REMARKS ON COORDINATIZATIONS OF A PROJECTIVE PLANE

By

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1. Summary. By making use of coordinate transformations, planar rings with (i) right zero, (ii) left zero, (iii) zero, (iv) right zero and right unity (or left unity, or unity), (v) left zero and left unity (or right unity, or unity), or (vi) zero and unity, etc., are constructed from a general planar ternary ring, so that, the projective planes induced by the old and the new planar rings are isomorphic. Such constructions lead to slightly different proofs of two interesting theorems by *Wesson* [3]. Discussions of planar ternary rings with one–sided zeros and one–sided unities are also given.

2. Introduction. As is well–known, a *projective plane* is a set P of "points" with certain subsets called "lines" such that:

P 1. *Any two distinct points are contained in exactly one line.*

P 2. *Any two distinct lines contain exactly one common point.*

P 3. *P contains at least four points, no three of which are on the same line.*

By definition, a *planar ternary ring* (PTR) is a pair $(S, < \;\; >)$ consisting of a set S with at least two elements and a ternary operation $< \;\; >$ which assigns to every ordered triple a, b, c of elements of S a unique element of S denoted by $<abc>$ such that for all a, b, c, d in S:

I. $<abx>=c$ *has a unique solution* x *in* S.

II. $<xab>=<xcd>$, $a \neq c$ *has a unique solution* x *in* S.

III. $<axy>=b$, $<cxy>=d$, $a \neq c$ *has a unique solution pair* (x, y) *consisting of elements of* S.

It has been shown by *Martin* [2] that

Proposition 1. *The uniqueness of solution pair in* III *follows from* I *and* II.

Here we would also like to point out the following for later use:

Proposition 2. *The uniqueness of the solution in* II *follws from* III.

Proof. Suppose we have two solutions $x=x_1$, $x=x_2$, $x_1 \neq x_2$ of $<xab>=<xcd>$, $a \neq c$; then $<x_1 ab>=<x_1 cd>=e$ and $<x_2 ab>=<x_2 cd>=f$. This shows that $<x_1 xy>=e$, $<x_2 xy>=f$, $x_1 \neq x_2$ has two solutions $(x=a, y=b)$ and $(x=c, y=d)$, $a \neq c$ which contradicts the uniqueness of solution in III.

It is also well–known that every planar ternary ring $(S, < \;\; >)$ induces a projective plane $(P, < \ >)$ which consists of points represented by (x, y), or (m), or (∞) for every x, y, m in S but ∞ not in S. The lines of the plane $(P, < \ >)$ are represented by $[m, b]$, or $[c]$, or $[\infty]$ for every m, b, c in S. The incidence relations are defined such that $[\infty]$ contains exactly (∞) and points (m) for all m in S, $[c]$ contains exactly (∞) and all points (c, y) for every y in S, and the line $[m, b]$ contains exactly (m) and all points (x, y) for which $y = <xmb>$.

A planar ternary ring is called [2] an *intermediate ternary ring* (ITR) if it also satisfies the following two conditions:

IV. $<amb> = <cmb> = d$, $a \neq c$, *implies* $<xmb> = d$ *for all* x *in* S.

V. $<mad> = <mcd> = b$, $a \neq c$, *implies* $<mxd> = b$ *for all* x *in* S.

The followings will also be used later:

Proposition 3. *Condition* IV *is equivalent to the following*:

IV′ *There exists an element* m_0 *of* S *and a permutation* $* : b \to b^*$ *on* S *such that* $<am_0 b^*> = b$ *for all* a, b *in* S.

Proof. IV $\to$ IV′: Since any two points (a, d) and (c, d) determine a line in the induced projective plane $(P, < \ >)$, condition IV means that all the points (x, d) (x variable and d fixed) are on a same line $[m, b]$. If all the points (x, g) $(g \neq d)$ are on the line $[n, q]$, the point of intersection of $[m, b]$ and $[n, q]$ can not be a point (e, f), because if (e, f) were the point of intersection, then $f = d = g$ which contradicts $g \neq d$. Thus $[m, b]$ and $[n, q]$ intersect at a point (m_0) and $m_0 = m = n$. Since $[n, q]$ is an arbitrary line containing all the points (x, g), every such line passes through the same point (m_0) on $[\infty]$, so that such a line can be represented as $[m_0, t^*]$, where t^* is obtained from $<xm_0 t^*> = t$ where (x, t) is an arbitrary point on the line (such a t^* is uniquely determined by IV). The correspondence $t \to t^*$ is evidently a permutation on S.

IV′ $\to$ IV: Suppose $<amb> = <cmb> = d$, $a \neq c$. By IV′ we have $<am_0 d^*> = <cm_0 d^*> = d$. Thus we have $<amb> = <am_0 d^*>$, $<cmb> = <cm_0 d^*>$, $a \neq c$. Consequently $m = m_0$ and $b = d^*$ by II. Therefore $<xmb> = <xm_0 d^*> = d$ for all x.

Proposition 4. *Condition* V *is equivalent to the following*:

V′. *There exists an element* a_0 *of* S *and a permutation* $* : d \to d^*$ *on* S *such that* $<a_0 md> = d^*$ *for all* m, d *in* S.

Proof. V $\to$ V′: In the induced projective plane $(P, < \ >)$, condition V means that all the lines $[x, d]$ (x variable and d fixed) pass through the same point (a, b). Suppose all the lines $[x, g]$, $g \neq d$ pass through the point (c, q). The line joining these two points can not be a line $[m, y]$, because if this were the line joining these points, then $y = d = g$ which contradicts $g \neq d$. Thus (a, b) and (c, q) are on a line $[a_0]$ through

(∞), and $a_0=a=c$. Since (c, q) is an arbitrary such point, every such point is on the same line $[a_0]$ and thus can be represented as (a_0, t^*), with t^* obtained from $<a_0xt>=t^*$, where $[x, t]$ is any line through the point. The correspondence $t\rightarrow t^*$ is evidently a permutation on S.

$\mathrm{V}'\rightarrow\mathrm{V}$: Suppose now that $<mad>=<mcd>=b$, $a\neq c$. By V′ we have $<a_0ad>=<a_0cd>=d^*$, $a\neq c$. Thus it follows that $m=a_0$ and $b=d^*$ by III. Consequently $<mxd>=<a_0xd>=d^*=b$ for all x in S.

An element u in S is called a *right zero* if $<xub>=b$ for all x, b in S, and an element v in S is called a *left zero* if $<vxb>=b$ for all x, b in S. An element z is said to be a *zero* if z is both a right zero and a left zero.

A left zero (or a right zero) is unique if it exists. It is shown by *Martin* [2] that:

Proposition 5. *If a* PTR $(S, <\ \ >)$ *has a right zero* u, *then* $<xab>=d$, $a\neq u$, *has a unique solution* x. *If* $(S, <\ \ >)$ *has a left zero* v, *then* $<axd>=b$, $a\neq v$ *has a unique solution* x.

From this, it follows that in a PTR, the existence of a right zero implies IV, and that of a left zero implies V. But IV does not imply the existence of a right zero, and V does not imply that of a left zero. In a PTR, the existence of a right zero and that of a left zero are independent conditions.

3. Coordinate transformations. To each element a in S assign a permutation $\sigma(a)$ on S (S may be finite or infinite). Let $\sigma'(a)$ be the inverse permutation of $\sigma(a)$.

Proposition 6. *The ternary system* $(S, \{\ \ \})$, *defined by* $\{amd\}=<amd>^{\sigma(a)}$ *for all* a, m, d *in* S, *is a planar ternary ring if and only if the ternary system* $(S, <\ \ >)$ *is a planar ternary ring.*

Proof. (1) As $\{abx\}=d$ if and only if $<abx>=d^{\sigma'(a)}$, condition I for $(S, \{\ \ \})$ follows from that for $(S, <\ \ >)$ and vice versa. (2) Similarly, since $\{axy\}=b$, $\{cxy\}=d$, $a\neq c$ if and only if $<axy>=b^{\sigma'(a)}$, $<cxy>=d^{\sigma'(c)}$, condition III for $(S, \{\ \ \})$ follows from that of $(S, <\ \ >)$ and vice versa. (3) Suppose II for $(S, <\ \ >)$ is satisfied; that is, $<xab>=<xcd>$, $a\neq c$ has a unique solution $x=x_1$. Then $\{x_1 ab\}=<x_1 ab>^{\sigma(x_1)}=<x_1 cd>^{\sigma(x_1)}=\{x_1 cd\}$; that is, $\{xab\}=\{xcd\}$, $a\neq c$ has a solution $x=x_1$. The uniqueness of the solution follows from I and III by Proposition 2. Assuming II for $(S, \{\ \ \})$, we can also prove II for $(S, <\ \ >)$ in the same way.

Proposition 7. *The ternary system* $(S, \{\ \ \})$, *defined by* $\{acb^{\sigma(c)}\}=<acb>$ *for all* a, b, c *in* S, *is a planar ternary ring if and only if the ternary system* $(S, <\ \ >)$ *is a planar ternary ring.*

Proof. For conditions I and II, proofs are similar to the proofs of I and III in

Proposition 6. Now assume that $<ax_1y_1>=b$, $<cx_1y_1>=d$, $a\neq c$ has a unique solution pair (x_1, y_1). Let $x=x_1$, $y=y_1{}^{\sigma(x_1)}$; then $\{axy\}=\{ax_1y_1{}^{\sigma(x_1)}\}=<ax_1y_1>=b$, and $\{cxy\}=\{cx_1y_1{}^{\sigma(x_1)}\}=<cx_1y_1>=d$. That is, $\{axy\}=b$, $\{cxy\}=d$, $a\neq c$ has a solution pair (x, y). The uniqueness of the solution follows from I and II by Proposition 1. Assuming that III holds for $(S, \{\quad\})$ we can show that III holds also for $(S, <\quad>)$ in the same way.

Proofs of these two propositions can be carried out more geometrically, if we make use of the projective plane $(P, <\quad>)$ induced by a planar ternary ring $(S, <\quad>)$.

In the induced projective plane $(P, <\quad>)$, assign new coordinates $(a, b^{\sigma(a)})$ to the point with the coordinates (a, b), but preserve all the old coordinates for all other kinds of elements. Instead of expressing the incidence of the point (a, b) with the line $[m, d]$ by $<amd>=b$, we use the relation $\{amd\}=b^{\sigma(a)}=<amd>^{\sigma(a)}$ in the new coordinates. Then the ternary system $(S, \{\quad\})$ gives a new coordinatization of the projective plane $(P, <\quad>)$, and therefore $(S, \{\quad\})$ is a planar ternary ring. This argument also shows that the projective plane induced by $(S, <\quad>)$ can also be seen as the projective plane induced by $(S, \{\quad\})$.

For Proposition 7, assign new coordinates $[c, b^{\sigma(c)}]$ to the line with the coordinates $[c, b]$ and preserve the old coordinates for all the other kinds of elements. Instead of expressing the incidence of the point (a, d) with the line $[c, b]$ by $<acb>=d$ in the old system, we use $\{acb^{\sigma(c)}\}=d$ in the new coordinate system. Then the ternary system $(S, \{\quad\})$ again gives a new coordinatization of the projective plane $(P, <\quad>)$, and thus $(S, \{\quad\})$ is a planar ternary ring. Again the projective plane induced by $(S, <\quad>)$ can also be seen as the projective plane induced by $(S, \{\quad\})$.

Thus in both cases, the projective planes induced by $(S, <\quad>)$ and $(S, \{\quad\})$ are isomorphic.

As special cases, consider $\sigma(a)\neq$identity for a fixed element a in S and $\sigma(x)=$ identity for all $x\neq a$. In Proposition 6, this gives a coordinate transformation on only one line $[a]$ and leaves the coordinates unchanged for every other element. In Proposition 7, this gives a coordinate transformation on only one pencil of lines with center (a) and leaves all other coordinates unchanged.

We can also consider a coordinate transformation only on the line $[\infty]$. Suppose $\sigma(\infty)=\sigma$ is a permutation on S and rename the point (m) as (m^σ). This gives rise to a PTR $(S, \{\quad\})$ defined by $\{am^\sigma d\}=<amd>$. Next, suppose $\rho(\infty)=\rho$ be a permutation on S; then a coordinate transformation on only one pencil with center (∞) can be obtained by renaming $[a]$ as $[a^\rho]$. This gives rise to a PTR $(S, \{\quad\})$ defined by $\{a^\rho md\}=<amd>$. The latter two cases are special instances of isotopisms [2].

Example 1. The following is a PTR satisfying IV :

$x=r$

$m \backslash b$	r	s	t
r	t	r	s
s	r	s	t
t	r	t	s

$x=s$

$m \backslash b$	r	s	t
r	t	r	s
s	t	r	s
t	s	r	t

$x=t$

$m \backslash b$	r	s	t
r	t	r	s
s	s	t	r
t	t	s	r

If we change the coordinates of lines of the pencil with center (r) by

$$[r, r] \to [r, t],\ [r, s] \to [r, r],\ [r, t] \to [r, s],$$

then we get a PTR with right zero r (r is not a left zero) as follows:

$x=r$

$m \backslash b$	r	s	t
r	r	s	t
s	r	s	t
t	r	t	s

$x=s$

$m \backslash b$	r	s	t
r	r	s	t
s	t	r	s
t	s	r	t

$x=t$

$m \backslash b$	r	s	t
r	r	s	t
s	s	t	r
t	t	s	r

We can get the former PTR from the latter by the coordinate transformation:

$$[r, r] \to [r, s],\ [r, s] \to [r, t],\ [r, t] \to [r, r].$$

Example 2. The following is a PTR satisfying V:

$x=r$

$m \backslash b$	r	s	t
r	s	r	t
s	s	r	t
t	s	r	t

$x=s$

$m \backslash b$	r	s	t
r	r	s	t
s	s	t	r
t	t	r	s

$x=t$

$m \backslash b$	r	s	t
r	r	t	s
s	s	r	t
t	t	s	r

If we change the coordinates on the line $[r]$ by

$$(r, r) \to (r, s),\ (r, s) \to (r, r),\ (r, t) \to (r, t),$$

then we get a PTR with left zero r (r is not a right zero) as follows:

$x=r$

$m \backslash b$	r	s	t
r	r	s	t
s	r	s	t
t	r	s	t

$x=s$

$m \backslash b$	r	s	t
r	r	s	t
s	s	t	r
t	t	r	s

$x=t$

$m \backslash b$	r	s	t
r	r	t	s
s	s	r	t
t	t	s	r

We can get the former PTR from the latter one by the same point coordinate transformation as above.

4. Planar ternary rings with one-sided zeros. Using coordinate transformations discussed in 3, we can construct a PTR with one-sided zeros from a general PTR by the following steps :

Proposition 8. *Let* $(S, <\ \ >)$ *be a planar ternary ring, and* u *be a fixed element of* S. *Then* (i) *the ternary system* $(S, \{\ \ \})$ *defined by* $<au\{amd\}>=<amd>$ *is a planar ternary ring with a right zero* u, *and* (ii) v *is a left zero of* $(S, \{\ \ \})$ *if and only if* $(S, <\ \ >)$ *is a planar ternary ring satisfying* V$'$ *with* $a_0=v$.

Proof. Although (i) can be checked directly, we prefer to relate its proof to Proposition 6. The set of points on the line $[a]$ is in one-to-one correspondence with the set of all lines in the pencil with the center (u) by assigning $(\infty)\to[\infty]$ and $(a, e)\to[u, b]$ which is incident with (a, e). In this way, a permutation $\sigma(a)$ on S is defined by $b=e^{\sigma(a)}$. Then $<aub>=e$. If $[m, d]$ is any line on (a, e), then $<amd>=e$. Thus $<amd>=<aub>$, and $\{amd\}=<amd>^{\sigma(a)}=b$ can be expressed as $<amd>=<au\{amd\}>$.

Putting $m=u$ in this relation, we have $<aud>=<au\{aud\}>$ which implies $\{aud\}=d$ for all a, d in S by I. Thus u is a right zero.

(ii) We know that $\{vmd\}=d$ for all m if and only if $<vud>=<vmd>$ for all m. By III, the latter holds if and only if $(S, <\ \ >)$ satisfies V$'$ with $a_0=v$ and $d^*=<vud>$. If v is the left zero of $(S, <\ \ >)$, we have $<vmd>=<vud>=d$; that is, V$'$ holds with $a_0=v$ and $d^*=d$.

Proposition 9. *Let* $(S, <\ \ >)$ *be a planar ternary ring, and* v *be a fixed element of* S. *For* a, m, b *in* S, *define* $\{amb\}=<amd>$, *where* d *is the unique solution of* $<vmd>=b$. *Then* (i) *the ternary system* $(S, \{\ \ \})$ *is a planar ternary ring with* v *as a left zero, and* (ii) u *is a right zero of* $(S, \{\ \ \})$ *if and only if* $(S, <\ \ >)$ *is a planar ternary ring satisfying* IV$'$ *with* $m_0=u$.

Proof. (i) By using the one-to-one onto correspondence between the set of points on the line $[v]$ and the set of lines in the pencil (m), rename the line $[m, d]$ in the pencil (m) as $[m, b]$ if and only if $[m, d]$ is on (v, b). This gives rise to a permutation $\sigma(m)$ on S with $d^{\sigma(m)}=b$. Then $<vmd>=b$ and $\{amd^{\sigma(m)}\}=<amd>$ is equivalent to $\{amb\}=<amd>$ where d is the unique solution of $<vmd>=b$. Proposition 7 then implies that $(S, \{\ \})$ is a planar ternary ring; this can also be checked directly. As $\{vmb\}=<vmd>=b$, v is a left zero.

(ii) Since $\{aub\}=b$ for all a, b in S if and only if $<aud>=b$ and $<vud>=b$ for all a, b in S, this means that IV′ holds with $u=m_0$ and $b^*=d$. If u is the right zero of $(S, <\ >)$, then $<aub>=b$ for all a, b in S; that is, IV′ holds with $m_0=u$ and $b^*=b$.

Now let u, v be two fixed elements in S. Then from Proposition 8 and Proposition 9, it follows that if we construct $(S, [\ ])$ from $(S, <\ >)$ by Proposition 8, and then construct $(S, \{\ \})$ from $(S, [\ ])$ by Propositition 9, the resulting $(S, \{\ \})$ is a planar ternary ring with right zero u and left zero v. It is obvious that we can also get a planar ternary ring with the same zeros if we interchange the order of applying the two constructions. Now, for convenience, we formulate one of these constructions in the following:

Proposition 10. *Let $(S, <\ >)$ be a planar ternary ring, and let u, v be two fixed elements in S. For any a, m, b in S, define the ternary operation $\{\ \}$ by the following steps:* (i) *determine a c such that $<vmc>=<vub>$,* (ii) *determine a d such that $<aud>=<amc>$, and* (iii) *define $\{amb\}=d$ (that is, define $\{\ \}$ by $<an\ \{amb\}>=<amc>$). Then $(S, \{\ \})$ is a planar ternary ring with the right zero u and the left zero v.*

By putting $z=u=v$ in Proposition 10, we can obtain a planar ternary ring with zero z. (See Theorem 3.3 in *Wesson* [3]). It is also interesting to note that if we apply the natural duality [2] to the PTR $(S, \{\ \})$ of Proposition 8 (that is, define $(S, [\ ])$ by $d=[mab]$ if and only if $b=\{amd\}$), then the resulting ternary system $(S, [\ ])$ is the PTR considered by *Wesson* in his Theorem 3.1 [3]. This is easily seen: from $<au\ \{amd\}>=<amd>$, it follows that $<aub>=<am\ [mab]>$.

Example 3. It is well-known that if S is a field, the system $(S, <\ >)$ with $<xyz>=xy+z$ is a planar ternary ring with zero 0 and unity 1. Now defining $[\ ]$ by $<au\ [amd]>=<amd>$, we obtain $[amd]=a(m-u)+d$. Next, defining $\{\ \}$ by $\{amb\}=[amd]$ where $[vmd]=b$, we obtain $\{amb\}=(a-v)(m-u)+b$. The system $(S, \{\ \})$ is then a planar ternary ring with the right zero u and the left zero v.

5. Planar ternary rings with one-sided zeros and one-sided unities. Suppose $(S, <\ >)$ is a planar ternary ring with right zero u (or left zero, or zero). An element

h of S is called a ***right unity associated with*** u if $<ahu>=a$ for all a in S. An element g of S is called a ***left unity associated with*** u if $<gmu>=m$ [for all m in S. An element e of S is called a ***unity associated with*** u if it is at the same time a right unity and a left unity associated with u. A planar ternary ring with a zero and a unity is called a *Hall ternary ring* (HTR) [2]; an HTR is frequently called a "planar ternary ring" in the literature [1].

Proposition 11. *If a* PTR S, $(<\ >)$ *with right zero* u *(or left zero, or zero) has at the same time a right unity* h *and left unity* g *associated with* u, *then* $h=g$ *and* h *is a unity associated with* u.

Proof. Since h is a right unity associated with u, we have $<ghu>=g$, and since g is a left unity associated with u, we have $<ghu>=h$. Therefore $h=g$.

Proposition 12. (i) *For a* PTR *with a right zero* u, *a right unity associated with* u *is unique, if it exists.* (ii) *For a* PTR *with a right zero* u, *a left unity associated with* u *is unique, if it exists.* (iii) *For a* PTR *with left zero* v, *a right unity associated with* v *is unique, if it exists.* (iv) *For a PTR with a left zero* v, *there may exist more than one left unity associated with* v.

Proof. (i) Suppose $h\neq h'$ are right unities associated with the right zero u of a PTR, then $<xhu>=<xh'u>=x$ for all x, and this contradicts II. (ii) If $g\neq g'$ are left unities associated with the right zero u of $(S, <\ >)$, then $<gmu>=<g'mu>=m$ for all m (so for $m\neq u$). This contradicts Proposition 5. (iii) Suppose $h\neq h'$ are right unities of a PTR $(S, <\ >)$ associated with the left zero v; then $<xhv>=<xh'v>=x$ for all x (so for $x\neq v$). This contradicts Proposition 5. (iv) The following is an example of a planar ternary ring $(S, <\ >)$ with a left zero r and two left unities s and t associated with the left zero r.

Example 4.

$x=r$

$m \backslash b$	r	s	t
r	r	s	t
s	r	s	t
t	r	s	t

$x=s$

$m \backslash b$	r	s	t
r	r	s	t
s	s	t	r
t	t	r	s

$x=t$

$m \backslash b$	r	s	t
r	r	t	s
s	s	r	t
t	t	s	r

Proposition 13. (i) *If* h *is a right unity of a* PTR *associated with the right zero* u, *then* $h\neq u$. (ii) *If* g *is a left unity of a* PTR *associated with the left zero* v,

then $g \neq v$. (iii) *For a* PTR *with a right zero* u, u *may be a left unity associated with itself.* (iv) *For a* PTR *with a left zero* v, v *may be a right unity associated with itself.*

Proof. (i) If $h=u$, then for any $x \neq u$ we have $<xuu>=<xhu>=x$ by the definition of right unity h associated with u. This contradicts $<xuu>=u$ which follows from the definition of the right zero u. (ii) If $g=v$, then for any $x \neq v$, we have $<vxv>=<gxv>=x$ by the definition of left unity associated with v. This contradicts $<vxv>=v$ which follows from the definition of the left zero v. (iii) The following is an example of a PTR in which r is a right zero and a left unity associated with itself:

Example 5.

$x=r$

$m \backslash b$	r	s	t
r	r	s	t
s	s	r	t
t	t	s	r

$x=s$

$m \backslash b$	r	s	t
r	r	s	t
s	r	t	s
t	r	t	s

$x=t$

$m \backslash b$	r	s	t
r	r	s	t
s	t	s	r
t	s	r	t

(iv) The following is an example of a PTR in which r is a left zero and a right unity associated with itself.

Example 6.

$x=r$

$m \backslash b$	r	s	t
r	r	s	t
s	r	s	t
t	r	s	t

$x=s$

$m \backslash b$	r	s	t
r	s	t	r
s	t	r	s
t	r	s	t

$x=t$

$m \backslash b$	r	s	t
r	t	s	r
s	r	t	s
t	s	r	t

Proposition 14. (i) *a) If a planar ternary ring* $(S, <\ \ >)$ *has a left zero* v *and a right unity* h *associated with the right zero* u, *or b) if* $(S, <\ \ >)$ *has a right zero* u *and a left unity* g *associated with the left zero* v, *then* $u=v$. (ii) *If* $(S,<\ \ >)$ *has a right unity* h *associated with the right zero* u *and a left unity* g *associated with the left zero* v, *then* $u=v$ *and* $g=h$. (iii) *A planar ternary ring* $(S,<\ \ >)$ *can have a right unity associated with the left zero* v *and a left unity* g *associated with the right zero* u *such that* $u \neq v$ *and* $g \neq h$.

Proof. (i) a) If h is a right unity associated with the right zero u, then $<vhu>=v$. On the other hand, as v is the left zero, we have $<vhu>=u$. Hence $u=v$. b) can be shown similarily. (ii) follows from (i) and Proposition 11. (iii) An example will be given in the next section.

6. Constructions of Hall ternary rings.

Proposition 15. *Suppose* $(S, <\ \ >)$ *is a* PTR *with a right zero* u *and left zero* v, *and let* k *be an element of* S *such that* $k \neq u$. *Define* $\{cmd\}=<amd>$, *where* a *is determined from* $<akv>=c$ *(such an* a *is uniquely determined by Proposition* 5, *as* $k \neq u$). *Then* (i) $(S,\{\ \ \})$ *is a* PTR *with right zero* u, *left zero* v, *and* k *as a right unity associated with the left zero* v. *(ii) If* g *is a left unity associated with the right zero* u *in* $(S, <\ \ >)$, *then* $f=<gkv>$ *is a left unity associated with the right zero* u *in* $(S, \{\ \ \})$.

Proof. (i) We note that $(S, \{\ \ \})$ is a new coordinatization of the projective plane induced by $(S, <\ \ >)$ obtained by the point coordinate transformation. $(a, b) \rightarrow (<akv>, b)$, $[m, d] \rightarrow [m, d]$. Thus $(S, \{\ \ \})$ is a PTR. 1) Because u is the right zero, $\{cud\}=<aud>=d$. 2) From $<akv>=v$ and $<vkv>=v$ (as v is the left zero) and $k \neq u$, we have $a=v$. Consequently, $\{vmd\}=<amd>=<vmd>=d$. 3) Because a is obtained from $<akv>=c$, we have $\{ckv\}=<akv>=c$.

(ii) We have that $\{fmu\}=<amu>$ where a is determined from $<akv>=f$. Since it is assumed that $<gkv>=f$, we have $a=g$ by Proposition 5. Then $\{fmu\}=<gmu>=m$, because g is a left unity associated with u in $(S, <\ \ >)$.

Proposition 16. *Suppose* $(S, <\ \ >)$ *be a* PTR *with a right zero* u *and a left zero* v. *Let* g *be an element of* S *such that* $g \neq v$. *Define* $\{and\}=<amd>$, *where* m *is determined from* $<gmu>=n$ *(*m *is uniquely determined by Proposition* 5, *as* $g \neq v$). *Then* (i) $(S, \{\ \ \})$ *is a* PTR *with right zero* u, *left zero* v, *and left unity* g *associated with* u. (ii) *If* k *is a right unity associated with* v *in* $(S, <\ \ >)$, *then* $h=<gku>$ *is a right unity associated with* v *in* $(S, \{\ \ \})$.

Proof. (i) We note that $(S, \{\ \ \})$ is a new coordinatization of the projective plane induced by $(S, <\ \ >)$, obtained by the line coordinate transformation: $[m, d] \rightarrow [<gmu>, d]$, $(a, b) \rightarrow (a, b)$. Thus $(S, \{\ \ \})$ is a PTR. 1) Because v is the left zero of $(S, <\ \ >)$, $\{vnd\}=<vmd>=d$. 2) In $\{aud\}=<amd>$, m is obtained from $<gmu>=u$. But $<guu>=u$ as u is the right zero. Thus $m=u$ by Proposition 5. Then $\{aud\}=<amd>=<aud>=d$. 3) Because m is determined from $<gmu>=<gmu>=n$.

(ii) In $\{ahv\}=<amv>$, m is determined from $<gmu>=h$. But we have also $<gku>=h$ by assumption. Hence $m=k$ by Proposition 5. Then $\{ahv\}=<akv>=a$ as k is a right unity associated with v in $(S, <\ \ >)$.

Starting from a PTR with a right zero u and a left zero v $(u \neq v)$, we can construct a PTR with right zero u, left zero v, right unity k associated with v and left unity f associated with u (left unity g associated with u and a right unity h associated with v) by combining Propositions 15 and 16. This gives examples for (iii) of Proposition 14.

In Proposition 15 (or 16), if $u=v$, then we obtain a PTR with zero u and a right unity k (or a left unity g) associated with the zero u.

Putting $u=v=z$, we can construct a PTR with zero z and unity $h=g$ (or $f=k$) by combining Propositions 15 and 16, because we have $h=<gku>=g$, as k is the right unity of $(S, <\quad>)$ associated with u (or $k=<gkv>=f$, as g is the left unity of $(S, <\quad>)$ associated with v). These are the constructions of Proposition 1, 2, and 3 of *Martin* [2].

Instead of carrying out the constructions of Propositions 15 and 16 one after another, we can also use the following construction:

Proposition 17. *Let* $(S, <\quad>)$ *be a* PTR *with a right zero* u *and a left zero* v. *Let* k, g *be two elements in* S *such that* $k \neq u$ *and* $g \neq v$. *Define* $\{cnd\}=<amd>$, *where* a *is determined from* $<akv>=c$ *and* m *is determined from* $<gmu>=n$. *Then* $(S, \{\quad\})$ *is a* PTR *with right zero* u, *left zero* v, $f=<gkv>$ *as left unity associated with the right zero* u, *and* $h=<gku>$ *as right unity associated with the left zero* v.

Proof. We note that $(S, \{\quad\})$ is a new coordinatization of the projective plane induced by $(S, <\quad>)$ obtained by the coordinate transformation: $(a, b) \rightarrow (<akv>, b)$ and $[m, d] \rightarrow [<gmu>, d]$. 1) By definition $\{cud\}=<amd>=<aud>=d$, since u is the right zero of $(S, <\quad>)$, and $m=u$ which follows from $<gmu>=u$ and $<guu>=u$. 2) By definition $\{vnd\}=<amd>=<vmd>=d$, because v is the left zero, and $a=v$ which follows from $<akv>=v$ and $<vkv>=v$. 3) We note that $a=g$ as $<akv>=f$ and $<gkv>=f$; hence $\{fnu\}=<amu>=<gmu>=n$. 4) Finally, noting that $m=k$ as $<gmu>=h$ and $<gku>=h$, we see that $\{chv\}=<amv>=<akv>=c$.

If we put $z=u=v$ in Proposition 17, we have $f=h=<gkz>$, and hence we obtain a PTR with zero z and unity h. This is the construction given in Theorem 4.1 of *Wesson* [3]. See also Theorem 19 of *Martin* [2].

If $u \neq v$, then $h=<gku> \neq <gkv>=f$ by I. In this case Proposition 17 also gives an example for (iii) of Proposition 14.

In Proposition 17, we can modify the definition of $\{\quad\}$ to obtain the following:

Proposition 18. *Let* $(S, <\quad>)$ *be a* PTR *with right zero* u *and left zero* v, *and let k, g be two elements of S such that $k \neq u$ and $g \neq v$. Define $\{cnd\} = \langle amd \rangle$ where a is determined from $\langle aku \rangle = c$ and m is determined from $\langle gmv \rangle = n$. Then $(S, \{\ \ \})$ is a PTR with right zero v, left zero u, $f = \langle gku \rangle$ as left unity associated with right zero v, and $h = \langle gkv \rangle$ as right unity associated with left zero u.*

Proof of this proposition is similar to that of Proposition 17.

We can also modify the definition of $\{\ \ \}$ in Propositions 15 and 17 to form the following propositions.

Proposition 19. *Let $(S, \langle\ \ \rangle)$ be a* PTR *with a right zero u, and let k be an element of S such that $k \neq u$. Define $\{cmd\} = \langle amd \rangle$, where a is determined from $\langle aku \rangle = c$. Then* (i) *$(S, \{\ \ \})$ is a* PTR *with right zero u and k as right unity associated with u*; (ii) *if k is a left unity associated with u in $(S, \langle\ \ \rangle)$, k is the unity associated with the right zero u in $(S, \{\ \ \})$.*

Proof. For assertion (i), the proof is similar to that of Proposition 15. For (ii), we have $\{kmu\} = \langle amu \rangle = \langle kmu \rangle = m$, because k is a left unity associated with u in $(S, \langle\ \ \rangle)$, so that $\langle kku \rangle = k$, and $\langle aku \rangle = k$ imply $a = k$. This shows that k is a left unity associated with u in $(S, \{\ \ \})$. This and (i) imply (ii).

Proposition 20. *Let $(S, \langle\ \ \rangle)$ be a* PTR *with a left zero v, and let g be an element of S such that $g \neq v$. Define $\{and\} = \langle amd \rangle$, where m is determined from $\langle gmv \rangle = n$. Then* (i) *$(S, \{\ \ \})$ is a* PTR *with left zero v and g as left unity associated with v.* (ii) *If g is a right unity associated with v in $(S, \langle\ \ \rangle)$, then g is the unity associated with left zero v in $(S, \{\ \ \})$.*

Proof. (ii). We have $\{agv\} = \langle amv \rangle = \langle agv \rangle = a$, because g is a right unity associated with v in $(S, \langle\ \ \rangle)$, so that $\langle ggv \rangle = g$, and $\langle gmv \rangle = g$ imply $m = g$.

Proposition 21. *Suppose, in Proposition 19, that v, g $(g \neq v)$ are, respectively, a left zero and a left unity associated with v in $(S, \langle\ \ \rangle)$; then v, g are also respectively, a left zero and a left unity associated with v in $(S, \{\ \ \})$ if and only if $u = v$ and $g = k$.*

Proof. The "only if" part follows from Proposition 14 (ii). The "if" part can be easily shown as follows : $\{kmu\} = \langle amu \rangle$ where $\langle aku \rangle = k$. But $\langle kku \rangle = k$. Hence $a = k$ by Proposition 5, and so $\{kmu\} = \langle kmu \rangle = m$ as k is a left unity associated with $u = v$.

Proposition 22. *Suppose, in Proposition 20, that u, k $(k \neq u)$ are, respectively, a right zero and a right unity associated with u in $(S, \langle\ \ \rangle)$; then u, k are also respectively, right zero and a right unity associated with u in $(S, \{\ \ \})$ if and only if $u = v$ and $g = k$.*

Proof. The "if" part can be shown as follows: $\{akv\}=<amv>$ where $<kmv>=k$. But $<kkv>=k$ as k is a right unity associated with $u=v$. Hence $m=k$ by Proposition 5. Then $\{akv\}=<akv>=a$.

Combining Proposition 8 and Proposition 19, and combining Proposition 9 and Proposition 20 we will have, respectively, the following:

Proposition 23. *Let* $(S, <\ \ >)$ *be a planar ternary ring, and let* u, h *be two distinct elements in* S. *For any given* c, m, b *in* S, *define a ternary operation* $\{\ \ \}$ *by the following steps*: i) *determine an* a *such that* $<auc>=<ahu>$; ii) *determine an* e *such that* $<aue>=<amb>$; iii) *define* $\{cmb\}=e$. *Then the ternary system* $(S, \{\ \ \})$ *obtained in this way is a planar ternary ring with* u *as a right zero and* h *as a right unity associated with* u.

Proposition 24. *Let* $(S, <\ \ >)$ *be a planar ternary ring, and let* v, g *be distinct elements in* S. *For elements* c, n, p *in* S, *define a ternary operation* $\{\ \ \}$ *by the following steps*: i) *determine* m, q *such that* $<gmq>=n$ *and* $<vmq>=v$; ii) *determine* b *such that* $<vmb>=p$; iii) *define* $\{cnp\}=<cmb>$. *Then the ternary system* $(S, \{\ \ \})$ *is a planar ternary ring with the left zero* v *and the left unity* g *associated with* v.

Substituting the Propositions 23 and 24 for Propositions 19 and 20, we have the following propositions corresponding to Propositions 21 and 22:

Proposition 25. *Suppose, in Proposition* 23, *that* v, g *are, respectively, a left zero and a left unity of* $(S, <\ \ >)$ *associated with* v. *Then* v, g *are also, respectively, a letf zero and a left unity of* $(S, \{\ \ \})$ *associated with* v *if and only if* $u=v$, $g=h$ *and there exists an element* a *such that* $<aum>=<amu>$ *for all* m.

Proof. The proof of the part "$u=v$" is the same as in Proposition 14 and is obvious. Now, if g is a left unity associated with v in $(S, \{\ \ \})$, then $\{gmu\}=m$ for all m; that is, the element a obtained from $<aug>=<ahu>$ satisfies $<aum>=<amu>$ for all m. Then we have also $<ahu>=<auh>$ and hence $g=h$. Conversely, if $g=h$ and $<aum>=<amu>$ for all m, then g is obviously a left unity associated with v.

Proposition 26. *In Proposition* 24, *suppose* u *and* h *are, respectively, a right zero and a right unity associated with* u *in* $(S, <\ \ >)$. *Then* u *and* h *are also, respectively, a right zero and a right unity associated with* u *in* $(S, \{\ \ \})$ *if and only if* $u=v$ *and* $g=h$.

Proof. We will omit the proof of the "$u=v$" part. Now, h is a right unity of $(S, \{\ \ \})$ associated with the right zero u $(=v)$ if and only if $\{ahv\}=a$ for all a. This means that $<gmq>=h$, $<vmq>=v$, $<vmb>=v$, and $<amb>=a$. From $<vmq>=$

v, and $<vmb>=v$, it follows that $b=q$ by I. Since h is the right unity of $(S, < \ \ >)$ associated with right zero u $(=v)$, $<ahv>=a$ for all a. Then $<amb>=a$ and $<ahv>=a$ yield that $<ahv>=<amb>$ for all a. Hence, it follows by II that $m=h$ and $b=v$. Then $<ghv>=g$ and $<ghv>=h$, hence $h=g$. Conversely, if $h=g$, then $<gmq>=g$, $<vmq>=v$, and $<ggv>=g$, $<vgv>=v$. From these relations, it follows that $m=g$, $q=v$ by III. Furthermore, from $<vgb>=v=<vgv>$, it follows that $b=v$. Consequently, $\{cgv\}=<cmb>=<cgv>=c$.

Suppose now that $e=h=g$ and $z=u=v$. If we construct $(S, [\ \ ])$ from $(S, < \ \ >)$ by Proposition 23, then construct $(S, \{\ \ \})$ from $(S, [\ \ ])$ by Proposition 24, the resulting $(S, \{\ \ \})$ is a HTR with the zero z and the unity e by Proposition 26. Actually, such a construction can be formulated as follows :

Proposition 27. *Let* $(S, < \ \ >)$ *be a planar ternary ring, and* e, z *be two distinct elements in* S. *For any elements* c, n, p *in* S, *define the ternary operation* $\{\ \ \}$ *by the following steps*: i) *determine an* a *such that* $<aze>=<aez>$; ii) *determine an* a_1 *such that* $<a_1zz>=<a_1ez>$; iii) *determine* m, q *such that* $<amq>=<azn>$ *and* $<a_1mq>=<a_1zz>$ (*since* $e\neq z$, *it follows from* i) *and* ii) *that* $a\neq a_1$ *by* III); iv) *determine a* b *such that* $<a_1zp>=<a_1mb>$; v) *determine an* a_2 *such that* $<a_2zc>=<a_2ez>$; vi) *determine an* r *such that* $<a_2zr>=<a_2mb>$; *finally*, viii) *define* $\{cnp\}=r$. *Then* $(S, \{\ \ \})$ *is a Hall ternary ring with zero* z *and unity* e.

In a PTR with a left zero v $(=u)$ and a left unity g associated with v constructed in Proposition 24, the condition that there is an element a such that $\{aum\}=\{amu\}$ for all m in S is not necessarily satisfied. For this condition to be satisfied, it is necessary and sufficient that there is a fixed element a such that $<am_1b_1>=<am_2q_2>$ for all solutions m_1, b_1, m_2 and q_2 obtained from : $<gm_1q_1>=v$, $<vm_1q_1>=v$, $<vm_1b_1>=m$, $<gm_2q_2>=m$, and $<vm_2q_2>=v$ for all m in S. The following example shows that this is not necessarily true :

Example 7.

$x=p$

$m \backslash b$	p	q	r	s	t
p	s	t	p	r	q
q	s	q	r	p	t
r	t	q	p	r	s
s	p	s	t	r	q
t	s	t	q	r	p

$x=q$

$m \backslash b$	p	q	r	s	t
p	p	t	s	q	r
q	t	s	p	q	r
r	s	q	p	t	r
s	t	s	q	r	p
t	q	p	t	s	r

$x=r$

$m \backslash b$	p	q	r	s	t
p	t	q	s	p	r
q	r	p	q	t	s
r	t	q	r	s	p
s	p	q	r	t	s
t	s	p	t	r	q

$x=s$

$m \backslash b$	p	q	r	s	t
p	q	r	p	s	t
q	p	q	t	r	s
r	t	p	s	q	r
s	t	s	q	p	r
t	t	p	s	r	q

$x=t$

$m \backslash b$	p	q	r	s	t
p	p	t	r	s	q
q	s	t	r	q	p
r	s	p	t	q	r
s	p	q	r	t	s
t	t	q	r	p	s

Take $v=p$ and $g=q$. Then it follows from $<qm_1 q_1>=p$, $<pm_1q_1>=p$ that $m_1=r$ and $q_1=r$. Thus we have the following solutions:

$m=p$	$m_1=r$	$b_1=r$	$m_2=r$	$q_2=r$	$a=arbitrary$
q	r	q	q	s	q
r	r	s	t	t	s
s	r	t	p	r	t
t	r	p	s	p	s

7. Alternative constructions of planar ternary rings with one-sided zeros and one-sided unities.

Proposition 28. *Suppose* $(S, <\ \ >)$ *is a* PTR, *and* h, u *are in* S *with* $h\neq u$. *Define a ternary operation* $\{\ \ \}$ *as follows*:

$$\begin{aligned}\{amb\}&=a \text{ if } <amb>=<ahu>\\ &=<ahu> \text{ if } <amb>=a\\ &=<amb> \text{ if } <amb>\neq<ahu>,\ <amb>\neq a.\end{aligned}$$

Then the system $(S, \{\ \ \})$ *is a planar ternary ring satisfying* $\{ahu\}=a$ *for all* a.

Proof. We note that $(S, \{\ \ \})$ is a new coordinatization of the projective plane induced by $(S, <\ \ >)$. It is obtained by the coordinate transformation which changes $(a, <ahu>)$ to (a, a), (a, a) to $(a, <ahu>)$ on every line $[a]$ and leaves coordinates of all other points unchanged. Thus $(S, \{\ \ \})$ is a PTR. Now $\{ahu\}=a$, since in the above definition $m=h$, $b=u$ and $<amb>=<ahu>$.

Proposition 29. *In Proposition* 28, *if* u *is a left zero of* $(S, <\ \ >)$, *then* $(S, \{\ \ \}$ *is a* PTR *with a left zero* u *and a right unity* h *associated with* u.

Proof. If $<umb>=<uhu>$, then $b=u$ and $\{umb\}=u=b$. If $\{umb\}=u$, then $b=u$ and $\{umb\}=<uhu>=u=b$. For other cases $\{umb\}=<umb>=b$.

Combining Proposition 9 and Proposition 28, we can construct a PTR with a left zero u and right unity h associated with u.

Applying the construction of Proposition 20 to the PTR with a left zero v and a right unity g associated with v thus obtained, we can obtain a PTR with a left zero v and a unity g associated with the left zerc v.

Proposition 30. *Suppose* $(S, <\ \ >)$ *is a* PTR *satisfying* $<ahu>= a$ *for all* a *in* S *and two fixed elements* $h\neq u$. *Define*

$$\{amb\}=<amb> \text{ if } m\neq u,$$
$$\{aub\}=<aud>, \text{ where } d \text{ is determined from } <bud>=b.$$

Then $(S, \{\ \ \})$ *is a* PTR *satisfying* $\{ahu\}=a$ *and* $\{aua\}=a$ *for all* a *in* S.

Proof. $(S, \{\ \ \})$ is obtained by changing the coordinates of the line through the points (u) and (a, a) to $[u, a]$. Now, since $h\neq u$, $\{ahu\}=<ahu>=a$ for all a by assumption, and $\{aua\}=<aud>$, where $<aud>=a$. Hence $\{aua\}=a$ for all a in S.

Proposition 31. *Suppose* $(S, <\ \ >)$ *is a* PTR *satisfying* $<ahu>=a$ *and* $<aua>=a$ *for all* a *in* S *and two fixed elements* $h\neq u$. *Define* $\{amd\}=b$, *where* b *is determined from* $<aub>=<amd>$. *Then the system* $(S, \{\ \ \})$ *is a* PTR *with right zero* u *and right unity* h *associated with* u.

Proof. $(S, \{\ \ \})$ is obtained by the coordinate transformation used in Proposition 8. Since $\{ahu\}=b$ is determined from $<aub>=<ahu>=a$, and $<aua>=a$, we have $b=a$, hence $\{ahu\}=a$ for all a. Now $\{aud\}=d$, because $<aub>=<aud>$ implies $b=d$.

Combining Propositions 28, 30, and 31, we can construct a PTR with right zero u and right unity h associated with u.

Proposition 32. *Let* $(S, <\ \ >)$ *be a* PTR, *and let* u *be a fixed element in* S. *Define*

$$\{amu\}=<amt>, \text{ where } t \text{ is determined from } <umt>=u,$$
$$\begin{aligned}\{amt\} &=<amu>, \text{ if } t \text{ satisfies } <umt>=u,\\ &=<amt>, \text{ if } <umt>\neq u.\end{aligned}$$

Then, $(S, \{\ \ \})$ *is a* PTR *satisfying* $\{umu\}=u$ *for all* m *in* S.

Proof. We note that $(S, \{\ \ \})$ is obtained by the coordinate transformation which changes $[m, t]$ through the point (u, u) to $[m, u]$, $[m, u]$ to $[m, t]$, and leaves the coordinates of other lines unchanged. Now $\{umu\}=<umt>=u$ for all m, because t is determined from $<umt>=u$.

Proposition 33. *Let* $(S, <\ \ >)$ *be a* PTR *satisfying* $<upu>=u$ *for all* p *and a fixed element* u. *Let* g *be an element of* S *distinct from* u. *Then, for a given* n, *there is a unique solution* m *satisfying* $<gmu>=n$.

Proof. By III, $<uxy>=u$, $<gxy>=n$, $g\neq u$ has a unique solution pair $x=m$,

$y=t$. Then $<umt>=u$. But we have $<umu>=u$ by assumption. Thus $t=u$ and $<gmu>=n$.

Proposition 34. *Let* $(S, <\ \ >)$ *be a* PTR *satisfying* $<umu>=u$ *for all* m *and a fixed element* u. *Let* g *be a fixed element of* S *distinct from* u. *Define*

$$\{gnt\}=m, \text{ where } m \text{ is determined from } <gnt>=<gmu>,$$
$$\{ant\}=<ant>, \text{ if } a\neq g.$$

Then $(S, \{\ \ \})$ *is a* PTR *satisfying* $\{umu\}=u$ *and* $\{gmu\}=m$ *for all* m *in* S.

Proof. $\{gnu\}=m=n$, because $<gmu>=<gnu>$ implies $m=n$ by Proposition 33. Now $\{unu\}=<unu>=n$ because $g\neq u$. The system is obtained by the coordinate transformation which changes $(g, <gmu>)$ to (g, m) on the line $[g]$ and leaves all the other coordinates unchanged.

Proposition 35. *In Proposition* 34, *if* u *is the left zero* (*then* $<umu>=u$ *is obviously satisfied*), *then the resulting system* $(S, \{\ \ \})$ *is a* PTR *with the left zero* u *and left unity* g *associated with* u.

Proof. $\{unt\}=<unt>=t$ for all t, as $g\neq u$ and u is the left zero of $(S, <\ \ >)$.

Combining the constructions of Propositions 9, 34, and 35, we can construct a PTR with left zero u and left unity g associated with u.

Proposition 36. *Suppose* $g\neq u$ *are in* S *and* $(S, <\ \ >)$ *is a* PTR *satisfying* $<umu>=u$ *and* $<gmu>=m$ *for all* m. *Define*

$$\{aub\}=<aut>, \text{ where } t \text{ is determined from } <gut>=b,$$
$$\{amb\}=<amb>, \text{ if } m\neq u.$$

Then, $(S, \{\ \ \})$ *is a* PTR *satisfying* $\{umu\}=u$ *for all* m, $\{gnu\}=n$ *for all* n, *and* $\{gub\}=b$ *for all* b.

Proof. 1) For. $m=u$, $\{uuu\}=<uut>$, where t is determined by $<gut>=u$. Since we have $<guu>=u$ by assumption, it follows that $t=u$. Thus $\{uuu\}=u$ as $<uuu>=u$. If $m\neq u$, then $\{umu\}=<umu>=u$ by assumption. 2) By definition $\{gub\}=<gut>=b$, as t is determined from $<gut>=b$. 3) If $n=u$, then $\{guu\}=u$ by 2). If $n\neq u$ then $\{gnu\}=<gnu>=n$ by assumption.

We note that $(S, \{\ \ \})$ is obtained by the coordinate transformation which changes the coordinates $[u, t]$ of the line through (u) and (g, m) to the coordinates $[u, m]$ and leaves all other coordinates unchanged.

Proposition 37. *Suppose* $g\neq u$ *are in* S, *and* $(S, <\ \ >)$ *is a* PTR *satisfying* $<umu>=u$ *for all* m, $<gnu>=n$, *for all* n, *and* $<gub>=b$ *for all* b. *Define* $\{amb\}=d$, *where* d *is determined from* $<aud>=<amb>$. *Then* $(S, \{\ \ \})$ *is a* PTR *with right zero u, left unity g associated with u, and for all m in S,* $\{umu\}=u$.

Proof. Note that $(S, \{\quad\})$ is obtained by the coordinate transformation used in Proposition 8. 1) As $<aud>=<aub>$ implies $d=b$, $\{aub\}=b$ for all a, b. 2) Because $d=<gud>=<gnu>=n$, $\{gnu\}=n$, 3) Since $<uud>=<umu>=u=<uuu>$ implies $d=u$, we have $\{umu\}=u$.

Combining Propositions 32, 34, 36, and 37, we can obtain a PTR with right zero u and left unity g associated with u. Furthermore if the construction in Proposition 19 is applied to this PTR with right zero u and a left unity k $(=g)$ associated with u, we can obtain a PTR with right zero u and unity k associated with u.

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