ON AN EXTENSION OF THE MITTAG-LEFFLER FUNCTION*

By

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1. INTRODUCTION

Ever since its introduction in 1903, the Mittag-Leffler function [1)]

$$E_\alpha(z)=\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(m\alpha+1)} \tag{1.1}$$

has received considerable attention of several writers. The fact that it not only furnishes an interesting extension of the exponential function but it also plays an important rôle in certain problems of frequent occurrence in mathematical physics has stimulated the development of its various generalizations in one and more variables. *Humbert and Agarwal* ([3], [4]) have studied the properties of a slightly more general function defined by

$$E_{\alpha,\beta}(x)=x^{\frac{\beta-1}{\alpha}}\sum_{m=0}^{\infty}\frac{x^m}{\Gamma(m\alpha+\beta)} \tag{1.2}$$

with the Laplace-Carson operational image

$$E_{\alpha,\beta}(x^\alpha)\supset\frac{p^{\alpha-\beta+1}}{p^\alpha-1}, \tag{1.3}$$

where the abbreviation $f(x)\supset\Phi(p)$ stands for the integral equation

$$\Phi(p)=p\int_0^\infty e^{-px}f(x)\,dx.$$

An extension of this function to two variables is due to *Humbert and Delerue* [5] who defined the function

$$\mathfrak{E}_{\alpha,\beta}(x,y)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^{m+\frac{\beta(n+1)-1}{\alpha}}y^n}{\Gamma[m\alpha+(n+1)\beta]\,\Gamma(n\beta+1)} \tag{1.4}$$

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1) For a detailed discussion of the various important properties and applications of the Mittag-Leffler function see *Erdélyi et al.* [1, pp. 206-212] and *Humbert* [2].

which has the image

$$(1.5)\qquad \mathfrak{E}_{\alpha,\beta}(x^{\alpha}, y^{\beta}) \supset\supset \frac{p^{\alpha+1}}{p^{\alpha}-1}\cdot\frac{q^{\beta}}{(p\,q)^{\beta}-1}$$

in two-dimensional symbolic calculus.

Subsequently, *Delerue and Blöndel* [6] gave several interesting properties of this generalization, while in a recent paper *Chak* [7] has considered the function

$$(1.6)\qquad \mathfrak{E}^{k}_{\alpha,\beta}(x, y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^{n+\frac{\beta(nk+1)-1}{\alpha}}\,y^{n}}{\Gamma[m\alpha+(nk+1)\beta]\,\Gamma(n\beta+1)}$$

which satisfies the operational relation

$$(1.7)\qquad \mathfrak{E}^{k}_{\alpha,\beta}(x^{\alpha}, y^{\beta}) \supset\supset \frac{p^{\alpha+\beta(k-1)+1}}{p^{\alpha}-1}\cdot\frac{q^{\beta}}{p^{k\beta}\,q^{\beta}-1},$$

$$Re(p) > 1,\ Re(q) > 1,$$

and reduces to the one studied in [5] and [6] when $k=1$.

In this paper we aim at the unification of the various observations of the earlier writers by first presenting a detailed and systematic analysis of the symmetrical generalization of (1.4) and (1.6) in the form

$$(1.8)\qquad \xi^{\nu;\sigma}_{\alpha,\beta,\lambda,\mu}[x, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{x^{m+\frac{\beta(\nu n+1)-1}{\alpha}}\,y^{n+\frac{\mu(\sigma m+1)-1}{\lambda}}}{\Gamma[m\alpha+(\nu n+1)\beta]\,\Gamma[n\lambda+(\sigma m+1)\mu]},$$

and then introducing further extensions of (1.8) in two and more arguments.

2. RELATIONSHIPS WITH KNOWN FUNCTIONS

The following relations between our function and the earlier ones are immediate consequences of the definition (1.8).

$$(2.1)\qquad \xi^{k,0}_{\alpha,\beta,\beta,1}[x, y] = \mathfrak{E}^{k}_{\alpha,\beta}(x, y).$$

$$(2.2)\qquad \xi^{1,0}_{\alpha,\beta,\beta,1}[x, y] = \mathfrak{E}_{\alpha,\beta}(x, y).$$

$$(2.3)\qquad \xi^{0,0}_{\alpha,\beta,\lambda,\mu}[x, y] = E_{\alpha,\beta}(x)\,E_{\lambda,\mu}(y).$$

$$(2.4)\qquad \xi^{0;0}_{\alpha,1,\lambda,1}[x, y] = E_{\alpha}(x)\,E_{\lambda}(y).$$

$$(2.5)\qquad \xi^{0;0}_{1,1,1,1}[x, y] = \exp(x+y).$$

$$(2.6)\qquad \xi^{0;0}_{2,2,2,2}[x^{2}, y^{2}] = \sinh x\ \sinh y.$$

$$(2.7)\qquad \xi^{0;0}_{2,1,2,1}[x^{2}, y^{2}] = \cosh x\ \cosh y.$$

$$(2.8)\qquad \xi^{0,0}_{\alpha,\alpha+1,\lambda,\lambda+1}[x, y] = E_{\alpha}(x)\,E_{\lambda}(y) - E_{\alpha}(x) - E_{\lambda}(y) + 1.$$

(2.9) $$\xi^{0,0}_{\frac{1}{2},1,\frac{1}{2},1}[\sqrt{x},\sqrt{y}]=4\pi^{-1}\,exp\,(-x-y)\,Erfc\,(-\sqrt{x})\,Erfc\,(-\sqrt{y}).$$

(2.10) $$\xi^{0,0}_{\alpha,\beta,\lambda,1}\,[x,0]=E_{\alpha,\beta}\,(x).$$

(2.11) $$\xi^{0,0}_{\alpha,1,\lambda,1}\,[x,0]=E_{\alpha}\,(x).$$

(2.12) $$\xi^{0,0}_{1,1,\lambda,1}\,[x,0]=exp\,(x).$$

(2.13) $$\xi^{0,0}_{\frac{1}{2},1,\lambda,1}\,[\sqrt{x},0]=2\pi^{-\frac{1}{2}}\,exp\,(-x)\,Erfc\,(-\sqrt{x}).$$

3. OPERATIONAL PROPERTIES

The image of our function is given by

(3.1) $$\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}\,[x^{\alpha},y^{\lambda}]\supset\supset\frac{1}{p^{\beta-1}\,q^{\mu-1}}\left(1-\frac{1}{p^{\alpha}\,q^{\mu\sigma}}\right)^{-1}\left(1-\frac{1}{p^{\nu\beta}\,q^{\lambda}}\right)^{-1},$$

provided $Re\,(\beta)>0$, $Re\,(\mu)>0$, $Re\,(p)>1$, and $Re\,(q)>1$.

Following the usual notations of the symbolic calculus, if we write $\underset{x}{\supset}$ for the image with respect to x, and $\underset{y}{\supset}$ with respect to y, then we readily have

(3.2) $$\xi^{\nu,0}_{\alpha,\beta,\lambda,\mu}\,[x^{\alpha},y^{\lambda}]\underset{x}{\supset}\frac{p^{\alpha-\beta+\frac{(\mu-1)\nu\beta}{\lambda}+1}}{p^{\alpha}-1}\,E_{\lambda,\mu}\left(\frac{y^{\lambda}}{p^{\nu\beta}}\right),$$

(3.3) $$\xi^{0,\sigma}_{\alpha,\beta,\lambda,\mu}\,[x^{\alpha},y^{\lambda}]\underset{y}{\supset}\frac{q^{\lambda-\mu+\frac{(\beta-1)\mu\sigma}{\alpha}+1}}{q^{\lambda}-1}\,E_{\alpha,\beta}\left(\frac{x^{\alpha}}{q^{\mu\sigma}}\right),$$

(3.4) $$\xi^{\nu,0}_{\alpha,\beta,\lambda,\mu}\,[x^{\alpha},-y^{\lambda}]\underset{x}{\supset}\frac{p^{\alpha-\beta+\frac{(\mu-1)\nu\beta}{\lambda}+1}}{p^{\alpha}-1}\,E_{\lambda,\mu}\left(-\frac{y^{\lambda}}{p^{\nu\beta}}\right)$$

and

(3.5) $$\xi^{0,\sigma}_{\alpha,\beta,\lambda,\mu}\,[-x^{\alpha},y^{\lambda}]\underset{y}{\supset}\frac{q^{\lambda-\mu+\frac{(\beta-1)\mu\sigma}{\alpha}+1}}{q^{\lambda}-1}\,E_{\alpha,\beta}\left(-\frac{x^{\alpha}}{q^{\mu\sigma}}\right).$$

Also we can easily prove that

(3.6) $$\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}\,[-x^{\alpha},y^{\lambda}]\supset\supset\frac{p^{\alpha-\beta+\nu\beta+1}e^{-\frac{i\pi}{\alpha}(1-\beta+\nu\beta)}}{p^{\alpha}\,q^{\mu\sigma}+1}\cdot\frac{q^{\lambda-\mu+\mu\sigma+1}}{p^{\nu\beta}q^{\lambda}\,e^{-\frac{i\pi\nu\beta}{\alpha}}-1}$$

and

(3.7) $$\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}\,[x^{\alpha},-y^{\lambda}]\supset\supset\frac{p^{\alpha-\beta+\nu\beta+1}}{p^{\alpha}\,q^{\mu\sigma}-1}\cdot\frac{q^{\lambda-\mu+\mu\sigma+1}\,e^{-\frac{i\pi}{\lambda}(1-\mu+\mu\sigma)}}{p^{\nu\beta}\,q^{\lambda}\,e^{-\frac{i\pi\mu\sigma}{\lambda}}+1}.$$

4. DIFFERENTIAL EQUATIONS

If β and λ are positive integers, then by direct differentiation it can be shown that the function

$$\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[x^\alpha, y^\lambda]$$

satisfies the partial differential equation

$$\frac{\partial^{\nu\beta+\lambda} f}{\partial x^{\nu\beta}\, \partial y^\lambda} = f \tag{4.1}$$

for all α, μ, ν, and σ such that $\lambda-\mu$ and $(\nu-1)\beta$ are non-negative integers.

Similarly, if α and μ are positive integers, then the function

$$\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[x^\alpha, y^\lambda]$$

will be a solution of

$$\frac{\partial^{\alpha+\mu\sigma} f}{\partial x^\alpha\, \partial y^{\mu\sigma}} = f \tag{4.2}$$

for all β, λ, ν, and σ provided that $\alpha-\beta$ and $\mu(\sigma-1)$ are non-negative integers.

The special case $\lambda=\beta$, $\mu-1=\sigma=0$ of (4.1) leads us to the known differential equation (17), p. 261 in [7] which, in turn, reduces to the one given earlier by *Delerue and Blöndel* [6, p. 49] for $\mathfrak{E}_{\alpha,\beta}(x^\alpha, y^\beta)$ when $\nu=1$.

5. ASSOCIATED INTEGRALS

Making use of *Hankel's* well-known generalization of the second Eulerian integral, viz.

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{-\infty}^{(0+)} t^{-z} e^t \, dt,$$

we obtain a contour integral representation of our function in the form

$$\xi^{\nu,0}_{\alpha,\beta,\lambda,\mu}[x, y] = x^{\frac{\beta-1}{\alpha}} \cdot \frac{1}{2\pi i}\int_{-\infty}^{(0+)} \frac{t^{\alpha-\beta+\frac{(\mu-1)\nu\beta}{\lambda}}}{t^\alpha - x} \cdot e^t E_{\lambda,\mu}(yt^{-\nu\beta})\, dt, \tag{5.1}$$

where $|arg(t)| \leqq \pi$ on the path of integration which is a loop that starts and ends at $-\infty$, and encircles the circular disc $|t| \leqq |x|^{\frac{1}{\alpha}}$ in the positive sense.

Similarly we have

$$\xi^{0,\sigma}_{\alpha,\beta,\lambda,\mu}[x, y] = y^{\frac{\mu-1}{\lambda}} \cdot \frac{1}{2\pi i}\int_{-\infty}^{(0+)} \frac{t^{\lambda-\mu+\frac{(\beta-1)\mu\sigma}{\alpha}}}{t^\lambda - y} \cdot e^t E_{\alpha,\beta}(xt^{-\mu\sigma})\, dt, \tag{5.2}$$

the integral being taken along a contour C which starts at $-\infty$, encircles the circular disc $|t| \leqq |y|^{\frac{1}{\lambda}}$ once counter-clockwise and returns to the starting point; $|arg(t)| \leqq \pi$ on C.

Next we multiply both sides of the identity

$$\frac{p^\alpha}{p^\alpha-1}=\frac{p^{2\alpha}}{p^{2\alpha}-1}+\frac{1}{p}\cdot\frac{1}{p^{\alpha-1}}\cdot\frac{p^{2\alpha}}{p^{2\alpha}-1}$$

by

$$p^{1-\beta+\frac{(\mu-1)\nu\beta}{\lambda}}E_{\lambda,\mu}\left(\frac{y^\lambda}{p^{\nu\beta}}\right)$$

and make use of (3.2) to get

$$\xi^{\nu,0}_{\alpha,\beta,\lambda,\mu}[x^\alpha,y^\lambda]=\xi^{\nu,0}_{2\alpha,\beta,\lambda,\mu}[x^{2\alpha},y^\lambda]+\int_0^x\frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}\xi^{\nu,0}_{2\alpha,\beta,\lambda,\mu}[s^{2\alpha},y^\lambda]\,ds. \tag{5.3}$$

Similar consequence of (3.3) is the formula

$$\xi^{0,\sigma}_{\alpha,\beta,\lambda,\mu}[x^\alpha,y^\lambda]=\xi^{0,\sigma}_{\alpha,\beta,2\lambda,\mu}[x^\alpha,y^{2\lambda}]+\int_0^y\frac{(y-t)^{\lambda-1}}{\Gamma(\lambda)}\xi^{0,\sigma}_{\alpha,\beta,2\lambda,\mu}[x^\alpha,t^{2\lambda}]\,dt. \tag{5.4}$$

From the elementary identity

$$\frac{1}{2b}\left[\frac{1}{a-b}-\frac{1}{a+b}\right]=\frac{1}{2a}\left[\frac{1}{a-b}+\frac{1}{a+b}\right]$$

it follows that

$$\frac{p^{\alpha-\beta+\nu\beta+1}q^{\lambda-\mu+\mu\sigma+1}}{p^\alpha q^{\mu\sigma}-1}\left[\frac{1}{p^{\nu\beta}q^\lambda-1}-\frac{1}{p^{\nu\beta}q^\lambda+1}\right]=\frac{1}{pq}\cdot\frac{p^{\alpha-\beta+\nu\beta+1}q^{\lambda-\mu+\mu\sigma+1}}{p^\alpha q^{\mu\sigma}-1}$$

$$\cdot\left[\frac{1}{p^{\nu\beta}q^\lambda-1}+\frac{1}{p^{\nu\beta}q^\lambda+1}\right]\cdot\frac{1}{p^{\nu\beta-1}q^{\lambda-1}},$$

and therefore, by considering the original of both sides in view of (3.1) and (3.7), we find that if

$$exp\left(\frac{i\pi\mu\sigma}{\lambda}\right)=1,$$

then

$$\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[x^\alpha,y^\lambda]-e^{\frac{i\pi}{\lambda}(1-\mu)}\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[x^\alpha,-y^\lambda] \tag{5.5}$$

$$=\int_0^x\int_0^y\left\{\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[s^\alpha,t^\lambda]+e^{\frac{i\pi}{\lambda}(1-\mu)}\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[s^\alpha,-t^\lambda]\right\}\cdot\frac{(x-s)^{\nu\beta-1}(y-t)^{\lambda-1}}{\Gamma(\nu\beta)\Gamma(\lambda)}ds\,dt.$$

By symmetry, we have

$$\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[x^\alpha,y^\lambda]-e^{\frac{i\pi}{\alpha}(1-\beta)}\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[-x^\alpha,y^\lambda] \tag{5.6}$$

$$=\int_0^x\int_0^y\left\{\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[s^\alpha,t^\lambda]+e^{\frac{i\pi}{\alpha}(1-\beta)}\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[-s^\alpha,t^\lambda]\right\}\cdot\frac{(x-s)^{\alpha-1}(y-t)^{\mu\sigma-1}}{\Gamma(\alpha)\Gamma(\mu\sigma)}ds\,dt,$$

provided

$$exp\left(\frac{i\pi\nu\beta}{\alpha}\right)=1.$$

Also since

$$\frac{1}{pq}=\frac{1}{pq}\cdot\frac{p^{\alpha-\beta+\nu\beta+1}\,q^{\lambda-\mu+\mu\sigma+1}}{(p^\alpha q^{\mu\sigma}-1)(p^{\nu\beta}q^\lambda-1)}\cdot\frac{(p^\alpha q^{\mu\sigma}-1)(p^{\nu\beta}q^\lambda-1)}{p^{\alpha-\beta+\nu\beta+1}\,q^{\lambda-\mu+\mu\sigma+1}}=\frac{1}{pq}\cdot\frac{p^{\alpha-\beta+\nu\beta+1}\,q^{\lambda-\mu+\mu\sigma+1}}{(p^\alpha q^{\mu\sigma}-1)(p^{\nu\beta}q^\lambda-1)}$$

$$\cdot\left[\frac{1}{p^{1-\beta}q^{1-\mu}}-\frac{1}{p^{1-\beta+\nu\beta}q^{\lambda-\mu+1}}-\frac{1}{p^{\alpha-\beta+1}q^{1-\mu+\mu\sigma}}+\frac{1}{p^{\alpha-\beta+\nu\beta+1}q^{\lambda-\mu+\mu\sigma+1}}\right],$$

it follows that

$$(5.7)\qquad xy=\int_0^x\int_0^y \xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[s^\alpha,t^\lambda]\left\{\frac{(x-s)^{1-\beta}(y-t)^{1-\mu}}{\Gamma(2-\beta)\Gamma(2-\mu)}-\frac{(x-s)^{\nu\beta-\beta+1}(y-t)^{\lambda-\mu+1}}{\Gamma(\nu\beta-\beta+2)\Gamma(\lambda-\mu+2)}\right.$$

$$\left.-\frac{(x-s)^{\alpha-\beta+1}(y-t)^{\mu\sigma-\mu+1}}{\Gamma(\alpha-\beta+2)\Gamma(\mu\sigma-\mu+2)}+\frac{(x-s)^{\alpha-\beta+\nu\beta+1}(y-t)^{\lambda-\mu+\mu\sigma+1}}{\Gamma(\alpha-\beta+\nu\beta+2)\Gamma(\lambda-\mu+\mu\sigma+2)}\right\}ds\,dt.$$

Next we use the identity

$$\frac{1}{q^{m+\mu-1}}\cdot\frac{p^{\alpha-\beta-n+1}}{p^\alpha-1}=\frac{1}{pq}\cdot\frac{p^{\alpha-\beta+\nu\beta+1}\,q^{\lambda-\mu+1}}{(p^\alpha-1)(p^{\nu\beta}q^\lambda-1)}\cdot\left[\frac{1}{p^{n-1}q^{m-1}}-\frac{1}{p^{n+\nu\beta-1}q^{m+\lambda-1}}\right],$$

and we have

$$(5.8)\qquad \frac{y^{m+\mu-1}}{\Gamma(m+\mu)}E_{\alpha,n+\beta}(x^\alpha)=\int_0^x\int_0^y \xi^{\nu,0}_{\alpha,\beta,\lambda,\mu}[s^\alpha,t^\lambda]\cdot\left\{\frac{(x-s)^{n-1}(y-t)^{m-1}}{\Gamma(n)\Gamma(m)}\right.$$

$$\left.-\frac{(x-s)^{n+\nu\beta-1}(y-t)^{m+\lambda-1}}{\Gamma(n+\nu\beta)\Gamma(m+\lambda)}\right\}ds\,dt.$$

Similarly

$$(5.9)\qquad \frac{x^{n+\beta-1}}{\Gamma(n+\beta)}E_{\lambda,m+\mu}(y^\lambda)=\int_0^x\int_0^y \xi^{0,\sigma}_{\alpha,\beta,\lambda,\mu}[s^\alpha,t^\lambda]\cdot\left\{\frac{(x-s)^{n-1}(y-t)^{m-1}}{\Gamma(n)\Gamma(m)}\right.$$

$$\left.-\frac{(x-s)^{n+\alpha-1}(y-t)^{m+\mu\sigma-1}}{\Gamma(n+\alpha)\Gamma(m+\mu\sigma)}\right\}ds\,dt.$$

From (3.1) we readily get

$$\xi^{\nu,0}_{\alpha,\beta,1,1}[x^\alpha,y]\supset\supset\frac{p^{\alpha-\beta+1}}{p^\alpha-1}+\frac{1}{pq}\cdot\frac{p^{\alpha-(\nu+1)\beta+2}}{p^\alpha-1}\cdot\frac{p^{\nu\beta}q}{p^{\nu\beta}q-1}$$

whence it follows that

$$(5.10)\qquad \xi^{\mu,0}_{\alpha,\beta,1,1}[x^\alpha,y]=E_{\alpha,\beta}(x^\alpha)+\int_0^x\int_0^y E_{\alpha,(\nu+1)\beta-1}\{(x-s)^\alpha\}\,J_0^{\nu\beta}(-s^{\nu\beta}t)\,ds\,dt,$$

where $J^\mu_\lambda(x)$ is *Wright's* generalized Bessel function defined by

$$J^\mu_\lambda(x)=\sum_{m=0}^\infty\frac{(-x)^m}{m!\,\Gamma(\lambda+\mu m+1)}\,.$$

Similarly we have

$$(5.11)\qquad \xi^{0,\sigma}_{1,1,\lambda,\mu}[x, y^\lambda]=E_{\lambda,\mu}(y^\lambda)+\int_0^x\int_0^y E_{\lambda,(\sigma+1)\mu-1}\{(y-t)^\lambda\}J_0^{\mu\sigma}(-st^{\mu\sigma})\,ds\,dt.$$

The operational relation (3.7) gives us

$$\xi^{\nu,0}_{\alpha,\beta,2,1}[x^\alpha, -y^2]\supset\supset\frac{p^{\alpha-\beta+1}}{p^\alpha-1}-\frac{1}{pq}\cdot\frac{p^{\alpha-\beta+1}}{p^\alpha-1}\cdot\frac{pq}{p^{\nu\beta}q^2+1},$$

and since [8, p. 63]

$$bei(2\sqrt{xy})\supset\supset\frac{pq}{p^2q^2+1},$$

we find that

$$(5.12)\qquad \xi^{\nu,0}_{\alpha,\frac{2}{\nu},2,1}[x^\alpha, -y^2]=E_{\alpha,\frac{2}{\nu}}(x^\alpha)-\int_0^x\int_0^y E_{\alpha,\frac{2}{\nu}}\{(x-s)^\alpha\}bei(2\sqrt{st})\,ds\,dt,$$

and similarly

$$(5.13)\qquad \xi^{0,\sigma}_{2,1,\lambda,\frac{2}{\sigma}}[-x^2, y^\lambda]=E_{\lambda,\frac{2}{\sigma}}(y^\lambda)-\int_0^x\int_0^y E_{\lambda,\frac{2}{\sigma}}\{(y-t)^\lambda\}\,bei(2\sqrt{st})\,ds\,dt.$$

The special case $\nu=1$ of (5.12) was derived earlier by *Delerue and Blöndel* [6, p. 46], and it may not be out of place to remark that my colleague, Dr. *Chak* does confirm the errors in his formula (6), p. 258 in [7] which was intended to generalize the aforesaid result of [6].

For $\lambda=n$, n being a positive integer, our formula (3.7) yields

$$\xi^{\nu,0}_{\alpha,n\beta,n,1}[x^\alpha, -y^n]\supset\supset\frac{p^{\alpha-n\beta+1}}{p^\alpha-1}-\frac{1}{pq}\cdot\frac{p^{\alpha-(\nu+n)\beta+2}}{p^\alpha-1}\cdot\frac{p^{\nu\beta}q}{p^{\nu n\beta}q^n+1},$$

and since

$$\frac{p^{\nu\beta}q}{(p^{\nu\beta}q)^n+1}\subset\subset-\frac{1}{n}\sum_{m=0}^{n-1}\frac{J_0^{\nu\beta}(-\omega^{2m+1}x^{\nu\beta}y)}{\omega^{(n-1)(2m+1)}},$$

where $\omega=e^{\frac{\pi i}{n}}$, it follows that

$$(5.14)\qquad \xi^{\nu,0}_{\alpha,n\beta,n,1}[x^\alpha,-y^n]=E_{\alpha,n\beta}(x^\alpha)+\frac{1}{n}\int_0^x\int_0^y E_{\alpha,(\nu+n)\beta-1}\{(x-s)^\alpha\}\sum_{m=0}^{n-1}\frac{J_0^{\nu\beta}(-\omega^{2m+1}s^{\nu\beta}t)}{\omega^{(n-1)(2m+1)}}\,ds\,dt.$$

It is not difficult to show that for $n=1$, (5.14) corresponds to our earlier result (5.10), nor does it seem worthwhile to record the various steps that lead us to its complementary formula

$$(5.15)\qquad \xi^{0,\sigma}_{n,1,\lambda,n\mu}[-x^n, y^\lambda]=E_{\lambda,n\mu}(y^\lambda)+\frac{1}{n}\int_0^x\int_0^y E_{\lambda,(\sigma+n)\mu-1}\{(y-t)^\lambda\}\sum_{m=0}^{n-1}\frac{J_0^{\mu\sigma}(-\omega^{2m+1}st^{\mu\sigma})}{\omega^{(n-1)(2m+1)}}\,ds\,dt$$

which would readily yield (5.11) when $n=1$ and x is replaced by $-x$.

Next we evaluate two infinite integrals associated with the generalized *Mittag-Leffler* function by making use of the symbolic calculus in two variables. Indeed it is known that if

$$\Phi(p, q) \subset\subset f(x, y),$$

then [8, p. 62]

$$\text{(i)} \qquad \sqrt{pq}\,\Phi(\sqrt{p}, \sqrt{q}) \subset\subset \frac{1}{4\pi x^{\frac{3}{2}} y^{\frac{3}{2}}} \int_0^\infty\int_0^\infty e^{-\frac{s^2}{4x}-\frac{t^2}{4y}} \cdot f(s, t)\, st\, ds\, dt,$$

and [9, p. 29]

$$\text{(ii)} \qquad \frac{1}{p^{a-1} q^{b-1}} \Phi\left(\frac{1}{p}, \frac{1}{q}\right) \subset\subset \int_0^\infty\int_0^\infty \left(\frac{x}{s}\right)^{\frac{a}{2}} \left(\frac{y}{t}\right)^{\frac{b}{2}} \cdot J_a(2\sqrt{xs}) J_b(2\sqrt{yt}) f(s, t)\, ds\, dt.$$

In order to apply (i), let us put

$$f(x, y) = \xi^{\nu, \sigma}_{\alpha, \beta, \lambda, \mu}[x^\alpha, y^\lambda]$$

so that

$$\sqrt{pq}\,\Phi(\sqrt{p}, \sqrt{q}) \subset\subset \xi^{\nu, \sigma}_{\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\lambda}{2}, \frac{\mu}{2}}[x^{\frac{\alpha}{2}}, y^{\frac{\lambda}{2}}],$$

and we get

$$(5.16) \qquad \xi^{\nu, \sigma}_{\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\lambda}{2}, \frac{\mu}{2}}[x^{\frac{\alpha}{2}}, y^{\frac{\lambda}{2}}] = \frac{1}{4\pi x^{\frac{3}{2}} y^{\frac{3}{2}}} \int_0^\infty\int_0^\infty e^{-\frac{s^2}{4x}-\frac{t^2}{4y}} \cdot \xi^{\nu, \sigma}_{\alpha, \beta, \lambda, \mu}[s^\alpha, t^\lambda]\, st\, ds\, dt,$$

provided $Re(x)>0$, $Re(y)>0$, $Re(\beta)>-1$, and $Re(\mu)>-1$.

On the other hand, if we let

$$f(x, y) = \xi^{\nu, \sigma}_{\alpha, \beta, \lambda, \mu}[(-x)^\alpha, (-y)^\lambda],$$

then (3.1) will give us

$$\Phi(p, q) = \frac{(-p)^{\alpha-\beta+\nu\beta+1}(-q)^{\lambda-\mu+\mu\sigma+1}}{[(-p)^\alpha(-q)^{\mu\sigma}-1]\,[(-p)^{\nu\beta}(-q)^\lambda-1]}$$

whence

$$\frac{1}{p^{a-1} q^{b-1}} \Phi\left(\frac{1}{p}, \frac{1}{q}\right) = (-)^{\alpha+\lambda+\mu\sigma+\nu\beta}\, \Phi(p, q)$$

provided $a=2\beta-\alpha-\nu\beta-1$, $b=2\mu-\lambda-\mu\sigma-1$.

By an appeal to (ii) we readily obtain an interesting extension of *Tricomi's* result [10] in the form

$$(5.17) \qquad (-)^{\alpha+\lambda+\mu\sigma+\nu\beta}\, \xi^{\nu, \sigma}_{\alpha, \beta, \lambda, \mu}[(-x)^\alpha, (-y)^\lambda] = \int_0^\infty\int_0^\infty \left(\frac{x}{s}\right)^{\beta-\frac{1}{2}\alpha-\frac{1}{2}\nu\beta-\frac{1}{2}} \left(\frac{y}{t}\right)^{\mu-\frac{1}{2}\lambda-\frac{1}{2}\mu\sigma-\frac{1}{2}}$$
$$\cdot J_{2\beta-\alpha-\nu\beta}(2\sqrt{xs}) J_{2\mu-\lambda-\mu\sigma}(2\sqrt{yt})\, \xi^{\nu, \sigma}_{\alpha, \beta, \lambda, \mu}[(-s)^\alpha, (-t)^\lambda]\, ds\, dt,$$

where, for convergence, $Re(\beta) > -\frac{1}{2}$ and $Re(\mu) > -\frac{1}{2}$.

6. MISCELLANEOUS RESULTS

For $\frac{\nu\beta}{\alpha}$ an even integer, our formulas (3.1), (3.6) and the elementary identity

$$\frac{p^{\alpha+1}\,q^{\mu\sigma+1}}{p^{\alpha}\,q^{\mu\sigma}+1}+\frac{p^{\alpha+1}\,q^{\mu\sigma+1}}{p^{\alpha}\,q^{\mu\sigma}-1}=\frac{2p^{2\alpha+1}\,q^{2\mu\sigma+1}}{p^{2\alpha}\,q^{2\mu\sigma}-1}$$

lead us to the interesting relation

$$\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[x^{\alpha}, y^{\lambda}]+e^{\frac{i\pi}{\alpha}(1-\beta)}\,\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[-x^{\alpha}, y^{\lambda}]=2\xi^{\nu,2\sigma}_{2\alpha,\beta,\lambda,\mu}[x^{2\alpha}, y^{\lambda}]. \tag{6.1}$$

Similarly we have

$$\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[x^{\alpha}, -y^{\lambda}]+e^{i\pi\left[\frac{1-\beta}{\alpha}-\frac{1-\mu}{\lambda}\right]}\,\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[-x^{\alpha}, y^{\lambda}]=2\xi^{\nu,2\sigma}_{2\alpha,\beta,\lambda,\mu}[x^{2\alpha}, -y^{\lambda}], \tag{6.2}$$

if $\frac{\nu\beta}{\alpha}$ is an odd integer, but $\frac{\mu\sigma}{\lambda}$ is even.

From (6.1) and (6.2) it follows that for $\frac{\nu\beta}{\alpha}$ an even integer,

$$\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[x, y]+e^{i\pi\left[\frac{1-\beta}{\alpha}\right]}.\,\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[-x, y]=2\xi^{\nu,2\sigma}_{2\alpha,\beta,\lambda,\mu}[x^{2}, y]\,; \tag{6.3}$$

and if $\frac{\nu\beta}{\alpha}$ is an odd integer,

$$\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[x, -y]+e^{i\pi\left[\frac{1-\beta}{\alpha}-\frac{1-\mu}{\lambda}\right]}\,\xi^{\nu,\sigma}_{\alpha,\beta,\lambda,\mu}[-x, y]=2\xi^{\nu,2\sigma}_{2\alpha,\beta,\lambda,\mu}[x^{2}, -y] \tag{6.4}$$

provided that $\frac{\mu\sigma}{\lambda}$ is an even integer.

In general, if we suppose that

$$1,\ \theta^{\alpha+\frac{\mu\nu\beta\sigma}{\lambda}},\ \theta^{2\left(\alpha+\frac{\mu\nu\beta\sigma}{\lambda}\right)},\ldots,\ \theta^{(n-1)\left(\alpha+\frac{\mu\nu\beta\sigma}{\lambda}\right)}$$

denote the n^{th} roots of unity, then

$$\frac{np^{n\alpha}\,q^{\mu n\sigma}}{p^{n\alpha}\,q^{\mu n\sigma}-1}=\sum_{k=0}^{n-1}\frac{p^{\alpha}\,q^{\mu\sigma}}{p^{\alpha}\,q^{\mu\sigma}-\theta^{k\left(\alpha+\frac{\mu\nu\beta\sigma}{\lambda}\right)}}=\sum_{k=0}^{n-1}\frac{\left(\frac{p}{\theta^{k}}\right)^{\alpha}\left(\theta^{\frac{k\nu\beta}{\lambda}}q\right)^{\mu\sigma}}{\left(\frac{p}{\theta^{k}}\right)^{\alpha}\left(\theta^{\frac{k\nu\beta}{\lambda}}q\right)^{\mu\sigma}-1}.$$

Now multiply both sides by

$$\frac{p^{\nu\beta-\beta+1}\,q^{\lambda-\mu+1}}{p^{\nu\beta}\,q^{\lambda}-1}$$

and make use of (3.1) to get

(6.5)
$$n\,\xi^{\nu,\,n\sigma}_{n\alpha,\,\beta,\,\lambda,\,\mu}\,[x^{n\alpha},\,y^{\lambda}]=\sum_{k=0}^{n-1}\theta^{k\left[1-\beta+(\mu-1)\frac{\beta}{\lambda}\right]}\cdot\xi^{\nu,\,\sigma}_{\alpha,\,\beta,\,\lambda,\,\mu}\,[(\theta^{k}x)^{\alpha},\,(\theta^{-\frac{k\nu\beta}{\lambda}}y)^{\lambda}].$$

The special case $\lambda=\beta,\sigma=\mu-1=0$ of (6.5) gives us the known result [7, p. 259]

(6.6)
$$n\mathfrak{E}^{\nu}_{n\alpha,\,\beta}\,[x^{n\alpha},\,y^{\beta}]=\sum_{k=0}^{n-1}\theta^{k(1-\beta)}\,\mathfrak{E}^{\nu}_{\alpha,\,\beta}\,[(\theta^{k}x)^{\alpha},\,(\theta^{-k\nu}y)^{\beta}],$$

where $1,\theta^{\alpha},\theta^{2\alpha},\cdots,\theta^{(n-1)\alpha}$ are the n^{th} roots of unity.

7. THE ASYMPTOTIC BEHAVIOR

From the definition (1.8) it follows that

(7.1)
$$\xi^{\nu,\,\sigma}_{\alpha,\,\beta,\,\lambda,\,\mu}\,[x,y]=O\left\{|x|^{\frac{\beta-1}{\alpha}}.|y|^{\frac{\mu-1}{\lambda}}\right\}$$

when $|x|$ and $|y|\to 0$.

Next we write

$$\xi^{\nu;\,0}_{\alpha,\,\beta,\,\lambda,\,\mu}[x,y]=\sum_{n=0}^{\infty}\frac{y^{n+\frac{\mu-1}{\lambda}}}{\Gamma(n\lambda+\mu)}E_{\alpha,\,(\nu n+1)\beta}(x),$$

and make use of the known fact that for $x\to\infty$,

(7.2)
$$E_{\alpha,\beta}(x)\sim\frac{1}{\alpha}\,exp\left(x^{\frac{1}{\alpha}}\right)+O\left(\frac{1}{x}\right).$$

We thus have

(7.3)
$$\xi^{\nu,\,0}_{\alpha,\,\beta,\,\lambda,\,\mu}\,[x,y]\sim\frac{1}{\alpha}\,exp\left(x^{\frac{1}{\alpha}}\right)E_{\lambda,\mu}(y)$$

when y is real and finite, and $x\to\infty$.

From (7.2) and (7.3) it follows that

(7.4)
$$\xi^{\nu,\,0}_{\alpha,\,\beta,\,\lambda,\,\mu}\,[x,y]\sim\frac{1}{\alpha\lambda}\,exp\left(x^{\frac{1}{\alpha}}+y^{\frac{1}{\lambda}}\right)$$

when both x and y are large.

In a similar manner it can be shown that for large real values of x and y,

(7.5)
$$\xi^{0;\,\sigma}_{\alpha,\,\beta,\,\lambda,\,\mu}\,[x,y]\sim\frac{1}{\alpha\lambda}\,exp\left(x^{\frac{1}{\alpha}}+y^{\frac{1}{\lambda}}\right).$$

Combining (7.4) and (7.5) we have

(7.6)
$$\xi^{\nu;\,\sigma}_{\alpha,\,\beta,\,\lambda,\,\mu}\,[x,y]\sim\frac{1}{\alpha\lambda}\,exp\left(x^{\frac{1}{\alpha}}+y^{\frac{1}{\lambda}}\right)$$

when both x and y are large.

These asymptotic behaviors of the generalized *Mittag–Leffler* function enable us to determine the domains of convergence of the various finite and infinite integrals of section 5 as well as to justify certain changes we have made earlier in the order of summation and integration.

8. FURTHER EXTENSIONS

The remarkable success of our generalization (1.8) inspires us to introduce a 'mild' extension of the image (3.1) in the form

$$\frac{\Gamma(a)\,\Gamma(b)}{p^{\beta-1}\,q^{\mu-1}}\left(1-\frac{1}{p^{\alpha}\,q^{\mu\sigma}}\right)^{-a}\left(1-\frac{1}{p^{\nu\beta}\,q^{\lambda}}\right)^{-b} \tag{8.1}$$

which corresponds to the original

$$\xi^{a,\,b,\,\nu,\,\sigma}_{\alpha,\,\beta,\,\lambda,\,\mu}\,[x^{\alpha},\,y^{\lambda}],$$

where

$$\xi^{a,\,b,\,\nu,\,\sigma}_{\alpha,\,\beta,\,\lambda,\,\mu}\,[x,\,y]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\Gamma(a+m)\,\Gamma(b+n)}{m\,!\;n\,!}\cdot\frac{x^{m+\frac{\beta(\nu n+1)-1}{\alpha}}\,y^{n+\frac{\mu(\sigma m+1)-1}{\lambda}}}{\Gamma\,[m\alpha+(\nu n+1)\,\beta]\;\Gamma\,[n\lambda+(\sigma m+1)\,\mu]}\,. \tag{8.2}$$

Obviously

$$\xi^{1,\,1,\,\nu,\,\sigma}_{\alpha,\,\beta,\,\lambda,\,\mu}\,[x,\,y]=\xi^{\nu,\,\sigma}_{\alpha,\,\beta,\,\lambda,\,\mu}\,[x,\,y].$$

The methods illustrated in the preceding sections will apply well to the function defined by (8.2), and in a forthcoming communication it may be of interest to present a detailed investigation of its n–dimensional extension

$$\xi^{a_i,\,\nu_i}_{\alpha_i,\,\lambda_i}\,[x_1,\,\cdots,\,x_n]=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{\Gamma\,[a_1+m_1]}{(m_1)\,!}\cdots \tag{8.3}$$
$$\cdots\frac{\Gamma\,[a_n+m_n]}{(m_n)\,!}\prod_{j=1}^{n}\frac{[x_j]^{f(j)}}{\Gamma\,[m_j\,\alpha_j+\sum\limits_{i\neq j}(\nu_i\,m_i+1)\,\lambda_i]}\,,$$

where, for the sake of brevity,

$$f(j)=m_j+\frac{\sum\limits_{i\neq j}[(\nu_i\,m_i+1)\,\lambda_i]-1}{\alpha_j}\,,\quad \sum_{i\neq j}\theta_i=\sum_{i=1}^{n}\theta_i-\theta_j,\ j=1,\,2,\,\cdots,\,n\,;$$

so that formally we have the image

$$\xi^{a_i,\,\nu_i}_{\alpha_i,\,\lambda_i}\,[x_1^{\alpha_1},\,\cdots,\,x_n^{\alpha_n}] \tag{8.4}$$

$$\supset\cdots\supset\frac{\Gamma(a_1)\cdots\Gamma(a_n)}{p_1^{\sum_{i\neq 1}\lambda_i-1}\cdots p_n^{\sum_{i\neq n}\lambda_i-1}}\cdot\left[1-\frac{1}{p_1^{\alpha_1}\,(p_2\cdots p_n)^{\nu_1\,\lambda_1}}\right]^{-a_1}\cdots\left[1-\frac{1}{(p_1\cdots p_{n-1})^{\nu_n\,\lambda_n}\,p_n^{\alpha_n}}\right]^{-a_n}$$

in the symbolic calculus of n variables.

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