A THEOREM OF PIECEWISE LINEAR APPROXIMATIONS II.

By

TATSUO HOMMA

(Received November 30, 1968)

§ 1. We showed in [1] a piecewise linear approximation theorem but there was an error in the proof. We shall correct it and prove the following Theorem.

Theorem *Let f: $P \to Q$ be a piecewise linear mapping of a compact polyhedron P onto a compact polyhedron Q and g: $P \to E^n$ a piecewise linear mapping of P into a euclidean n–space E^n such that*

$$n > dim\ Q + 2 \max_{q \in Q} dim\ f^{-1}(q).$$

Then for any $\varepsilon > 0$ there is a non-degenerate piecewise linear map h: $P \to E^n$ such that

a) $d(h, g) < \varepsilon$

b) $h | f^{-1}(q)$ *is a homeomorphism for any* $q \in Q$

c) if $q_1 \neq q_2 \in Q$ *and* $h(f^{-1}(q_1)) \cap h(f^{-1}(q_2)) \neq \phi$ *then there are polyhedral cells* C_1 *and* C_2 *of* P *satisfying* $C_1 \subset f^{-1}(q_1)$, $C_2 \subset f^{-1}(q_2)$ *and*

$$h(C_1) = h(C_2) = h(f^{-1}(q_1)) \cap h(f^{-1}(q_2)).$$

We shall assume that all polyhedra and complexes are contained in a euclidean space E^l. Let K be a finite complex and $g: |K| \to E^n$ be a continuous map of underlying space $|K|$ into E^n such that for any simplex $\xi \in K$, $g | \xi \to E^n$ is linear. Then we shall say $g: K \to E^n$ is a *semi-simplicial* (or *SS*) map. It is clear that if $g: K \to E^n$ is semi-simplicial $g: |K| \to E^n$ is *piecewise linear* (or *PL*). Throughout this paper we shall assume that $f: K \to H$ is a simplicial map of a finite complex K onto a finite complex H such that

$$n > dim\ H + 2 \max_{q \in |H|} dim\ f^{-1}(q)$$

and $g: K \to E^n$ is a semi-simplicial map of K into a euclidean n-space E^n.

If $\xi \in K$ and $\eta \in H$ such that $f(\xi) = \eta$. We denote $\xi = [a_{v_i}]$, $i = 0, 1, \cdots, m(v, \xi)$ and $\eta = [v]$, where

a_{v_i}, v are vertices of ξ and η respectively,

$f(a_{v_i}) = v$,

$m(v, \xi) = dim(f^{-1}(v) \cap \xi)$.

Since $f | \xi \to \eta$ is linear. It is easy to show the following;

Proposition 1. *For any point q of the interior $\mathring{\eta}$ of η, $f^{-1}(q)\cap\xi$ is a polyhedral convex m-cell which is the intersection of ξ and a hyper plane parallel to the hyper plane spanned by the vectors $\{a_{v_i}-a_{v_0}\}$ $0\leq i\leq m(v,\xi)$, $v\in\eta$.*

We denote the dimension m of $f^{-1}(q)\cap\xi$ by $d_f(\xi)$. Since $f|\xi\rightarrow\eta$ is onto and q is an interior point. Proposition 1 implies;

Proposition 2.

$$d_f(\xi)=\sum_{v\in\eta} m(v,\xi)=dim\ \xi-dim\ \eta\leqq \max_{q\in|H|} dim\ f^{-1}(q).$$

Under the assumption of Proposition 1, if $h:K\rightarrow E^n$ is a non-degenerate SS-map of K into E^n, $h|\xi\rightarrow E^n$ is a non-degenerate and linear. Therefore the following is clear;

Proposition 1′. *$h(f^{-1}(q)\cap\xi)$ is a polyhedral convex $d_f(\xi)$-cell which is the intersection of $h(\xi)$ and a hyper plane parallel to the hyper plane $\tilde{E}^{d_f(\xi)}$. spanned by vectors*

$$\{h(a_{v_i})-h(a_{v_0})\},\quad 0\leq i\leq m(v,\xi),\quad v\in\eta.$$

A pair (a,b) is a set of two points a and b satisfying $(a,b)=(b,a)$. a and b are called *vertices of the pair* (a,b). *A* set of pairs $S=\{(a_1,b_1),(a_2,b_2),\cdots,(a_k,b_k)\}$ is *cyclic* if $b_1=a_2, b_2=a_3,\cdots,b_k=a_1$. *A* finite set T of points of E^n is *in pairwisely general position*, if, for any set S of pairs of points in T such that

1) S does not contain cyclic subset,
2) the number $N(S)$ of pairs of $S\leqq n$,

the set of vectors $\{b-a \mid (a,b)\in S\}$ is linearly independent.

Proposition 3. *If S is a set of pairs of points of E^n such that*

1) *S does not contain cyclic subset,*
2) *$N(S)\leqq n$*

and if $V(S)$ is the set of vertices of pairs of S. Then for any $\varepsilon>0$ there is a map $h|V(S)\rightarrow E^n$ such that

$$d(h,1)<\varepsilon$$

and $\{h(b)-h(a)|(a,b)\in S\}$ is linearly independent.

Proof. We can order S in the order $\{(a_1,b_1),(a_2,b_2),\cdots,(a_k,b_k)\}$ so that $b_i\notin\{a_1,b_1,a_2,b_2,\cdots,b_{i-1},a_i\}$, $i=1,2,\cdots;k$. We can inductively construct $h:V(S)\rightarrow E^n$ such that

i) $d(h,1)<\varepsilon$
ii) the vector $h(b_i)-h(a_i)$ is not contained the vector space spanned by vectors

$\{h(b_1)-h(a_1), \cdots, h(b_{i-1})-h(a_{i-1})\}$.

If a finite set T' of E^n is in pairwisely general position, there is a $\delta > 0$ such that for any map $h: T' \to E^n$, satisfiymg $d(h, 1) < \delta$, $h(T')$ is in pairwisely general position. Therefore by Proposition 3 it is easy to prove.

Proposition 4. *If T is a finite set of points of E^n. For any $\varepsilon > 0$ there is a map $h: T \to E^n$ such that*

$$d(h, 1) < \varepsilon$$

and $h(T)$ is in pairwisely general position.

Let η be a simplex of H and ξ, ξ' be simplexes of K such that $f(\xi)=f(\xi')=\eta$. Then we can write

$$\eta=[v],$$
$$\xi=[a_{v_i}], \qquad 0 \leqq i \leqq m(v, \xi)$$
$$\xi'=[a'_{v_i}], \qquad 0 \leqq i \leqq m(v, \xi')$$

where $f(a_{v_i})=v$, $f(a'_{v_i})=v$. Moreover we may assume that $\xi \cap \xi'=[a''_{v_i}]$, $a''_{v_i}=a_{v_i}=a_{v_i}$, $0 \leqq i \leqq m(v, \xi \cap \xi')$

Let $A_v(\xi)=\{(a_{v_0}, a_{v_i})\} \quad 0 \leqq i \leqq m(v, \xi)$,

$A_v(\xi')=\{(a'_{v_0}, a'_{v_i})\} \quad 0 \leqq i \leqq m(v, \xi')$ and

$$A_v(\xi, \xi')=\begin{cases} A_v(\xi) \cup A_v(\xi)' & \text{if } a_{v_0}=a'_{v_0}. \\ A_v(\xi) \cup (a_{v_0}, a'_{v_0}) \cup A_v(\xi') & \text{if } a_{v_0} \neq a'_{v_0}. \end{cases}$$

It is clear that $A_v(\xi, \xi')$ does not contain any cyclic subset and $VA_v(\xi, \xi') \cap VA_{v'}(\xi, \xi')=\phi$ if $v \neq v' \in \eta$.

Proposition 5. *$A(\xi, \xi')=\bigcup_{v \in \eta} A_v(\xi, \xi')$ has no cyclic subset and the number $NA(\xi, \xi')$ of vertices of $A(\xi, \xi')$ is less than or equal to n.*

Proof. Since $A_v(\xi, \xi')$ has no cyclic subset. $A(\xi, \xi')$ has no cyclic subset.

$$\begin{aligned} NA(\xi, \xi') &= \sum_{v \in \eta} NA_v(\xi, \xi') \\ &\leq \sum_{v \in \eta} (NA_v(\xi)+NA_v(\xi')+1) \\ &= \sum_{v \in \eta} m(v, \xi)+\sum_{v \in \eta} m(v, \xi')+dim\ \eta+1 \\ &= d_f(\xi)+d_f(\xi')+dim\ \eta+1 \\ &\leqq 2\max_{q \in |H|} dim\ f^{-1}(q)+dim\ H+1 \leqq n. \end{aligned}$$

Lemma 1. *For any $\varepsilon>0$ there is a an SS-map $h: K \to E^n$ such that $d(h, g)<\varepsilon$ and $h|f^{-1}(q) \to E^n$ is a homeomorphism for any $q \in |H|$.*

Proof. From Proposition 4 we can choose $h: K \to E^n$ such that $d(h, g)<\varepsilon$ and $\{h(a) \mid a \in VK=K^0\}$ is in pairwisely general position. Since we have

$$n>dim\ H+2 \max_{q\in |H|} dim\ f^{-1}(q)$$
$$\geqq dim\ H+ \max_{q\in |H|} dim\ f^{-1}(q) \geqq dim\ K.$$

h is a non-degenerate map. Let q be any point of $|H|$. Then there is a simplex η of H such that $q\in\mathring{\eta}$. $h|f^{-1}(q)\cap\xi$ is a linear homeomorphism for any $\xi\in K$. Then it is sufficient to show that

$$h(f^{1}(q)\cap\xi)\cap h(f^{-1}(q)\cap\xi')=h(f^{-1}(q)\cap\xi\cap\xi')$$

for any $\xi,\xi'\in K$ such that $f(\xi)=f(\xi')=\eta$. Let

$$q=\sum_{v\in\epsilon}\mu_v v,\ \sum_{v\in\eta}\mu_v=1\ \mu_v>0 \text{ and let } p\in f^{-1}(q)\cap\xi \text{ and } p'=f^{-1}(q)\cap\xi'.$$

Then we have the following;

$$p=\sum_{v\in\eta}\Sigma\{\lambda_{v_i}a_{v_i}|\ 0\leqq i\leqq m(v,\xi)\}$$
$$p'=\sum_{v\in\eta}\Sigma\{\lambda'_{v_i}a'_{v_i}|0\leqq i\leqq m(v,\xi')\}$$
$$\sum_{v\in\eta}\Sigma\{\lambda_{v_i}|0\leqq i\leqq m(v,\xi)\}=\sum_{v\in\eta}\Sigma\{\lambda'_{v_i}|0\leqq i\leqq m(v,\xi')\}=1$$
$$\lambda_{v_i}\geqq 0\quad \lambda'_{v_i}\geqq 0 \quad\text{and}$$
$$\Sigma\{\lambda_{v_i}|0\leqq i\leqq m(v,\xi)\}=\Sigma\{\lambda'_{v_i}|0\leqq i\leqq m(v,\xi')\}=\mu_v.$$

Put $r=\sum_{v\in\eta}\mu_v a_{v_0}$ and $r'=\sum_{v\in\eta}\mu_v a'_{v_0}$. Then we have

$$h(p)-h(r)=\sum_{v\in\eta}\Sigma\{\lambda_{v_i}(h(a_{v_i})-h(a_{v_0}))|1\leqq i\leqq m(v,\xi)\}$$
$$h(p')-h(r')=\sum_{v\in\eta}\Sigma\{\lambda'_{v_i}(h(a'_{v_i})-h(a'_{v_0}))|1\leqq i\leqq m(v,\xi')\}$$

Therefore if $h(p)=h(p')$,

$$0=h(p')-h(p)=(h(p')-h(r'))-(h(p)-h(r))+(h(r')-h(r))$$
$$=\sum_{v\in\eta}(\sum_i\lambda_{vi}(h(a'_{v_i})-h(a'_{v_0}))-\sum_i\lambda_{v_i}(h(a_{v_i})-h(a_{v_0}))+\mu_v(h(a'_{v_0})-h(a_{v_0})))$$

By Proposition 5 the set of vectors $\{h(b)-h(a)|(b,a)\in A(\xi,\xi')\}$ is linearly independent. Therefore $h(p')-h(r')=h(p)-h(r)=h(r')-h(r)=0$. and consequently $h(p)=h(r)=h(r')=h(p')$. Furthermore $h|\xi, h|\xi\cap\xi', h|\xi'$ are one-to-one, then $p=r=r'=p'$. Thus $h(f^{-1}(q)\cap\xi)\cap h(f^{-1}(q\cap\xi')=h(f^{-1}(q)\cap\xi\cap\xi')$ and we have proved Lemma 1.

We denote the origin of E^n by o.

Proposition 6. *If $h:K\to E^n$ is an SS-map such that $o\cup h(VK)$ is in pairwisely general position. Then for any simplexes ξ,ξ' of K and any point q of $f(\xi)$ such that $h(f^{-1}(q)\cap\xi)\cap h(\xi')\neq\phi$, there is one and only one q' of $f(\xi')$ such that*

$$h(f^{-1}(q)\cap\xi)\cap h(\xi')=h(f^{-1}(q)\cap\xi)\cap h(f^{-1}(q')\cap\xi')$$

and $h(f^{-1}(q)\cap\xi)\cap h(\xi')$ is a polyhedral convex cell of dimension $\leqq d_f(\xi\cap\xi')$,

Proof. Let $f(\xi)=\eta=[v]$, $f(\xi')=\eta'=[v']$, $\xi=[a_{v_i}]$, $0\leqq i\leqq m(v,\xi)$, $\xi'=[a'_{v'_i}]$, $0\leqq i\leqq m(v',\xi')$, $\xi\cap\xi'=[a''_{v''_i}]$, $0\leqq i\leqq m(v'',\xi\cap\xi')$ where

$$f(a_{vi})=v',\quad f(a'_{v'_i})=v',\quad f(a''_{v''_i})=v''$$

$$a''_{v''_i}=a_{v_i}=a'_{v'_i} \quad \text{for} \quad v''=v=v' \quad 0\leqq i\leqq m(v'', \xi\cap\xi').$$

If $q=\sum_{v\epsilon\eta}\mu_v v, \quad \sum_{v\epsilon\eta}\mu_v=1, \quad \mu_v\geqq 0$ and $p, \tilde{p}$ are points of $f^{-1}(q)\cap\xi$ and $p', \tilde{p}'$ are points of ξ' such that

$$h(p)=h(p'), \qquad h(\tilde{p})=h(\tilde{p}').$$

Then we have

$$p=\sum_{v\epsilon\eta}\sum_i\lambda_{v_i}a_{v_i}, \quad \sum_{v\epsilon\eta}\sum_i\lambda_{v_i}=1, \quad \lambda_{vi}\geqq 0$$

$$\tilde{p}=\sum_{v\epsilon\eta}\sum_i\tilde{\lambda}_{v_i}a_{v_i}, \quad \sum_{v\epsilon\eta}\sum_i\tilde{\lambda}_{v_i}=1, \quad \tilde{\lambda}_{v_i}\geqq 0$$

$$p'=\sum_{v'\epsilon\eta'}\sum_i\lambda'_{v'_i}a_{v'_i}, \quad \sum_{v\epsilon\eta'}\sum_i\lambda'_{v'_i}=1, \quad \lambda'_{v'_i}\geqq 0$$

$$\tilde{p}'=\sum_{v'\epsilon\eta'}\sum_i\tilde{\lambda}'_{v'_i}a'_{v'_i}, \quad \sum_{v\epsilon\eta'}\sum_i\tilde{\lambda}'_{v'_i}=1, \quad \tilde{\lambda}_{v'_i}\geqq 0.$$

Since $\sum_{0\leq i}\lambda_{v_i}=\mu_v=\sum_{0\leq i}\tilde{\lambda}_{v_i}, \quad v\epsilon\eta$. We have

$$p=\sum_{v\epsilon\eta}\mu_v a_{v_0}+\sum_{v\epsilon\eta}\sum_{1\leq i}\lambda_{v_i}(a_{v_i}-a_{v0})$$

$$\tilde{p}=\sum_{v\epsilon\eta}\mu_v a_{v_0}+\sum_{v\epsilon\eta}\sum_{1\leq i}\tilde{\lambda}_{v_i}(a_{v_i}-a_{v_0})$$

Let $\mu'_{v'}=\sum_{0\leq i}\lambda'_{v'_i}, \quad \tilde{\mu}'_{v'}=\sum_{\leq i}\tilde{\lambda}'_{v'_i} \quad v'\epsilon\eta'$.

Then $p'=\sum_{v'\eta}\mu'_{v'}a'_{v'_0}+\sum_{v'\epsilon\eta'}\sum_{1\leqq i}\lambda'_{v'_i}(a'_{v'i}-a'_{v'_0})$

$$\tilde{p}'=\sum_{v'n'}\tilde{\mu}_{v'}a'_{v'_0}+\sum_{v'\epsilon\eta'}\sum_{1\leq i}\tilde{\lambda}'_{v'_i}(a'_{v_i}-a'_{v_0}).$$

Therefore

$$h(\tilde{p})-h(p)=\sum_{v\epsilon\eta}\sum_{1\leq i}(\tilde{\lambda}_{v_i}-\lambda_{v_i})\ (h(a_{v_i})-h(a_{v_0}))$$

$$h(\tilde{p}')-h(p')=\sum_{v\epsilon\eta'}(\tilde{\mu}'_{v'}-\mu'_{v'})\,h(a'_{v'_0})$$
$$+\sum_{v'\epsilon\eta'}\sum_{1\leq i}(\tilde{\lambda}'_{v'_i}-\lambda'_{v'_i})(h(a'_{v'_i})-h(a'_{v'_0})).$$

The set of pairs $\Delta=\{(h(a_{v_0}), h(a_{v_i}))\}\cup\{(h(a'_{v'_0}), h(a'_{v'_i}))\}\cup\{o, h(a'_{v'_0}))\}$ has no cyclic subset and moreover

$$N\Delta=\sum_{v\epsilon\eta}m(v,\xi)+\sum_{v'\epsilon\eta'}m(v',\xi')+dim\ \eta'+1$$
$$\leq 2\max_{q\epsilon|H|}dim\ f^{-1}(q)+dim\ H+1\leqq n.$$

Then the set of vectors $\{h(a_{v_i})-h(a_{v_0})\}\cup\{h(a'_{v'_i})-h(a'_{v'_0})\}\cup\{h(a'_{v'_0})\}$ is linearly independent. Hence by the conditions $h(p)=h(p')$ and $h(\tilde{p})=h(\tilde{p}')$ we have formulas

$$\mu'_{v'}=\tilde{\mu}'_{v'}, \quad v'\epsilon\eta',$$

$$\tilde{\lambda}_{v_i}-\lambda_{v_i}=\tilde{\lambda}'_{v'_i}-\lambda'_{v'_i} \quad \text{if} \quad v''=v=v' \quad 0\leq i\leq m(v'', \xi\cap\xi'),$$

$\tilde{\lambda}_{v_{\bar{i}}} - \lambda_{v_i} = 0 \quad \tilde{\lambda}'_{v'_i} - \lambda'_{v'_i} = 0$ if otherwise.

Let $q' = \sum_{v' \in \eta'} \mu'_{v'} v'$. Then it is clear that

$$h(f^{-1}(q) \cap \xi) \cap h(\xi') = h(f^{-1}(q) \cap \xi) \cap h(f^{-1}(q') \cap \xi').$$

and this is a polyhedral convex cell in a hyper plane parallel to $E^{a_f(\xi \cap \xi')}$ spanned by vectors $\{h(a''_{v''_i}) - h(a''_{v''_0})\}$ $v'' \in \xi \cap \xi'$, $1 \leq i \leq m(v'', \xi \cap \xi')$. We have proved Proposition 6.

§2. Let ξ be any simplex and v be any vertex of H. Then we denote the simplex $f^{-1}(v) \cap \xi$ by ξ_v. Let ξ and ξ' be any simplex of K such that $f(\xi) = f(\xi')$. Then we denote the number $NV(\xi_v \cap \xi'_v)$ of vertices $V(\xi_v \cap \xi'_v)$ by $\alpha_v(\xi, \xi')$ and $\sum_{v \in VH} \alpha_v(\xi, \xi')$ by $\alpha(\xi, \xi')$. If $\xi_1, \xi_1', \xi_2, \xi_2'$ are simplexes of K such that $\eta_i = f(\xi_i) = f(\xi'_i)$, $i = 1, 2$, we denote $\alpha_v(\xi_1, \xi'_1) + \alpha_v(\xi_2, \xi'_2)$ by $\alpha_v(\xi_1, \xi_1' : \xi_2, \xi_2')$ and $\sum_{v \in VH} \alpha_v(\xi_1, \xi_1' : \xi_2, \xi_2')$ by $\alpha(\xi_1, \xi_1' : \xi_2, \xi_2')$. We define the number $\beta_v = \beta_v(\xi_1, \xi_1' : \xi_2, \xi_2')$ by the following formulas;

$$\beta_v = 0, \qquad \text{if } (\xi_{1v} \cup \xi'_{1v}) \cap (\xi_{2v} \cup \xi'_{2v}) = \phi$$

$$\beta_v = NV((\xi_{1v} \cup \xi'_{1v}) \cap (\xi_{2v} \cup \xi'_{2v})) - 1 \quad \text{if } (\xi_{1v} \cup \xi'_{1v}) \cap (\xi_{2v} \cup \xi'_{2v}) \neq \phi.$$

Put $\beta(\xi_1, \xi'_1 : \xi_2, \xi'_2) = \sum_{v \in VH} \beta_v(\xi_1, \xi'_1 : \xi_2, \xi'_2)$.

Let

$$\xi_{1v} = [a_{v_j}], \qquad 0 \leq j \leq m(v, \xi_1),$$

$$\xi'_{1v} = [a'_{v_j}], \qquad 0 \leq j \leq m(v, \xi'_1)$$

$$\xi_{2v} = [b_{v_j}], \qquad 0 \leq j \leq m(v, \xi_2)$$

$$\xi'_{2v} = [b'_{v_j}], \qquad 0 \leq j \leq m(v, \xi'_2)$$

$$\xi_{1v} \cap \xi'_{1v} = [a''_{v_j}], \qquad 0 \leq j \leq \alpha_v(\xi_1, \xi'_1) - 1$$

$$\xi_{2v} \cap \xi'_{2v} = [b''_{v_j}], \qquad 0 \leq j \leq \alpha_v(\xi_2, \xi'_2) - 1,$$

where

$$a''_{v_j} = a_{v_j} = a'_{v_j}, \qquad 0 \leq j \leq \alpha_v(\xi_1, \xi'_1) - 1$$

$$b''_{v_j} = b_{v_j} = b'_{v_j}, \qquad 0 \leq j \leq \alpha_v(\xi_2, \xi'_2) - 1.$$

A vertex v of $\xi_1, \xi'_1, \xi_2, \xi'_2$ is *free* if it is contained in only one of $\xi_1, \xi'_1, \xi_2, \xi'_2$. A pair x of $A(\xi_1, \xi'_1)$ and $A(\xi_2, \xi'_2)$ is *free* if at least one vertex of x is free. Hereafter we shall assume that a_{v_0} *(similarly* $a'_{v_0}, b_{v_0}, b'_{v_0}$*) is free if and only if all vertices of* ξ_{1v} *(*$\xi'_{1v}, \xi_{2v}, \xi'_{2v}$ *respectively) are free. A* pair x is *linearly dependent on a set* Γ of pairs if $x \cup \Gamma$ is cyclic.

Proposition 7. $NA(\xi_i, \xi_i') \leq n - \alpha(\xi_i, \xi'_i)$, $i = 1, 2$.

Proof. Let $NA(\xi_i, \xi'_i)$ is the number of pairs of $A(\xi_i, \xi_i')$, $i = 1, 2$. Then

$$NA(\xi_i, \xi'_i) = \sum_{v \in \eta_i} NA_v(\xi_i, \xi'_i)$$

$$= \sum_{v \in \eta_i} (dim\ \xi_{iv} + dim\ \xi'_{iv} - dim\ (\xi_{iv} \cap \xi'_{iv}))$$

$$= \sum_{v \in \eta_i} (dim\ \xi_{iv} + dim\ \xi'_{iv} - \alpha_v(\xi_i, \xi'_i) + 1)$$

$$= d_f(\xi_i) + d_f(\xi'_i) - \alpha(\xi_i, \xi'_i) + dim\ \eta_i + 1$$

$$\leq 2 \max_{q\in|H|} dim\ f^{-1}(q)+dim\ H+1-\alpha(\xi_i, \xi'_i)\leqq n-\alpha(\xi_i, \xi'_i).$$

Proposition 8. *There are non-free pairs $x_1, x_2, \cdots, x_{\beta v}$ of $A_v(\xi_1, \xi'_1)$ such that $(A_v(\xi_1, \xi'_1)-x_1\cup x_2\cup\cdots\cup x_{\beta v})\cup A_v(\xi_2, \xi'_2)$ has no cyclic subset and any x_i, $i=1, 2, \cdots, \beta_v$, is linearly dependent on non-free pairs of $A_v(\xi_1, \xi'_1)\cup A_v(\xi_2, \xi'_2)$*

Proof. In the case that $\beta_v=0$. $A_v(\xi_i, \xi'_i)$, $i=1, 2$, has no cyclic subset and $A_v(\xi_1, \xi'_1)$ and $A_v(\xi_2, \xi'_2)$ have at most one common vertex. Then $A_v(\xi_1, \xi'_1)\cup A_v(\xi_2, \xi'_2)$ has no cyclic subset. In the case that $\beta_v=1$ let $a\cup a'=VA_v(\xi_1, \xi'_1)\cap VA_v(\xi_2, \xi'_2)$ $(a\neq a')$. 1) If $a\cup a'=a_{v_0}\cup a'_{v_0}$, put $x_1=(a_{v_0}, a'_{v_0})$. 2) If $a\cup a'\neq a_{v_0}\cup a'_{v_0}$. We can assume without loss of generality that $a=a_{v_i}$ $(i\neq 0)$. Let $x_1=(a_{v_0}, a_{v_i})$. From the condition that if a_{v_0} is free all vertices of ξ_{1v} are free, x_1 is not free and $a\cup a'=a_{v_0}\cup a_{v_i}$. Then there is a cycle consisting of x_1 and non-free pairs of $A_v(\xi_1, \xi'_1)\cup A_v(\xi_2, \xi'_2)$. In the case that $\beta_v\geq 2$ there is a common vertex a of $A_v(\xi_1, \xi'_1)$ and $A_v(\xi_2, \xi'_2)$ such that $a\neq a_{v_0}, a'_{v_0}$. Suppose $a=a_{v_i}$ $(i\neq 0)$ and put $x_1=(a_{v_0}, a_{v_i})$, Then the number $NV(A_v(\xi_1, \xi'_1)-x_1)\cap A_v(\xi_2, \xi'_2))$ of common vertices of $A_v(\xi_1, \xi'_1)-x_1$ and $A_v(\xi_2, \xi'_2)$ is β_v-1. Inductively we can choose $x_2, \cdots, x_{\beta v-1}$ as similarly as x_1 in 2) and choose $x_{\beta v}$ as similarly as x_1 in 1). We have proved Proposition 8.

Let $\tilde{A}_v(\xi_1, \xi'_1:\xi_2, \xi'_2)=(A_v(\xi_1, \xi'_1)-x_1\cup\cdots\cup x_{\beta_v})\cup A_v(\xi_2, \xi'_2)$ and $\tilde{A}(\xi_1, \xi'_1:\xi_2, \xi'_2) = \bigcup_{v\in VH}\tilde{A}_v(\xi_1, \xi'_1:\xi_2, \xi'_2))$, where if $v\notin\eta_1$ $A_v(\xi_1, \xi'_1)-x_1\cup\cdots\cup x_{\beta_v}=\phi$ *and* if $v\notin\eta_2$ $A_v(\xi_2, \xi'_2)=\phi$. Then Proposition 7 and Proposition 8 implies the following;

Proposition 9. *$N\tilde{A}(\xi_1, \xi'_1:\xi_2, \xi'_2)\leqq 2n-\alpha(\xi_1, \xi'_1:\xi_2, \xi'_2)-\beta(\xi_1, \xi'_1:\xi_2, \xi'_2)$.*

If $h:K\rightarrow E^n$ is an *SS-map* and p_1, p'_1, p_2, p'_2 are points in $\xi_1, \xi'_1, \xi_2, \xi'_2$ respectively such that

$$f(p_1)=f(p'_1)=q_1=\sum_{v\in\eta_1}\lambda_v v$$
$$f(p_2)=f(p'_2)=q_2=\sum_{v\in\eta_2}\mu_v v$$
$$h(p_1)=h(p_2),\ h(p'_1)=h(p'_2).$$

Let

$$p_1=\sum_{v\in\eta_1}\Sigma\{\lambda_{v_i} a_{v_i}\mid 0\leqq i\leqq m(v, \xi_1)\} \quad (1)$$
$$p'_1=\sum_{v\in\eta_1}\Sigma\{\lambda'_{v_i} a'_{v_i}\mid 0\leq i\leq m(v_1\xi'_1)\} \quad (1')$$
$$p_2=\sum_{v\in\eta_2}\Sigma\{(\mu_{v_i} b_{v_i}\mid 0\leq i\leqq m(v, \xi_2)\} \quad (2)$$
$$p'_2=\sum_{v\in\eta_2}\Sigma\{(\mu'_{v_i} b'_{v_i}\mid 0\leq i\leq m(v, \xi'_2)\} \quad (2')$$

Then from $h(p_1)=h(p_2)$ and $h(p'_1)=h(p'_2)$ we have.

$$\sum_{v\in\eta_1}(\lambda_v h(a_{v_0})+\sum_{1\leq i}\lambda_{v_i}(h(a_{v_i})-h(a_{v_0})))$$
$$=\sum_{v\in\eta_2}(\mu_v h(b_{v_0})+\sum_{1\leq i}\mu_{v_i}(h(b_{v_i})-h(b_{v_0}))) \qquad (1'')$$
$$\sum_{v\in\eta_1}(\lambda_v(h(a'_{v_0})+\sum_{1\leq i}\lambda'_{v_i}(h(a'_{v_i})-h(a'_{v_0})))$$
$$=\sum_{v\in\eta_1}(\mu_v h(b'_{v_0})+\sum_{1\leq i}\mu'_{v_i}(h(b'_{v_i})-h(b'_{v_0}))) \qquad (2'')$$

By $(1'')-(2'')$ we have

$$\sum_{v\in\eta_2}(\lambda_v(h(a_{v_0})-h(a'_{v_0}))+\sum_{1\leq i}\lambda_{v_i}(h(a_{v_i})-h(a_{v_0}))-\sum_{1\leq i}\lambda'_{v_i}h(a'_{v_i})-h(a'_{v_0})))$$
$$=\sum_{v\in\eta_2}(\mu_v(h(b_{v_0})-h(b'_{v_0}))+\sum_{1\leq i}\mu_{v_i}(h(b_{v_i})-h(b_{v_0}))-\sum_{1\leq i}\mu'_{v_i}(h(b'_{v_i})-h(b'_{v_0}))). \qquad (3)$$

Lemma 2. *If $h:K\to E^n$ is in pairwisely genera l position and $\alpha(\xi_1,\xi_1:\xi_2,\xi'_2)+\beta(\xi_1\xi'_1,:\xi_2,\xi'_2)\geqq n$. Then the coefficients of free vertices in* (1), (1′), (2), (2′) *are* 0.

Proof. By Proposition 9

$$NA(\xi_1,\xi_2,:\xi'_1,\xi'_2)\leqq 2n-\alpha(\xi_1,\xi'_1:\xi_2,\xi'_2)-\beta(\xi_1,\xi'_1,\xi_2,\xi'_2)\leqq n.$$

Then the vectors in (3) corresponing to $\tilde{A}(\xi_1,\xi'_1:\xi_2,\xi'_2)$ are linearly independent and any vector in (3) corresponding to $\bigcup_{v\in VH}\{x_1,\cdots,x_{\beta v}\}$ is linearly dependent on vectors corresponding to non-free pairs of $A(\xi_1,\xi'_1)\cup A(\xi_2,\xi'_2)$. Therefore the coefficients of free vertices in (1), (1′), (2), (2′) are 0. We have proved Lemma 2.

§3 Let $a=\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}$ and $b=\begin{pmatrix}b_1\\b_2\\\vdots\\b_n\end{pmatrix}$ be n-dimensional vectors. Then the $2n$-denensional vector $\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}a_1\\\vdots\\a_n\\b_1\\\vdots\\b_n\end{pmatrix}$ is called *a vector of type* 0) with *top vector a* and *bottom vector b.*

If $a=b$, $b=0$, $a=0$, we say that the vector $\begin{pmatrix}a\\b\end{pmatrix}$ has *type* 1), *type* 2), *type* 2′) respectively. A pair of vectors, $\left\{\begin{pmatrix}a\\0\end{pmatrix},\begin{pmatrix}0\\a\end{pmatrix}\right\}$ is called *a pair of type* 3).

Proposition 10. *If $A=(a_{ij})$ is a $(2n,r)$ matrix, which consists of $N(i)$ vectors of type i), $i=0,1,2,2'$, and $N(3)$ pairs of type* 3) *such that*

$$N(0)+N(1)+N(2)+N(2')+2N(3)=r\leqq 2n$$
$$N(1)+N(3)\leq n$$
$$N(2)+N(3)\leq n$$
$$N(2')+N(3)\leq n$$

Then for any $\varepsilon>0$ *there is a* $(2n, r)$ *matrix* $A'=(a'_{ij})$ *consisting of vectors and pairs with the same types and the same numbers as* $A=(a_{ij})$ *such that*

1) *column vectors of* A' *are linearly independent.*

2) $|a_{ij}-a'_{ij}|<\varepsilon$.

Proof. Any minor determinant of A is an analytic function of $\{a_{ij}\}$. Then it is sufficient to show the existence of such an A' satisfymg only the condition 1). Furthermore the vectors of type 0) can be arbitrarily moved. Then it is sufficient to prove the existence of linearly independent vectors of types 1), 2), 2') and pairs of type 3) with number $N(1)$, $N(2)$, $N(2')$ and $N(3)$ respectively. We denote by s_i the n-dimensional vector whose i-th element is 1 and whose other elements are 0. If

$N(1)+N(2)+N(3)\leq n$ and $N(1)+N(2')+N(3)\leq n$.

Let $E_{1i}=\binom{s_i}{s_i}$ $\quad 1\leq i\leq N(1)$,

$E_{2j}=\binom{s_j}{0}$ $\quad N(1)<j\leq N(1)+N(2)+N(3)$,

$E_{2'j'}=\binom{0}{s_{j'}}$ $\quad N(1)<j'\leq N(1)+N(2')+N(3)$.

Then $\{E_{1i}, E_{2j}, E_{2'j'}\}$ are the required vectors.

If $N(1)+N(2)+N(3)>n$.

Let $E_{1i}=\binom{s_i}{s_i}$ $\quad 0\leq i\leq N(1)+N(2)+N(3)-n$,

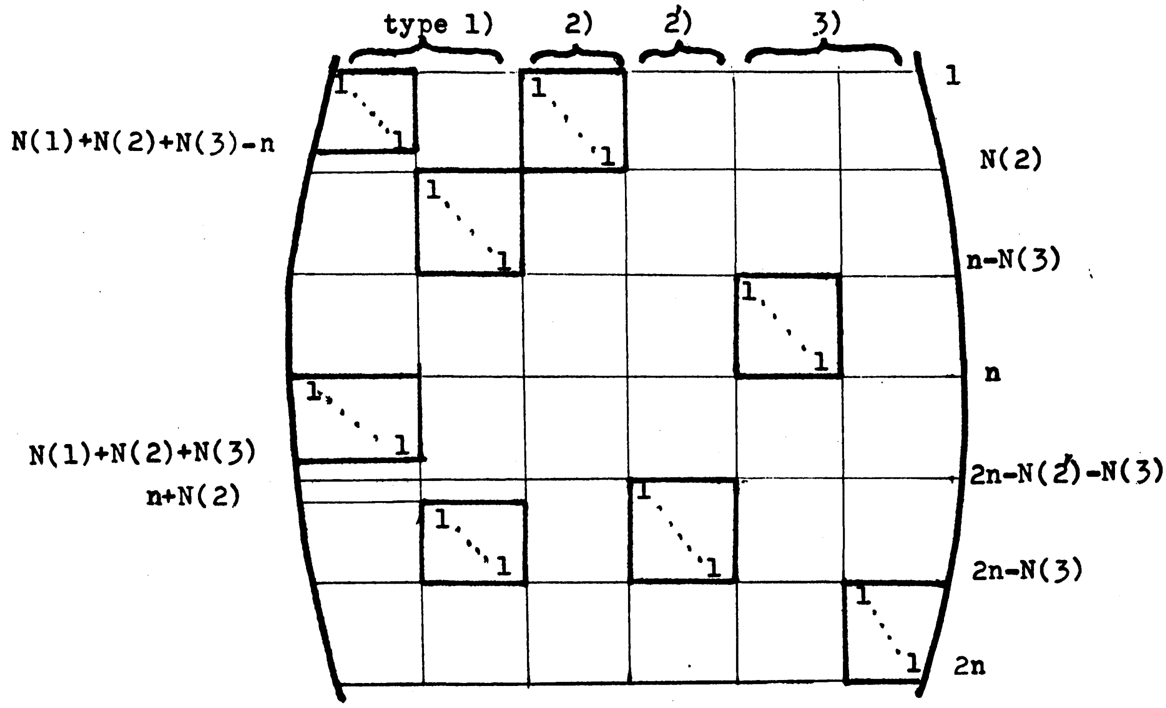
$E_{1'i'}=\binom{s_{i'}}{s_{i'}}$ $\quad N(2)<i'\leq n-N(3)$,

$E_{2j}=\binom{s_j}{0}$ $\quad 0\leq j\leq N(2)$

$E_{2'j'}=\binom{0}{s_{j'}}$ $\quad n-N(2')-N(3)<j'\leq n-N(3)$

$E_{3k}=\binom{s_k}{0}$ $\quad n-N(3)<k\leq n$

$E_{3'k'}=\binom{0}{s_k}$ $\quad n-N(3)<k'\leq n$.



Then $\{E_{1i}, E_{1'i'}, E_{2j}, E_{2'j'}, E_{3k}, E_{3'k'}\}$ are the required linearly independent vectors. Similarly we can prove it in the case that $N(1)+N(2')+N(3)>n$. Therefore we have proved Proposition 10.

We can restate the formulas (1″) and (2″) as follows:

$$\sum_{v\in\eta_1}\left(\lambda_v\begin{pmatrix}h(a_{v_0})\\ h(a'_{v_0})\end{pmatrix}+\sum_{1\leq i}\lambda_{v_i}\begin{pmatrix}h(a_{v_i})-h(a_{v_0})\\ 0\end{pmatrix}+\sum_{1\leq i}\lambda'_{v_i}\begin{pmatrix}0\\ h(a'_{v_i})-ha'_{v_0}\end{pmatrix}\right)$$

$$=\sum_{v\in\eta_2}\left(\mu_v\begin{pmatrix}h(b_{v_0})\\ h(b'_{v_0})\end{pmatrix}+\sum_{1\leq i}\mu_{v_i}\begin{pmatrix}h(b_{v_i})-h(b_{v_0})\\ 0\end{pmatrix}+\sum_{1\leq i}\mu'_{v_i}\begin{pmatrix}0\\ h(b'_{v_i})-h(b_{v_0})\end{pmatrix}\right) \tag{4}$$

We denote by $A_{vh}(\xi_1, \xi'_1)$ and $A_{vh}(\xi_2, \xi'_2)$ the sets of $2n$-dimensional vectors

$$\left\{\begin{pmatrix}h(a_{v_0})\\ h(a'_{v_0})\end{pmatrix}, \begin{pmatrix}h(a_{v_i})-h(a_{v_0})\\ 0\end{pmatrix}, \begin{pmatrix}0\\ h(a'_{v_{i'}})-h(a'_{v_0})\end{pmatrix}\right\} \qquad \begin{matrix}1\leq i\leq m(v, \xi_1)\\ 1\leq i'\leq m(v, \xi'_1)\end{matrix}$$

$$\left\{\begin{pmatrix}h(b_{v_0})\\ h(b'_{v_0})\end{pmatrix}, \begin{pmatrix}h(b_{v_i})-h(b_{v_0})\\ 0\end{pmatrix}, \begin{pmatrix}0\\ h(b''_{v_{i'}})-h(b'_{v_0})\end{pmatrix}\right\} \qquad \begin{matrix}1\leq i\leq m(v, \xi_2)\\ 1\leq i'\leq m(v, \xi'_2)\end{matrix}$$

Vectors $\begin{pmatrix}h(a)\\ h(a')\end{pmatrix}, \begin{pmatrix}h(b)-h(b')\\ 0\end{pmatrix}, \begin{pmatrix}0\\ h(c)-h(c')\end{pmatrix}$ are *free*, if pairs $(a, a'), (b, b'), (c, c')$ are free respectively. We define the number $\gamma_v(\xi_1, \xi'_1, \xi_2, \xi'_2)$ by the formulas

$$\gamma_v=0, \qquad \text{if} \quad (\xi_{1v}\cap\xi_{2v})\cup(\xi_{1v}\cap\xi'_{2v})=\phi$$
$$\gamma_v=NV(\xi_{1v}\cap\xi_{2v})+NV(\xi'_{1v}\cap\xi'_{2v})-1, \quad \text{if} \quad (\xi_{1v}\cap\xi_{2v})\cup(\xi'_{1v}\cap\xi'_{2v})\neq\phi.$$

Then it is clear that $\gamma_v(\xi_1, \xi'_1 : \xi_2, \xi'_2) \leqq \beta_v(\xi_1, \xi_2 : \xi_2, \xi'_2)$.

Proposition 11. *There are $\gamma_v(\xi_1, \xi'_1, \xi_2, \xi'_2)$ non-free vectors $\{x_1, x_2, \cdots, x_{\gamma_v}\}$ of $A_{vh}(\xi_1, \xi'_1)$ such that x_i is linearly dependent on non-free vectors of $(A_{vh}(\xi_1, \xi'_1)-x_1\cup\cdots\cup x_{\gamma_v})\cup A_{vh}(\xi_2, \xi'_2)$.*

Proof. If $\gamma_v=0$ this proposition is trivial. If $\gamma_v=1$ then $(\xi_{1v}\cup\xi_{2v})\cup(\xi'_{1v}\cap\xi'_{2v})=a\cup a'$, $a\neq a'$.

1) If $a\cup a'=a_{v_0}\cup a'_{v_0}$. We can write

$$a=a_{v_0}=b_{v_j}, \qquad a'=a'_{v_0}=b'_{v_{j'}}.$$

Let $x_1=\begin{pmatrix} h(a_{v_0}) \\ h(a'_{v_0}) \end{pmatrix}$. Then we have

$$x_1=\begin{pmatrix} h(b_{v_j})-h(b_{v_0}) \\ 0 \end{pmatrix}+\begin{pmatrix} h(b_{v_0}) \\ h(b'_{v_0}) \end{pmatrix}+\begin{pmatrix} 0 \\ h(b'_{v_{j'}})-h(b'_{v_0}) \end{pmatrix}.$$

Since b_{v_j} and $b'_{v_{j'}}$ are non-free vertices. b_{v_0} and b'_{v_0} are non-free vertices and then x_1 is the sum of non-free vectors $\begin{pmatrix} h(b_{v_j})-h(b_{v_0}) \\ 0 \end{pmatrix}$, $\begin{pmatrix} h(b_{v_0}) \\ h(b'_{v_0}) \end{pmatrix}$, $\begin{pmatrix} 0 \\ h(b'_{v_{j'}})-h(b'_{v_0}) \end{pmatrix}$.

2) If $a\cup a'\neq a_{v_0}\cup a'_{v_0}$. Then we may assume $a=a_{v_i}=b_{v_j}$, $i\neq 0$.

Put $x_1=\begin{pmatrix} h(a_{v_i})-h(a_{v_0}) \\ 0 \end{pmatrix}$.

If $a'=a_{v\tilde{i}}=b_{v\tilde{j}}$, we have

$$x_1=\begin{pmatrix} h(b_{v_j})-h(b_{v_0}) \\ 0 \end{pmatrix}-\begin{pmatrix} h(b_{v\tilde{j}})-h(b_{v_0}) \\ 0 \end{pmatrix}+\begin{pmatrix} h(a_{v\tilde{i}})-h(a_{v_0}) \\ 0 \end{pmatrix}.$$

If $a'=a'_{v_{i'}}=b'_{vj'}$, we have

$$x_1=\begin{pmatrix} h_v(b_{v_j})-h(b_{v_0}) \\ 0 \end{pmatrix}+\begin{pmatrix} h(b_{v_0}) \\ h(b'_{v_0}) \end{pmatrix}+\begin{pmatrix} 0 \\ h(b'_{v_{j'}})-h(b'_{v_0}) \end{pmatrix}$$
$$-\begin{pmatrix} 0 \\ h(a'_{v_{i'}})-h(a'_{v_0}) \end{pmatrix}-\begin{pmatrix} h(a_{v_0}) \\ h(a'_{v_0}) \end{pmatrix}.$$

Therefore x_1 is the sum of non-free vectors of $(A_{vh}(\xi_1, \xi'_1)-x_1)\cap A_{vh}(\xi_2, \xi'_2)$. If $\gamma_v \geq 0$, we may choose $x_1, \cdots, x_{r_v-1}$ as similarly as x_1 in 2) and x_{r_v} as in 1) or 2). We complete the proof of Proposition 11.

We denote the set of above vectors by

$\tilde{A}_{vh}(\xi_1, \xi'_1) = A_{vh}(\xi_1, \xi'_1) - x_1 \cup x_2 \cup \cdots \cup x_{r_v}$ $\tilde{A}_{vh}(\xi_2, \xi'_2) = A_{vh}(\xi_2, \xi'_2)$

$\tilde{A}_{vh}(\xi_1, \xi'_1 : \xi_2, \xi'_2) = \tilde{A}_{vh}(\xi_1, \xi'_1) \cup \tilde{A}_{vh}(\xi_2, \xi'_2)$.

Lemma 3 *If $h: K \to E^n$ is an SS-map and*

$$\alpha(\xi_1, \xi'_1 : \xi_2, \xi'_2) + \beta(\xi_1, \xi'_1,' : \xi_2, \xi'_2) \leq n.$$

Then for any $\varepsilon > 0$. there is an SS-map. $h' : K \to E^n$ such that $d(h, h') < \varepsilon$ and $\bigcup_{v \in \eta_1 \cup \eta_2} \tilde{A}_{vh'}(\xi_1, \xi'_1 : \xi_2, \xi'_2)$ are linearly independent. Consequently the coefficients of free veetors in (1), (1′), (2), (2′) *are* 0.

Switch of base points. If $\xi_{1v} \cap \xi_{2v} \neq \phi$. There is a vertex $a = a_{v_i} = b_{v_j}$. Define a set of vectors $A'_{vh}(\xi_1, \xi'_1)$: as follows;

If $i=0$. $A'_{vh}(\xi_1, \xi'_1) = \tilde{A}_{vh}(\xi_1, \xi'_1)$.

If $i \neq 0$. We change the vectors of $\tilde{A}_{vh}(\xi_1, \xi_1')$ by.

$$x = \begin{pmatrix} h(a_{v_i}) - h(a_{v_0}) \\ 0 \end{pmatrix} \to \quad -x = \begin{pmatrix} h(a_{v_0}) - h(a_{v_i}) \\ 0 \end{pmatrix}$$

$$y = \begin{pmatrix} h(a_{v_0}) \\ h(a'_{v_0}) \end{pmatrix} \to \quad y + x = \begin{pmatrix} h(a_{v_i}) \\ h(a'_{v_0}) \end{pmatrix}$$

$$z = \begin{pmatrix} h(a_{v_{i'}}) - h(a_{v_0}) \\ 0 \end{pmatrix} \to \quad z - x = \begin{pmatrix} h(a_{v_{i'}}) - h(a_{v_i}) \\ 0 \end{pmatrix}.$$

By the above modification we get $A'_{vh}(\xi_1, \xi_1')$ such that $A'_{vh}(\xi_1, \xi'_1)$ is equivalent to $\tilde{A}_{vh}(\xi_1, \xi'_1)$ and the base point of top vectors of $A'_{vh}(\xi_1, \xi'_1)$ is $h(a) = h(a_{v_i})$. Similarly we get $A'_{vh}(\xi_2, \xi'_2)$ such that $A'_{vh}(\xi_2, \xi'_2)$ is equivalent to $\tilde{A}_{vh}(\xi_2, \xi'_2)$ and the base point of top vectors of $A'_{vh}(\xi_2, \xi'_2)$ is $h(a) = h(b_{v_j})$. Put

$$A'_{vh}(\xi_1, \xi'_1 : \xi_2, \xi'_2) = A'_{vh}(\xi_1, \xi'_1) \cup A'_{vh}(\xi_2, \xi'_2).$$

Then $A'_{vh} = A'_{vh}(\xi_1, \xi', : \xi_2, \xi'_2)$ is equivalent to $\tilde{A}_{vh}(\xi_1, \xi'_1) \cup \tilde{A}_{vh}(\xi_2, \xi'_2)$ and the base point of its top vectors is $h(a)$. Next we shall define $A''_{vh}(\xi_1, \xi'_1 : \xi_2, \xi'_2)$ by switch of the base point of bottom vectors of $A'_{vh}(\xi_1, \xi'_1 : \xi_2, \xi'_2)$. If $a'_{v_0} = b'_{v_0}$ put $A''_{vh} = A'_{vh}$. assume that $a'_{v_0} \neq' b_{v_0}$. Since

$$x = \begin{pmatrix} 0 \\ h(a'_{v_0}) - h(b'_{v_0}) \end{pmatrix} = \begin{pmatrix} h(a) \\ h(a'_{v_0}) \end{pmatrix} - \begin{pmatrix} h(a) \\ h(a'_{v_0}) \end{pmatrix}.$$

Change $\begin{pmatrix} h(a) \\ h(a'_{v_0}) \end{pmatrix}$ to $x=\begin{pmatrix} 0 \\ h(a'_{v_0})-h(b'_{v_0}) \end{pmatrix}$ and

$$y=\begin{pmatrix} 0 \\ h(a'_{v_i})-h(a'_{v_0}) \end{pmatrix} \text{ to } y+x=\begin{pmatrix} 0 \\ h(a'_{v_i})-h(b'_{v_0}) \end{pmatrix}.$$

Consequently we get A''_{vh} which is equivalent to A'_{vh} and then to $\tilde{A}_{vh}$ and the base points of top vectors and bottom vectors are $h(a)$ and $h(b'_{v_0})$ respectively. We denote the set of vertices of top vectors and bottom vectors of A_{vh} by $V_T A_{vh}$ and $V_B A_{vh}$. It is clear that

$$V_T \tilde{A}_{vh}=V_T A''_{vh}$$
$$V_B \tilde{A}_{vh}=V_B A''_{vh}.$$

Furthermore if $(\xi_{1v}\cup\xi_{2v})\cap(\xi'_{1v}\cup\xi'_{2v})\ni c$ we can easily modify A''_{vh} to A'''_{vh} so that all base points of top and bottom vectors of A'''_{vh} are $h(c)$ and $A'''_{vh}\sim\tilde{A}_{vh}$, $V_T\tilde{A}_{vh}=V_T A'''_{vh}$, $V_B\tilde{A}_{vh}=V_B A'''_{vh}$.

Proof of Lemma 3. We shall define an *SS-map* $\tilde{h}:K\to E^n$ and a set of vectors $B_{v\tilde{h}}$. At first we assume that 1) $(\xi_{1v}\cap\xi_{2v})\cup(\xi'_{1v}\cap\xi'_{2v})\neq\phi$. Moreover if 1.1) $(\xi_{1v}\cup\xi_{2v})\cap(\xi'_{1v}\cup\xi'_{2v})=\phi$. Put $\tilde{h}=h$ and $B_{v\tilde{h}}=A''_{vh}$. Then $B_{v\tilde{h}}$ has only vectors of types 0), 2) and 2′) which can be arbitrarily approximated. Furthermore it is easy to see that the numbers of vectors of $B_{v\tilde{h}}$ satisfy the following relations;

$$N(v,2)+N(v,3)=N(v,2)\leq dim\,\xi_{1v}+dim\xi_{2v}+1$$
$$N(v,2')+N(v,3)=N(v,2')\leq dim\,\xi'_{1v}+dim\,\xi'_{2v}+1$$
$$N(v,1)+N(v,3)=0\leq\alpha_v+\beta_v.$$

Secondly if 1.2) $(\xi_{1v}\cup\xi_{1v})\cap(\xi'_{1v}\cup\xi'_{2v})\neq\phi$. Put $h=\tilde{h}$ and $B_{v\tilde{h}}=A'''_{vh}$. Then $B_{v\tilde{h}}$ has only vectors of types 1), 2), 2′) and pairs of type 3) which are arbitrarily approximated. It is clear that

$$N(v,2)+N(v,3)\leq dim\ \xi_{1v}+dim\ \xi_{2v}+1$$
$$N(v,2')+N(v,3)\leq dim\ \xi'_{1v}+dim\ \xi'_{2v}+1.$$

Furthermore numbers of vertices satify the following

$$\alpha_v=NV(\xi_{1v}\cap\xi'_{1v})+NV(\xi_{2v}\cap\xi'_{2v})$$
$$\geqq\tilde{N}V(\xi_{1v}\cap\xi'_{1v})+\tilde{N}V(\xi_{2v}\cap\xi'_{2v}).$$

where $\tilde{N}V(\xi_{iv}\cap\xi'_{iv})$, $i=1,2$ is the number of vectices in $\xi_{iv}\cap\xi'_{iv}$ but not in $\xi_{jv}\cap\xi'_{jv}$, $j\neq i$.

$$\beta_v=NV(\,(\xi_{1v}\cup\xi'_{1v})\cap(\xi_{1v}\cup\xi'_{2v})\,)-1$$

$$=NV(\xi_{1v}\cap\xi_{2v})\cup(\xi'_{1v}\cap\xi'_{2v})\cup(\xi_{1v}\cap\xi'_{2v})\cup(\xi'_{1v}\cap\xi_{2v}))-1$$
$$\geq\tilde{N}N(\xi_{1v}\cap\xi'_{2v})+\tilde{N}V(\xi'_{1v}\cap\xi_{2v}).$$

Then $\alpha_v+\beta_v\geq\tilde{N}V(\xi_{1v}\cap\xi'_{1v})+\tilde{N}V(\xi_{2v}\cap\xi'_{2v})+\tilde{N}V(\xi_{1v}\cap\xi'_{2v})+\tilde{N}V(\xi'_{1v}\cap\xi_{2v})$
$\geq N(v,3)+1=N(v,3)+N(v,1).$

Next we assume that 2) $(\xi_{1v}\cap\xi_{2v})\cup(\xi'_{1v}\cap\xi'_{2v})=\phi$. 2.1) If $(\xi_{1v}\cap\xi'_{2v})\cup(\xi'_{1v}\cap\xi_{2v})=\phi$. Put $\tilde{h}=h$ and $B_{v\tilde{h}}=\tilde{A}_{vh}$. Then it is clear that

$$N(v,2)+N(v,3)\leq dim\ \xi_{1v}+dim\ \xi_{2v}$$
$$N(v,2')+N(v,3)\leq dim\ \xi'_{1v}+dim\ \xi'_{2v}$$
$$N(v_1 1)+N(v,3)\leq\alpha_v=\alpha_v+\beta_v.$$

2.2) If $(\xi_{1v}\cap\xi'_{2v})\cup(\xi'_{1v}\cap\xi_{2v})\neq\phi$ and then we assume that $d\in\xi_{1v}\cap\xi'_{2v}$. Define $\tilde{h}$ as follows :

$$\tilde{h}(a)=0 \qquad \text{if } a=d$$
$$\tilde{h}(a)=h(a) \qquad \text{if } a\neq d,\ a\in VK.$$

Put $B_{v\tilde{h}}=\left\{\begin{pmatrix}h(a_{v_0})\\0\end{pmatrix},\cdots,\begin{pmatrix}h(a_{vm(v,\xi_1)})\\0\end{pmatrix},\begin{pmatrix}0\\h(a'_{v_0})\end{pmatrix},\cdots\right.$.

$\begin{pmatrix}0\\h(a'_{vm(v,\xi'_1)})\end{pmatrix},\begin{pmatrix}h(b_{v_0})\\0\end{pmatrix},\cdots,\begin{pmatrix}h(b_{vm(v,\xi_2)})\\0\end{pmatrix},$

$\left.\begin{pmatrix}0\\h(b'_{v_0})\end{pmatrix},\cdots,\begin{pmatrix}0\\h(b'_{vm(v,\xi'_2)})\end{pmatrix}\right\}.$ Then it is clear that $B_{v\tilde{h}}$ is equivalent to $\tilde{A}_{v\tilde{h}}$ and $B_{v\tilde{h}}$ has only vectors of types 2), 2′) and pairs of 3). Moreover

$$N(v,2)+N(v,3)\leq dim\ \xi_{1v}+dim\ \xi_{2v}+1$$
$$N(v,2')+N(v,3)\leq dim\ \xi'_{1v}+dim\ \xi'_{2v}+1$$
$$N(v,{}_1)+N(v,3)\leq\alpha_v+\beta_v$$

Then we have

$$\sum_{v\in\eta_1\cup\eta_2}(N(v,2)+N(v,3))\leq n$$
$$\sum_{v\in\eta_1\cup\eta_2}(N(v,2')+N(v,3))\leq n$$
$$\sum_{v\in\eta_1\cup\eta_2}(N(v,1)+N(v_1 2))\leq\sum_{v\in\eta_1\cup v\eta_2}(\alpha_v+\beta_v)$$

$\leq \alpha(\xi_1, \xi'_1 : \xi_2, \xi'_2) + \beta(\xi_1, \xi'_1 : \xi_2, \xi'_2) \leq n.$

Therefore by Proposition 10 we have an SS-map $h' : K \to E^n$ such that $B_{h'} = \bigcup_{v \in \eta_1 \cup \eta_2} B_{vh'}$ is linearly independent. Therefore $\tilde{A}_{h'} = \bigcup \tilde{A}_{h'}(\xi_1, \xi'_1 : \xi_2, \xi'_2)$ is linearly independent. We have proved Lemma 3.

§4 Let K' and H' be first derived subdivisions of K and H respectively such that $f : K' \to H'$ is simplicial, Let denote by $\hat{\xi}$ the barycenter of a simplex ξ. If X is a subset of $|K|$ and ξ is a minimum simplex of K containing X. We denote ξ by $\eta(X)$. It is clear that $\eta(\hat{\xi}) = \xi$.

Proposition 12. *Let $\xi_1, \xi_2, \xi_3, \xi_4$ be simplexes of K' such that any vertex of them is non-free (i.e. belongs to at least two of them). Then if a vertex a does not belongs to ξ_i and belongs to ξ_j, for all $j \neq i$. The join $a * \xi_i$ is a simplex of K'.*

Proof. Sappose that $a \notin \xi_1$ and $a \in \xi_2, \xi_3, \xi_3$. Let b be a vertex of ξ_1. Then there is a simplex ξ_j $(j \neq 1)$ such that b is a vertex of ξ_j. Therefore $\eta(a) < \eta(b)$ or $\eta(b) < \eta(a)$. Thus $a * \xi_1 \in K'$.

Proposition 13. *If $\xi_1, \xi_2, \xi_3, \xi_4$ are simplexes of K' such that any vertex of them is non- free. Then there are two simplexes ζ_1 and ζ_2 of K such that*

$$\zeta_1 \cup \zeta_2 \supset \xi_1 \cup \xi_2 \cup \xi_3 \cup \xi_4.$$

Proof. It is sufficient to prove that for any ξ_i there is ξ_j $(j \neq i)$ such that $\eta(\xi_i) < \eta(\xi_j)$. Since $\hat{\eta}(\xi_i)$ is non-free and then a vertex of ξ_j $(i \neq j)$. $\eta(\xi_i) < \eta(\xi_j)$.

The following proposition is clear:

Proposition 14. *If $h : K \to E^n$ be a non-degenerate SS map. Then there is an $\varepsilon > 0$ such that if h'* is an SS-map of K' into E^n satisfying $d(h, h') < \varepsilon$ *then $h' | (K' | \xi)$ is an isomorphism for any $\xi \in K$, where $K' | \xi$ is the subcomplex of K' whose underlying space is ξ.*

Proof of Theorem Let $h : K' \to E^n$ be an SS-map which satisfies the conditions of $h : K \to E^n$ in Lemma 2 and the conditions of $h' : K \to E^n$ in Lemma 3 for any such sinplexes $\xi_1, \xi'_1, \xi_2, \xi'_2$ of K' that

$$\eta_1 = f(\xi_1) = f(\xi'_1), \quad \eta_2 = f(\xi_2) = f(\xi'_2)$$

Furthermore if $p_1 \in \xi_1$, $p'_1 \in \xi_1$, $p_2 = \xi_2$, $p_2' \in \xi'_2$ and $f(p_1) = f(p'_1)$, $f(p_2) = f(p'_2)$, $h(p_1) = h(p_2)$, $h(p'_2) = h(p'_2)$. Then by Lemma 2, Lemma 3, and Proposition 12 we may assume that any vertex of $\xi_1, \xi'_1, \xi_2, \xi'_2$ belongs to just two of them or all of them. Let p be any point of a simplex ξ. Then by barycentric coordinate we can wirte

$$p=\sum_{v\epsilon\xi}\lambda(v,\xi)\,v,\quad \sum_{v\epsilon\xi}\lambda(v,\xi)=1,\quad \lambda(v,\xi)\geq 0.$$

From the linear independentness of $\tilde{A}(\xi_1, \xi'_1:\xi_2, \xi'_2)=\bigcup_{v\epsilon\eta_1\cup\eta_2}((A_v(\xi_1, \xi'_1)-x_1\cup\cdots\cup x_{\beta v})\cup A_v(\xi_2, \xi'_2))$ (Proposition 8 and Lemma 2) and that of $\tilde{A}_h(\xi_1, \xi'_1:\xi_2, \xi'_2)=\bigcup_{v\epsilon\eta_1\cup\eta_2}\tilde{A}_{vh}(\xi_1, \xi_2':\xi_2,\xi'_2)$ (Lemma 3) it is easy to see that

1) $\lambda(v, \xi_1)-\lambda(v, \xi'_1)=\lambda(v, \xi_2)-\lambda(v, \xi'_2)$ if $v\epsilon\xi_1\cap\xi'_1\cap\xi_2\cap\xi'_2$

2) $\lambda(v, \xi_1)=\lambda(v, \xi'_1)$ if $v\epsilon\xi_1\cap\xi'_1-\xi_2\cap\xi'_2$

3) $\lambda(v, \xi_2)=\lambda(v, \xi'_2)$ if $v\epsilon\xi_2\cap\xi'_2-\xi_1\cap\xi'_1$

4) $\lambda(v, \xi_1)=\lambda(v, \xi_2)$ if $v\epsilon\xi_1\cap\xi_2-\xi'_1\cap\xi'_2$

5) $\lambda(v, \xi_1')=\lambda(v, \xi'_2)$ if $v\epsilon\xi'_1\cap\xi'_2-\xi_1\cap\xi_2$

6) $\lambda(v, \xi_1)=\lambda(v, \xi'_2)=0$ if $v\epsilon\xi_1\cap\xi'_2-\xi'_1\cap\xi_2$

7) $\lambda(v, \xi'_1)=\lambda(v_1\, \xi_2)=0$ if $v\epsilon\xi'_1\cap\xi_2-\xi_1\cap\xi'_2$.

By Proposition 13 there are two simplexes ζ_1, ζ_2 such that $\zeta_1\cup\zeta_2\supset\xi_1\cup\xi'_1\cup\xi_2\cup\xi'_2$. If $\xi_1\cup\xi_2\subset\zeta_1$ by Propositun 14 $p_1=p_2$. Then we may assume that $\xi_1\cup\xi_2$ is contained neither in ζ_1 nor ζ_2 and $\xi'_1\cup\xi'_2$ is similar. By the conditions 6) and 7) we may assume that

$$\xi_1\cap\xi'_2-\xi'_1\cap\xi_2=\xi'_1\cap\xi_2-\xi_1\cap\xi'_2=\phi$$

and then $\xi_1\cup\xi'_1\subset\zeta_1,\ \xi_2\cup\xi'_2\subset\zeta_2.$

Thus we have

$$p_1=\Sigma\ \{\lambda(v, \xi_1)\,v\mid v\epsilon\xi_1-\xi'_1\}+\Sigma\ \{\lambda(v, \xi_1)\,v\mid v\epsilon\xi_1\cap\xi'_1\}$$
$$p'_1=\Sigma\ \{\lambda(v, \xi'_1)\,v\mid v\epsilon\xi'_1-\xi_1\}+\Sigma\ \{\lambda(v, \xi'_1)\,v\mid v\epsilon\xi_1\cap\xi'_1\}$$
$$p_2=\Sigma\ \{\lambda(v, \xi_2)\,v\mid v\epsilon\xi_2-\xi'_2\}+\Sigma\{\lambda(v, \xi_2)\,v\mid v\epsilon\xi_2\cap\xi'_2\}$$
$$p'_2=\Sigma\ \{\lambda(v, \xi'_2)\,v\mid v\epsilon\xi'_2-\xi_2\}+\Sigma\ \{\lambda(v, \xi'_2)\,v\mid v\epsilon\xi_2\cap\xi'_2\}.$$

If $v\epsilon\xi_1-\xi'_1$ then $v\epsilon\xi_1\cap\xi_2$ and if $v\epsilon\xi_2-\xi'_2$ then $v\epsilon\xi_1\cap\xi_2$. By 2) and 3) we have

$$\Sigma\ \{\lambda(v, \xi_1)\,v\mid v\epsilon\xi_1-\xi'_1\}=\Sigma\ \{\lambda(v, \xi_2)\,v\mid v\epsilon\xi_2-\xi'_2\}.$$

Similarly we have

$$\Sigma\ \{\lambda(v, \xi'_1)\,v\mid v\epsilon\xi'_1-\xi_1\}=\Sigma\ \{\lambda(v, \xi'_2)\,v\mid v\epsilon\xi'_2-\xi_2\}.$$

Since $h(p_1)=h(p_2)$, $h(p'_1)=h(p'_2)$. From above formulas

8) $$\Sigma\ \{\lambda(v, \xi_1)\,h(v)\mid v\epsilon\xi_1\cap\xi'_1\}=\Sigma\ \{\lambda(v, \xi_2)\,h(v)\mid v\epsilon\xi_2\cap\xi'_2\}$$

9) $$\Sigma\ \{\lambda(v, \xi'_1)\,h(v)\mid v\epsilon\xi_1\cap\xi'_1\}=\Sigma\ \{\lambda(v, \xi'_2)\,h(v)\mid v\epsilon\xi_2\cap\xi'_2\}.$$

From the conditions 1), 2), 3), 4), 5) we have

$$\Sigma\ \{\lambda(v,\xi_1)\,|\,v\epsilon\xi_1-\xi'_1\}=\Sigma\ \{\lambda(v,\xi_2)\,|\,v\epsilon\xi_2-\xi'_2\}$$
$$=\Sigma\ \}\lambda(v,\xi'_1)\,|\,v\epsilon\xi'_1-\xi_1\}=\Sigma\ \{\lambda(v,\xi_2)\,|\,v\epsilon\xi'_2-\xi_2\}.$$

We denote this number by ν and put

$$q=\frac{1}{\nu}\Sigma\ \{\lambda(v,\xi)\,v\,|\,v\epsilon\xi_1-\xi'_1\}=\frac{1}{\nu}\Sigma\ \{\lambda(v,\xi_2)\,v\,|\,v\epsilon\xi_2-\xi'_2\}$$

$$r_1=\frac{1}{1-\nu}\Sigma\ \{\lambda(v,\xi_1)\,v\,|\,v\epsilon\xi_1\cap\xi'_1\}$$

$$r_2=\frac{1}{1-\nu}\Sigma\ \{\lambda(v,\xi_1)\,v\,|\,v\epsilon\xi_2\cap\xi'_2\}$$

$$q'=\frac{1}{\nu}\Sigma\ \{\lambda(v,\xi'_1)\,v\,|\,v\epsilon\xi'_1-\xi_1\}=\frac{1}{\nu}\ \{\lambda(v,\xi'_2)\,v\,|\,v\epsilon\xi'_2-\xi_2\}$$

$$r'_1=\frac{1}{1-\nu}\Sigma\ \{\lambda(v,\xi'_1)\,v\,|\,v\epsilon\xi_1\cap\xi'_1\}$$

$$r'_2=\frac{1}{1-\nu}\Sigma\ \{\lambda(v,\xi'_2)\,v\,|\,v\epsilon\xi_2\cap\xi'_2\}.$$

Then $p_1=\nu q+(1-\nu)\,r_1$; $p'_1=\nu q'+(1-\nu)\,r_1'$
$p_2=\nu q+(1-\nu)\,r_2,\quad p'_2=\nu q'+(1-\nu)\,r'_2.$

Furthemore

$$q,\ q'\ \epsilon\xi_1\cap\xi_2,$$
$$r_1,\ r'_1\ \epsilon\xi_1\cap\xi'_1 \qquad r_2,\ r'_2\ \epsilon\cap\xi'_2.$$

By 8). 9) we have

$$h(r_1)=h(r_2) \qquad h(r'_1)=h(r'_2)$$

If $u_1=\mu p_1+(1-\mu)\,p'_1,\ u_2=\mu p_2+(1-\mu)\,p'_2 \qquad 1>\mu>0.$

Then u_1, u_2 are points of ζ_1, ζ_2 respectively and

$$u_1=\nu\,(\mu q+(1-\mu)\,q')+(1-\nu)\,(\mu r_1+(1-\mu)\,r'_1)$$
$$u_2=\nu\,(\mu q+(1-\mu)\,q')+(1-\nu)\,(\mu r_2+(1-\mu)\,r'_2).$$

Therefore

$$h(u_1)=\nu h\,(\mu q+(1-\mu)\,q')+(1-\nu)\,h\,(\mu r_1+(1-\mu)\,r'_1)$$
$$h(u_2)=\nu h\,(\mu q+(1-\mu)\,q')+(1-\nu)\,h\,(\mu r_2+(1-\mu)\,r'_2).$$

Since $h(r_1)=h(r_2)$ $h(r'_1)=h(r'_2)$. We have $h(u_1)=h(u_2)$. Therefore we have proved the condition c) of Theorem. Lemma 1 implies b) and we have completed the proof of Theorem.

REFERENCES

[1] Tatsuo Homma. ***A Theorem of Piecewise Linear approximations.*** The Yokohama Mathematical Jaurnal. Vol XIV, (1966). 47–54.

[2] E. C. Zeeman. ***Seminar on Combinatorial Topology.*** Institut Des Hautes Etudes Scientifiques (1963).

Department of Mathematics
Yokohama City University