{l_0}\},F\}=0$, $V\{k,Q,M,\{i_{l_0}\},\Sigma aG\}=0$, $l^\circ=w(F+\Sigma aG)$. Hence in view of D. 2. we may assume $l^\circ=1$ and therefore by L. 3. and (3d') also $V\{k,Q,M_1,\{i_l\},F+\Sigma aG\}=0$.

Because $C\{F+G(x_j/a)\}\subset C\{F+\Sigma aG\}$, therefore from the above $V\{k,Q,M_1,\{i_{l'}\},F+G(x_j/a)\}=1$, $l'=w(F+G(x_j/a))$; therefore by (3d) and (5d) also $V\{k,Q,M_1,\{i_{l'}\},F+\Sigma aG\}=1$ and analogical by D. 2. we may replace here l' by l what gives a contradiction with the above.

In the case (2°) $E_{cm}=F+\Sigma aG$, $m<s$, and let M_1 be the description of A_c. By the inductive assumption $F+\Sigma aG\epsilon P(k,Q,M_1)$.

If for some $\{i_l\}\supset\{l_{w(F+G)}\}$, $l+p(F+G)\leqslant k$, for each $j\leqslant k$ and each $M\epsilon Q$ such that $M/\{i_l\}=M_1/\{i_l\}$ we have $F+G(x_j/a)\bar{\epsilon}P(k,Q,M)$, then in view of the above, D. 3.–4., (d), for each $H\epsilon C\{F+G(x_j/a)\}=C\{F+\Sigma aG\}$ wehave $N(k,Q,H)$ and $V\{k,Q,M,\{i_{l'}\},F+G(x_j/a)\}=0$, $l'=w(F+G(x_j/a))$; hence by (3d'), D. 2. we receive $V\{k,Q,M,\{i_l\},F\}=0$ and $V\{k,Q,M,\{i_l\},j,G(x_j/a)\}=0$. Therefore by L. 3, and (5d') also $V\{k,Q,M_1,\{i_l\},F\}=0$ and $V\{k,Q,M_1,\{i_l\},\Sigma aG\}=0$; thus by (3d') $V\{k,Q,M_1,\{i_l\},F+\Sigma aG\}=0$, what in view of the above D. 2. gives a contradiction.

Therefore for each $\{i_l\}\supset\{i_{w(F+G)}\}$, $l+p(F+G)\leqslant k$, for some $j\leqslant k$ and for some $M\epsilon Q$, $M/\{i_l\}=M_1/\{i_l\}$: $F+G(x_j/a)\epsilon P(k,Q,M)$. This M is the description of the needed A_t, what proves T. 2. in the case (2°).

The above closed the inductive proof of T. 2.

From T. 1. and T. 2. follows:

T. 3. If E is a thesis, then $E\epsilon P$.

D. 12. A set U of formulas of the first-order functional calculus is consistent respectively to B and $A\epsilon B$ iff there exists at least one formula E such that $A\cup U$, $B\not\vdash E$, where $A\cup U$ means that we assume the proof rule (11) also for elements of U and $k\geqslant n(F)$ for $F\epsilon U$.

The reader may define more general notion than in D. 12. by adding to each element of B a set of considered formulas.

In the same way as T. 2. we may prove the generalization of it:

T. 4. If Q is the family of all description of elements of $B, k=v(B)=v(Q), M$ is the description of A_t and for each $E \epsilon U, F \epsilon E(k, Q, M), E_{i1}, \cdots, E_{in_i}$ is a formal proof of E in $A_t \cup U$ respectively to $B, k \geqslant n(E_{is}), s=1, \cdots, n_i$, and $i=1, 2, \cdots$, then $E_{ts} \epsilon P(k, Q, M)$, s. foot note 3).

T. 4. is especially an interesting generalization of the known theorem that if M is a model for axioms of a given theory, then M is a model for their conclusions.

To each $A \epsilon B$, $k=v(B)$, we correspond a set J_A and to B we correspond the family B_J of all $J'_A s$ in the following way:

D. 13. *A* family B_J is called a family of generalized proper prime ideals respectively to B iff there exists such double sequence of formulas $E_1^{A_1}, E_2^{A_1}, \cdots$ that J_{A_1} is the set of all elements of the sequence $E_1^{A_1}, E_2^{A_2}, \cdots$ $A_1=A \cup U$ or $A_1 \epsilon B$, B_J is the family of all J_{A_1} and the following conditions are satisfied: [9]

1. If for some d and certain formulas F and G we have $E_d^{A_1}=F+G$, then there exist $i, j<d$ such that $E_i^{A_1}=F$ and $E_j^{A_1}=G$,
2. For each $d=1, 2, \cdots$ there exists $A_2 \epsilon B$ such that: $A_2, B \not\vdash E_1^{A_1}+\cdots+E_d^{A_1}$,
3. If a formula $E \bar{\epsilon} J_{A_1}$, then there exists d such that for each $A_2 \epsilon B, A_2, B \vdash E_1^{A_1}+\cdots+E_d^{A_1}+E$ and if $E=\Pi a F$, then $E_d^{A_1}=(\Pi a F)'$,
4. If $E_d^{A_1}=\Pi a E$, then $(\{i_l\})(\exists A_2)(\exists j)\{(\{i_l\} \supset \{i_{w(E)}\}) \wedge (l+p(E) \leqslant k) \rightarrow (A_2 \epsilon B) \wedge (j \leqslant k) \wedge (A_2/\{i_l\}=A_1/\{i_l\}) \wedge (E(x_j/a) \epsilon J_{A_2})\}$,

   If $E_d^{A_1}=(\Pi a F)'$, then $(\{i_l\})(A_2)(j)\{(A_2 \epsilon B) \wedge (\{i_l\} \supset \{i_{w(E)}\}) \wedge (l+p(E) \leqslant k) \wedge (j \leqslant k) \wedge (A_2/\{i_l\}=A_1/\{i_l\}) \rightarrow (E(x_j/a) \bar{\epsilon} J_{A_2})\}$. [10]
5. If $A_1/\{i_l\}=A_2/\{i_l\}$, $\{i_{w(E)}\} \subset \{i_l\}$, then: $E \epsilon J_{A_1}$ iff $E \epsilon J_{A_2}$.

In the following we consider only B closed under suitable permutations described on p. 7 [8)–15)], i.e. we consider only described there rules; we assume also:

(*) if k is finite, then for each different $\{i_l\}$, j and each $A_1 \epsilon B$ there exists $A_2 \epsilon B$ such that $A_1/\{i_l\}=A_2/\{i_l\}$ and $A_1/\{i_l\}, j \neq A_2/\{i_l\}, j/$.

T. 5'. If $A_1=A \cup U$, $A_1, B \not\vdash E_0''$ and $k \geqslant n(E_0+F_1+\cdots+F_n)^n$ for all $F_1, \cdots, F_n \epsilon U$, then there exists a family B_J of generalized proper prime ideals respectively to B such that $E_0'' \epsilon J_{A_1}$.

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9) We leave for readers the modification of D. 13. in which does not occur the double sequence; the property 1. may be also omitted, s. [3].

10) We may assume also that j satisfies conditions formulated in T. 5. The reader may formulate analogical properties for the existential quantifier.

11) From the proof follows that k may be less than given above.

Proof: Because we have a denumerable number of all significant parts of considered formulas, therefore we may order them in a double sequence

(1) $G_1^{A2}, G_2^{A2}, \cdots$

in such way that it satisfies the following conditions:

(1a) all formulas of each sequence of (1) are different,

(2a) $G_1^{A1}=E_0$, for $A_1=A \cup U$,

(3a) if for some d and certain formulas F and G we have $G_d^{A2}=F+G$, then there exist $i, j<d$ such that $G_i^{A2}=F$ and $G_j^{A2}=G$,

(4a) if for some d and some formula E we have $G_d^{A2}=\Pi aE$, then $G_{d'+1}^{A_3^0}=E''(x_j/a)$ and $j \bar{\epsilon} \{i_l\}$, where:

(1°) if k is infinite, then $A_3^0=A_2, G_{d+1}^{A2}, \cdots, G_{d'}^{A2}$ are all substitutions $E''(x_j/a)$, for $j \epsilon \{i_l\}$ and we enlarge all lines to the length d' e.g. as below in (2°) and $l=W(G_1^{A2}+\cdots+G_{d'}^{A2})$,

(2°) if k is finite, then for each $\{i_l\} \supset \{i_{w(E)}\}, l+p(E) \leqslant k$, we choose $A_3^{\circ} \epsilon B$ as at (*), i.e. a new line, such that $A_3^{\circ}/\{i_l\}=A_2/\{i_l\}$ and we assume that $G_1^{A^{\circ}3}, \cdots, G_{d'}^{A^{\circ}3}$ aie only such formulas belonging to $G_1^{A2}, \cdots, G_d^{A2}$ in which occur indices of $\{i_l\}$ and afterwards all substitutions $E''(x_j/a)$, for $j \epsilon \{i_{w(l)}\}$ and we enlarge all lines to the length d' by adding to them e.g. E_d with an even number of negations,

(5a) if non (4a), then G_{d+1}^{A2} is the first formula which is not included into $G_1^{A2}, \cdots, G_d^{A2}$,

Let

(2) $E_1^{A2}, E_2^{A2}, \cdots$

be a subsequence of the sequence (1) defined in the following way:

(1b) $E_1^{A1}=G_1^{A1}=E_0''$, for $A_1=A \cup U$,

(2b) E_{d+1}^{A2} is the first element after E_d^{A2} of the double sequence (1) such that for some $A_2^0 \epsilon B: A_2^0, B \not\vdash E_1^{A2}+\cdots+E_{d+1}^{A2}$ and if there exists d such that for each $A_2^0 \epsilon B$: $A_2^{\circ}, B \vdash E_1^{A2}+\cdots+E_d^{A2}+\Pi aF$, then in (4a)(1°), (2°) we add $E_{d'}=(\Pi aF)'$

(3b) If $A_2/\{i_l\}=A_3/\{i_l\}, \{i_{w(E)}\} \subset \{i_l\}$, then ahead: $E=E_j^{A2}$ iff $E=E_j^{A3}$. Here we restrict ourself to formulas with a finite length of significant parts; if we do not restrict ourself to such formulas we need use all finite sequences of formulas belonging to ideals as in [3], s. footnote 9).

We shall prove the family B_J of all $J_{A_2}, A_2 \epsilon B$ or $A_2=A_1$, where J_{A_2} is the set of all elements of the sequence $E_1^{A2}, E_2^{A2}, \cdots$, — is a family of proper prime ideals respectively to B; in this purpose it suffices to prove conditions 1.–5. of D. 13.

Immediately conclusions from the definition of the sequence (2) are 1. – 3., 5. s. [3], [11].

We prove 4.:

(1′) let $E_d^{A2} = \Pi a E$, for some E,

(2′) for each d there exists $A_2^{\bullet} \epsilon B : A_2^{\bullet}, B \not\vdash E_1^{A2} + \cdots + E_d^{A2}$

(3′) there exists such m that $G_m^{A2} = E_d^{A2} = \Pi a E$.

(4′) $G_{d41}^{A^\circ 3} = E''(x_j/a)$ and $j \bar{\epsilon} \{i_l\}$, where:

(1°) if k is infinite, then $A_3^0 = A_2$ and $l = w(G_1^{A2} \cdots G_d^{A2})$, s. (4a) (1°),

(2°) if k is finite, then for cach $\{i_l\} \supset \{i_{w(E)}\}$, $l + p(E) \leqslant k$ we choose $A_3^0 \epsilon B$, determined in (4a), (2°), i. e. a new line, such that $A_3/\{i_l\} = A_2/\{i_l\}$ and we assume that $G_1^{A^\circ 3}, \cdots, G_d^{A^\circ 3}$ are only such formulas belonging to $G_1^{A2}, \cdots, G_d^{A2}$ [or with two negations] in which occur indices of $\{i_l\}$, s. footnote 7),

(5′) for some $\{i_l\} \supset \{i_{w(E)}\}$, $l + p(E) \leqslant k$, for each $A_3 \epsilon B$ and $j \leqslant k$ if $A_3/\{i_l\} = A_2/\{i_l\}$, then $E(x_j/a) \bar{\epsilon} J_{A_3} - a$ contrary assumption,

(6′) for some $\{i_l\} \supset \{i_{w(E)}\}$, $l + p(E) \leqslant k$, for each $A_3 \epsilon B$ and $j \leqslant k$ if $A_3/\{i_l\} = A_2/\{i_l\}$, then there exists c such that for each $A_4 \epsilon B : A_4, B \vdash E_1^{A3} + \cdots + E_c^{A3} + E(x_j/a)$,

(7′) In the case (4′) (1°) we take $A_3 = A_2$ and thus: for each $A_4 \epsilon B$: $A_4, B \vdash E_1^{A2} + \cdots + E_c^{A2} + E(x_j/a)$, $c = d$, $j \epsilon \{i_l\}$ and some $j \bar{\epsilon} \{i_{w(E_1^{A2} \cdots E_c^{A2})}\}$,

(8′) for each $A_4 \epsilon B$: $A_4, B \vdash E_1^{A2} + \cdots + E_d^{A2} + \Pi a E$ —rule (16),

(9′) for each $A_4 \epsilon B$: $A_4, B \vdash E_1^{A2} + \cdots + E_d^{A2}$ —contradiction,

(10′) in the case (4′), (2°), we take $A_3 = A_3^0$ and thus; for earch $A_4 \epsilon B$; $A_4 . B \vdash E_1^{\mathring{A}3} + \cdots + E_c^{\mathring{A}3} + E(x_j/a)$, $c = d$, $j \epsilon \{i_l\}$ and some $j \bar{\epsilon} \{i_{w(E_1^{\mathring{A}3} + \cdots + E_c^{\mathring{A}3})}\}$,

(11′) for each $A_4 \epsilon B$: $A_4, B \vdash E_1^{\mathring{A}3} + \cdots + E_d^{\mathring{A}3} + \Pi a E$ —rule (16)

(12′) for each $A_4 \epsilon B$: $A_4, B \vdash E_1^{\mathring{A}3} + \cdots + E_d^{\mathring{A}3}$ —contradiction

The second sentence of 4. follows immediately from (17) considering the line described in (4a) (2°); q. e. d.

According to the above T. 5′. also holds with B composed of one non-determined element A and k infinite; thus it also holds for usual theses.

D. 14 $A^\circ \epsilon B [r]$ iff $(\{r\}$ is a permutation of k upon $r) \wedge (r \leqslant k) \wedge$

$(\exists A) \{(A \epsilon B) \wedge (A^\circ = A/\{r\})\} \wedge (r$ is a finite number$)$

Let B_J^0 be the set of all $J_A | \{i_r\}$, for a given r and $A_1 \epsilon B$ or $A_1 = A \cup U$.

The set B_J^0 we enlarge to a family B_J^r of proper prime ideals respectively to B^r where B^r results from $B[r]$ by adding to the last set lines described in (4a), (2°) and (4'), (2°) of T. 5'; of course, the fullfiling of $B[r]$ is according to T. 5'.

First of all we must note that if $A/\{i_r\}=A^\circ$ and $A^\circ, B^r \vdash H(\{r\}/\{i_r\})$ then $A, B \vdash H$.

The proof of the above sentence is inductive according to the length of the formal proof of the considered formula.

By virtue of the above, if we establish $r \geqslant n(E)$ for all considered formulas, then the following lemmas hold for $B, B^r, B_J, B_J^r, U | \{i_r\}$ respectively; proofs are almost identical with ones given in [3], [11].

L. 4. $E \epsilon J_{A_1}$ iff $E' \bar{\epsilon} J_{A_1}$.

L. 5. $E, F \epsilon J_{A_1}$ iff $E+F \epsilon J_{A_1}$.

T. 5. If $A_1 = A \cup U, A \epsilon B$ and $A_1, B \vdash E_0$, then there exists a family B_J of generalized proper prime ideals respectively to B such that $E_0 \epsilon J_{A_1}$.

T. 6. If $A_1 = A \cup U, A \epsilon B$, then for each family B_J of generalized proper prime ideals respectively to B there exists such family Q of models of the power $k = v(B)$ that for each $A_2 = A_1$ or $A_2 \epsilon B$ and the description M (which create the set Q) of negations of atomic formulas belonging J_{A_2} and each formula E of considered formulas, i. e. $n(E) \leqslant k$:

(1) If $E \bar{\epsilon} J_{A_2}$, then $(\{i_l\})\{((\{i_l\} \supset \{i_{w(E)}\}) \wedge (l+p(E) \leqslant k) \rightarrow V\{k, Q, M, \{i_l\}, E\} = 1\}$.

(2) If $E \epsilon J_{A_2}$, then $(\{i_l\})\{((\{i_l\} \supset \{i_{w(E)}\}) \wedge (l+p(E) \leqslant k) \rightarrow V\{k, Q, M, \{i_l\}, E\} = 0$.

(3) $E \bar{\epsilon} J_{A_2}$ iff $E \epsilon P(k, Q, M)$.

Proof: We prove implications (1), (2) together by induction on the length of the formula E.

In view of the assumptions, if E is an indecomposable formula, then (1), (2) hold.

Let (1) and (2) hold for formulas of the length $< m$; we shall prove it for m.

We consider three cases:

(1') $E = F'$, (2) $E = F+G$, (3') $E = \Pi a F$.

In the cases (1') and (2') we obtain immediately (1) and (2) in view of the inductive assumptions, L. 4., L. 5., (2d) and (3d); the strict proof is almost identical with [3], [11].

In the case (3') in view of the definition of Q, the inductive assumption (4d), D. 13. 4.–5. and L. 4. we obtain in both cases:

$E\bar{\epsilon}J_{A_2}$ iff $\Pi aF\bar{\epsilon}J_{A_2}\rightarrow(\{i_l\})(A_2^0)(j)\ \{(A_2^0\epsilon B)\wedge(\{i_l\}\supset\{i_{w(F)}\})\wedge(l+p(F)\leqslant k)\wedge(j\leqslant k)$ $\wedge(A_2^0/\{i_l\}=A_2/\{i_l\})\rightarrow(F(x_j/a)\bar{\epsilon}J_{A_2^0})\}\rightarrow(\{i_l\})(M_1)(j)\{(M_1\epsilon Q)\wedge(\{i_l\}\supset\{i_{w(F)}\})\wedge(l+p(F)\leqslant k)$ $\wedge(j\leqslant k)\wedge M_1/\{i_l\}=M/\{i_l\})\rightarrow V\{k,Q,M_1,\{i_l\},j,F(x_j/a)\}=1\}\rightarrow(\{i_l\})\ \{(\{i_l\}\supset\{i_{w(F)}\})\wedge$ $(l+p(F)\leqslant k)\rightarrow V\{k,Q,M,\{i_l\},\Pi aF\}=1\}\rightarrow(\{i_l\})\{(\{i_l\}\supset\{i_{w(E)}\})\wedge(l+p\,E)\leqslant k)\rightarrow V\{k,Q,$ $M,\{i_l\},E\}=1\}$.

$E\epsilon J_{A_2}$ iff $\Pi aF\epsilon J_{A_2}\rightarrow(\{i_l\})(\exists A_2^0)(\exists j)(\{i_l\}\supset\{i_{w(F)}\})\wedge(l+p(F)\leqslant k)\rightarrow(j\leqslant k)\wedge(A_2^0/\{i_l\}=$ $A_2/\{i_l\})\wedge(F(x_j/a)\epsilon J_{A_2})\}\rightarrow(\{i_l\})(\exists M_1)(\exists j)\{(\{i_l\}\supset\{i_{w(F)}\})\wedge(l+p(F)\leqslant k)\rightarrow(j\leqslant k)\wedge(M_1/\{i_l\}$ $=M/\{i_l\})\wedge V\{k,Q,M_1,\{i_l\},j,F(x_j/a)\}=0\}\rightarrow(\{i_l\})\ \{(\{i_l\}\supset\{i_{w(F)}\})\wedge(l+p(F)\leqslant k)\rightarrow V\{k,$ $Q,M,\{i_l\},\Pi aF\}=0\}\rightarrow(\{i_l\})\ \{(\{i_l\}\supset\{i_{w(E)}\})\wedge(l+p(E)\leqslant k)\rightarrow V\{k,Q,M,\{i_l\},E\}=0\}$, what proves (1) and (2).

Therefore we proved also $N(k,Q,M)$ for each considered formulas H. Hence by (1) and (2) we obtain (3); q. e. d.

Of course, T. 6. may be proved in a similar way for other truncated satisfiability definition considered in my papers [4], [6] - [9], [13], [14].

From T. 1., T. 3., T. 4., T. 5. and T. 6. follows:

Genralization of Gödel–Skolem–Löwenheim's therem: - If U is a consistent set of formulas respectively to B and $A_1=A\cup U, A\epsilon B, A_1, B\not\vdash E_0$ then there exists such family Q of models of the power $k=v(B)$ and a correspondence between $A\epsilon B$ and $M^A\epsilon Q$ such that for each $A_2\,(A_2=A_1$ else):

1. If $A_2, B\vdash E$, then $E\epsilon P(k,Q,M^{A_2})$,
2. $V\{k,Q,M^{A_1},E\}=0, k\vdash E$ iff $E\epsilon P\{k\}$,
3. In 1. - 2. we mey only assume $v(Q)\geqslant n(E_0)$

If B is not determind and k infinite, then in 1. - 2. we may assume Q one–elementing.

Generalizations of Gödel–Herbrand's theorem: [12] E ia a thesis iff $E\epsilon P$ iff $\Vdash E$.

Completenese theorem: All considered Boolean calculi ($k\vdash E$) with finite truncated general and existential quantifiers are complete. (According to the above theorem they approximate the first–order functional calculus), s. [15], [16].

The same theorems may be proved by analogical generalizations to methods used in [2], [21] and others.

Generalizations of Gentzen's sequent proof rules according to the above are given in [7], [8], s. also [18].

12) A homogeny formulation of both above theorems is given at [12]

Including of the above theorems in Tarski's systems theory will be an interesting topic of [10] ; in such way will be presented the above generalized theorem in the systems theory, s. [12]. [19]. According to 1–0–Bernoullie's sequences at [16] there are given finite and asymptotic infinite probabilistic models of regarded here generalized models with the generalizations of satisfiability definition.

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