ON AN nTH ORDER DIFFERENTIAL EQUATION AND HERMITE POLYNOMIALS

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In a recent note *M. S. Klamkin* [1] has pointed out that the following nth order differential eqution

$$\sum_{s=0}^{n} (-1)^s \binom{n}{s} x^s D^{n-s} y=0 \tag{1}$$

can be solved by transforming (1) into linear equation with constant coefficients by means of the identity

$$(i\sqrt{2})^n \sum_{s=0}^{n} (-1)^s \binom{n}{s} x^s D^{n-s} y=e^{x^2/2} H_n(iD/\sqrt{2}) ye^{-x^2/2}, \tag{2}$$

where $H_n(x)$ is the Hermite polynomial defined by

$$H_n(x)=\sum_{r=0}^{[n/2]} \frac{(-1)^r n!(2x)^{n-2r}}{(n-2r)!r!} . \tag{3}$$

The object of the present note is to point out that the identity (2) of *Klamkin* is equivalent to the following operational formula of Hermite polynomials due to *Burchnall* [2]:

$$\sum_{s=0}^{n} (-1)^n \binom{n}{s} H_s(x) D^{n-s} y=e^{x^2} D^n (e^{-x^2} y). \tag{4}$$

For, changing x into $x\sqrt{2}$ we obtain from (2)

$$\sum_{s=0}^{u} (-1)^s \binom{n}{s} (2x)^s D^{n-s} y=e^{x^2} \sum_{r=0}^{[n/2]} \frac{n!}{(n-2r)!\,r!} D^{n-2r} (ye^{-x^2}). \tag{5}$$

Next it follows from (3) and (4) that

$$e^{x^2} D^n (ye^{-x^2})=\sum_{s=0}^{n} (-n)^s \binom{n}{s} (D^{n-s} y).\sum_{r=0}^{[s/2]} \frac{(-1)^r s!(2x)^{s-2r}}{(s-2r)!\,r!}$$

$$=\sum_{r=0}^{[n/2]} \frac{(-1)^r n!}{(n-2r)!\,r!} \sum_{s=0}^{n-2r} \binom{n-2r}{s} (-2x)^{n-2r-s} D^s y.$$

Thus we have

$$\frac{1}{n!} e^{x^2} D^n (ye^{-x^2})=\sum_{r=0}^{[n/2]} \frac{(-1)^r}{r!} v_{n-2r},$$

where

$$v_{n-2r}=\sum_{s=0}^{n-2r}\binom{n-2r}{s}\frac{(-2x)^{n-2r-s}}{(n-2r)!}D^s y.$$

Now we know [3] that if u_n and v_n be two sequences, $n=0, 1, 2, \cdots$, then one of the relations

$$u_n=\sum_{r=0}^{[n/2]}\frac{(-1)^r}{r!}v_{n-2r}\,;\ v_n=\sum_{r=0}^{[n/2]}\frac{1}{r!}u_{n-2r}$$

implies the other.

Consequently we find

$$\sum_{s=0}^{n}\binom{n}{s}\frac{(-2x)^{n-s}}{n!}D^s y=\sum_{r=0}^{[n/2]}\frac{1}{r!}\cdot\frac{1}{(n-2r)!}e^{x^2}D^{n-2r}(ye^{-x^2})$$

i. e. $$\sum_{s=0}^{n}(-1)^s\binom{n}{s}(2x)^{n-s}D^{n-s}y=e^{x^2}\sum_{r=0}^{[n/2]}\frac{n!}{(n-2r)!\,r!}D^{n-2r}(ye^{-x^2}),$$

which is evidently (5).

REFERENCES

[1] M. S. Klamkin, Amer. Math. Monthly, vol. 74 (1967), pp. 1215–1216.
[2] J. L. Burchnall, Quart. Jour. Math. (Oxford), vol. 12 (1941), pp. 9–11.
[3] G. Szego, ***Orthogonal polynomials,*** Amer. Math. Soc. Collo. Publ. vol. XXIII (1959), p. 385.

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