ALMOST-CONTINUOUS MAPPINGS

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1. INTRODUCTION

The object of the present paper is to introduce a new class of mappings called almost-continuous mappings. This class contains the class of continuous mappings and is contained in the class of weakly-continuous mappings (see definition 2.3 below). Almost-continuous mappings turn out to be the natural tool for studying almost-compact spaces (A space is said to be almost-compact if each open cover has a finite subfamily whose closures cover the space) of Alexandroff and Urysohn as also nearly-compact spaces (A space is said to be nearly-compact if every open cover has a finite subfamily the interiors of the closures of whose members cover the space) in as much as every almost-continuous image of an almost-compact space is almost-compact and every almost-continuous open image of a nearly-compact space is nearly-compact [1]. Various properties of such mappings have been discussed in section 2. Section 3 is concerned with almost-open and almost-closed mappings obtained as generalisations of open and closed mappings respectively. In the last section, the notion of almost-quasi-compact mappings has been introduced and the relations of such mappings with other types of mappings introduced in sections 2 and 3 have been investigated.

A set A is called *regularly-open*, if it is the interior of its own closure or equivalently, if it is the interior of some closed set. A is called *regularly-closed*, if it is the closure of its own interior or equivalently, if it is the closure of some open set.

2. ALMOST-CONTINUOUS MAPPINGS

Definition 2.1. *A mapping $f: X \to Y$ is said to be almost-continuous at a point $x \epsilon X$, if for every neighbourhood M of $f(x)$ there is a neighbourhood N of x such that $f(N) \subset M^{-0}$. It is easy to see that the neighbourhoods M and N can be replaced by open neighbourhoods.*

Remark 2.1. It is clear that if $f: X \to Y$ is continuous at a point $x \epsilon X$, then it is almost-continuous at x. But the converse of this statement may not be true, as the following example shows.

Example 2.1. Let R be the set of real numbers and let $\mathfrak{T}$ consist of ϕ, R and the complements of all countable subsets of R. Let $X = \{a, b\}$ and let $\mathfrak{T}^* = \{X, \phi, \{a\}\}$. Let $f: (R, \mathfrak{T}) \to (R, \mathfrak{T}^*)$ be defined as follows: $f(x) = \begin{cases} a \text{ if } x \text{ is rational,} \\ b \text{ if } x \text{ is irrational.} \end{cases}$ Then f is almost-continuous at each point of R, but f is not continuous at $x \epsilon R$ if x is rational.

Theorem 2.1. *For a mapping $f: X \to Y$, the following are equivalent*:

(*a*) *f is almost-continuous at $x \in X$.*

(*b*) *For each regularly-open neighbourhood M of $f(x)$, there is a neighbourhood N of x such that $f(N) \subset M$.*

(*c*) *For each net $\{x_\lambda\}_{\lambda \in D}$ converging to x, the net $\{f(x_\lambda)\}_{\lambda \in D}$ is eventually in every regularly open set containing $f(x)$.*

Proof. (*a*) $\Rightarrow$ (*b*). If f is almost-continuous at x and M is a regularly-open neighbourhood of $f(x)$, then there is a neighbourhood N of x such that $f(N) \subset M^{-0} = M$.

(*b*) $\Rightarrow$ (*c*). Let $\{x_\lambda\}_{\lambda \in D}$ be a net converging to x and let U be any regularly-open set containing $f(x)$. Since f is almost-continuous, there is an open set M containing x such that $f(M) \subset U$. Now, since M is an open set containing x and the net $\{x_\lambda\}_{\lambda \in D}$ converges to x, therefore there is a $\lambda_0 \in D$ such that $\lambda \geqslant \lambda_0 \Rightarrow x_\lambda \in M$. The set D is directed by '$\geqslant$'. Thus, for all $\lambda \geqslant \lambda_0, f(x_\lambda) \in f(M) \subset U$. Hence the net is eventually in U.

(*c*) $\Rightarrow$ (*a*). Suppose that f is not almost-continuous at x. Then there is an open set V containing $f(x)$ such that for every open set U containing x, $f(U) \cap (Y \sim V^{-0}) \neq \phi$. This implies that $U \cap f^{-1}(Y \sim V^{-0}) \neq \phi$ for every open set U containing x. The family $\mathfrak{U}$ of all open sets U containing x is directed by set inclusion. For each $U \in \mathfrak{U}$ choose a point x_U belonging to $U \cap f^{-1}(Y \sim V^{-0})$. Then $\{x_U\}_{U \in \mathfrak{U}}$ is a net in X which converges to x and is such that no $f(x_U)$ is in V^{-0}. Thus $\{f(x_U)\}_{U \in \mathfrak{U}}$ is not eventually in the regularly-open set V^{-0}, which is a contradiction.

Definition 2.2. *A mapping $f: X \to Y$ is said to be almost-continuous if it is almost-continuous at each point x of X.*

Remark 2.2. An almost-continuons mapping may fail to be continuous. The mapping f of example 2.1 is an almost-continuous mapping which is not continuous. The following is another example of such a mapping.

Example 2.2. Let $(R, \mathfrak{T})$ be the space of example 2.1 and let $\mathfrak{U}$ denote the usual topology for R. Let i be the identity mapping of $(R, \mathfrak{U})$ onto $(R, \mathfrak{T})$. Then i is almost-continuous but not continuous (at any point !).

Remark 2.3. The inverse of an almost-continuous one-to-one mapping may fail to be almost-continuous. In fact, the inverse of the mapping i of example 2.2 is not almost continuous (at any point !).

Theorem 2.2. *For a mapping $f: X \to Y$, the following are equivalent*:

(*a*) *f is almost-continuous.*

(*b*) *Inverse image of every regularly-open subset of Y is an open subset of X.*

(*c*) *Inverse image of every regularly-closed subset of* Y *is a closed subset of* X.

(*d*) *For each point* x *of* X *and for each regularly-open neighbourhood* M *of* $f(x)$, *there is a neighbourhood* N *of* x *such that* $f(N) \subset M$.

(*e*) $f^{-1}(A) \subset [f^{-1}(A^{-0})]^0$ *for every open subset* A *of* Y.

(*f*) $[f^{-1}(B^{-0})]^- \subset f^{-1}(B)$ *for every closed subset* B *of* Y.

(*g*) *For any point* $x \in X$ *and for any net* $\{x_\lambda\}_{\lambda \in D}$ *which converges to* x, *the net* $\{f(x_\lambda)\}_{\lambda \in D}$ *is eventually in each regularly-open set containing* $f(x)$.

Proof. (*a*) ➡ (*b*). Let U be any regularly-open subset of Y and let $x \in f^{-1}(U)$. Then $f(x) \in U$. Therefore there exists an open set V in X such that $x \in V$ and $f(V) \subset \bar{U}^0 = U$. Thus, $x \in V \subset f^{-1}(U)$ and therefore $f^{-1}(U)$ is a neighbourhood of x. Hence $f^{-1}(U)$ is open.

(*b*) ➡ (*c*). Let A be any regularly-closed subset of Y. Then $Y \sim A$ is regularly-open and therefore $f^{-1}(Y \sim A)$ is open, that is, $X \sim f^{-1}(A)$ is open. Hence $f^{-1}(A)$ is closed.

(*c*) ➡ (*d*). Since M is regularly-open, therefore $Y \sim M$ is regularly-closed, and consequently $f^{-1}(Y \sim M)$ is closed, i. e., $f^{-1}(M)$ is open. Also, $x \in f^{-1}(M) = N$ (say). Then N is a neighbourhood of x such that $f(N) \subset M$.

(*d*) ➡ (*e*). Let $x \in f^{-1}(A)$. Then $\bar{A}^0$ is a regularly-open neighbourhood of $f(x)$, since A is open. Then, there exists an open neighbourhood N of x such that $f(N) \subset A^{-0}$. Thus, $x \in N \subset f^{-1}(A^{-0})$. This means that $x \in [f^{-1}(A^{-0})]^0$. Hence $f^{-1}(A) \subset [f^{-1}(A^{-0})]^0$.

(*e*) ➡ (*f*). Since $Y \sim B$ is open, therefore $f^{-1}(Y \sim B) \subset [f^{-1}(\overline{Y \sim B})^0]^0$. This implies that $[X \sim f^{-1}(\overline{Y \sim B})^0]^- \subset f^{-1}(B)$, i. e., $[f^{-1}(B^{0-})]^- \subset f^{-1}(B)$.

(*f*) ➡ (*g*). Let N be any regularly-open set containing $f(x)$. Then, $Y \sim N$ being closed, $[f^{-1}((Y \sim N)^{0-})]^- \subset f^{-1}(Y \sim N)$. Since $Y \sim N$ is regularly-closed, therefore $[f^{-1}(Y \sim N)]^- \subset X \sim f^{-1}(N)$. This means that $f^{-1}(N) \subset [f^{-1}(N)]^0$. Thus $f^{-1}(N)$ is an open set containing x. Since the net $\{x_\lambda\}_{\lambda \in D}$ converges to x, therefore there exists $\lambda_0 \in D$ such that for all $\lambda \geqslant \lambda_0$ (D is directed by '$\geqslant$') $x_\lambda \in f^{-1}(N)$. This means that $f(x_\lambda) \in N$ for all $\lambda \geqslant \lambda_0$, i. e., the net $\{f(x_\lambda)\}_{\lambda \in D}$ is eventually in N.

(*g*) ➡ (*a*). By using (*c*) of theorem 2.1, it is clear that f is almost-continuous.

This completes the proof of the theorem.

Definition 2.3. *A mapping* $f : X \rightarrow Y$ *is said to be weakly-continuous if for each point* $x \in X$ *and each neighbourhood* V *of* $f(x)$, *there exists a neighbourhood* U *of* x *such that* $f(U) \subset \bar{V}$ [2]. *It is easy to see that the 'neighbourhoods' in the definition can be replaced by 'open neighbourhoods'.*

Remark 2.4. Obviously, every almost-continuous mapping is weakly-continuous. But a weakly-continuous mapping may fail to be almost-continuous. The following is an example.

Example 2.3. Let $(R, \mathfrak{T})$ be the space of example 2.1. Let $X=\{a, b, c\}$ and let $\mathfrak{T}^*=\{\phi, X, \{a\}, \{c\}, \{a, c\}\}$. Let f be the mapping of $(R, \mathfrak{T})$ into $(X, \mathfrak{T}^*)$ defined as follows:

$f(x)=\begin{cases} a \text{ if } x \text{ is rational,} \\ b \text{ if } x \text{ is irrational.}\end{cases}$ Then f is a weakly-continuous mapping which is not almost-continuous (at any rational point).

However, we have the following:

Theorem 2.3. *If $f: X \rightarrow Y$ is a weakly-continuous open mapping, then f is almost-continuous.*

Proof. Let $x \in X$ and let M be any neighbourhood of x. Since f is weakly-continuous, there is an open neighbourhood N of x such that $f(N) \subset \bar{M}$. Since f is open, therefore $f(N)$ is open. Then $f(N) \subset M^{-0}$ and consequently f is almost-continuous.

Corollary 2.1. *An open mapping is almost-continuous iff it is weakly-continuous.*

Definition 2.4. *A space is said to be semi-regular if for each point x of the space and each open set U containing x, there is an open set V such that $x \in V \subset V^{-0} \subset U$* [3].

Theorem 2.4. *If f is an almost-continuous mapping of a space X into a semi-regular space Y, then f is continuous.*

Proof. Let $x \in X$ and let A be an open set containing $f(x)$. Since Y is semi-regular, there is an open subset M of Y such that $f(x) \in M \subset M^{-0} \subset A$. Now, since f is almost-continuous, therefore there is an open subset U of X containing x such that $f(x) \in f(U) \subset M^{-0}$. Thus U is an open set containing x such that $f(U) \subset A$. Thus f is continuous at x. Since x is arbitrary, it follows that f is continuous.

Theorem 2.5. *If f is an open continuous mapping of X onto Y and if g is a mapping of Y into Z, then $g \circ f$ is almost-continuous iff g is almost-continuous.*

Proof. First, let $g \circ f$ be almost-continuous. Let A be a regularly-open subset of Z. Since $g \circ f$ is almost-continuous, therefore $(g \circ f)^{-1}(A)$ is open, that is, $f^{-1}(g^{-1}(A))$ is open. Also, f is open. Therefore $f[f^{-1}(g^{-1}(A))]$ is open, that is, $g^{-1}(A)$ is open and consequently g is almost-continuous.

Now, let g be almost-continuous and let S be any regularly-open subset of Z. Then $g^{-1}(A)$ is an open subset of Y. Since f is continuous, therefore $f^{-1}(g^{-1}(A))$

is an open subset of X, i.e., $(g \circ f)^{-1}(A)$ is an open subset of X. Hence $g \circ f$ is almost-continuous.

Theorem 2.6. *Every restriction of an almost-continuous mapping is almost-continuous.*

Proof. Let f be an almost-continuous mapping of X into Y and let A be any subset of X. For any regularly-open subset S of Y, $(f/A)^{-1}(S) = A \cap f^{-1}(S)$. But, f being almost-continuous, $f^{-1}(S)$ is open and hence $A \cap f^{-1}(S)$ is a relatively open subset of A, i.e., $(f/A)^{-1}(S)$ is an open subset of A. Hence f/A is almost-continuous.

Theorem 2.7. *Let f map X into Y and let x be a point of X. If there exists a neighbourhood N of x such that the restriction of f to N is almost-continuous at x, then f is almost-continuous at x.*

Proof. Let U be any regularly-open set containing $f(x)$. Since f/N is almost-continuous at x, therefore, there is an open set V_1 such that $x \in N \cap V_1$ and $f(N \cap V_1) \subset U$. The result now follows from the fact that $N \cap V_1$ is a neighbourhood of x.

Corollary 2.1. *Let f map X into Y and let $\{G_\lambda : \lambda \in \Lambda\}$ be an open cover of X. If for each $\lambda \in \Lambda$, f/G_λ is almost-continuous at each point of G_λ, then f is almost-continuous.*

Theorem 2.8. *If f is a mapping of X into Y and $X = X_1 \cup X_2$, where X_1 and X_2 are closed and f/X_1 and f/X_2 are almost-continuous, then f is almost-continuous.*

Proof. Let A be a regularly-closed subset of Y. Then, since f/X_1 and f/X_2 are both almost-continuous, therefore $(f/X_1)^{-1}(A)$ and $(f/X_2)^{-1}(A)$ are both closed in X_1 and X_2 respectively. Since X_1 and X_2 are closed subsets of X, therefore $(f/X_1)^{-1}(A)$ and $(f/X_2)^{-1}(A)$ are also closed subsets of X. Also, $f^{-1}(A) = (f/X_1)^{-1}(A) \cup (f/X_2)^{-1}(A)$. Thus $f^{-1}(A)$ is the union of two closed sets and is therefore closed. Hence f is almost-continuous.

Theorem 2.9. *If f is a mapping of X into Y and $X = X_1 \cup X_2$, and if f/X_1 and f/X_2 are both almost-continuous at a point x belonging to $X_1 \cap X_2$, then f is almost-continuous at x.*

Proof. Let U be any regularly-open set containing $f(x)$. Since $x \in X_1 \cap X_2$ and $f/X_1, f/X_2$ are both almost-continuous at x, therefore there exist open sets V_1 and V_2 such that $x \in X_1 \cap V_1$ and $f(X_1 \cap V_1) \subset U$, and $x \in X_2 \cap V_2$ and $f(X_2 \cap V_2) \subset U$. Now, since $X = X_1 \cup X_2$, therefore $f(V_1 \cap V_2) = f(X_1 \cap V_1 \cap V_2) \cup f(X_2 \cap V_1 \cap V_2) \subset f(X_1 \cap V_1) \cup f(X_2 \cap V_2) \subset U$. Thus, $V_1 \cap V_2 (= V)$ is an open set containing x such that $f(V) \subset U$ and hence f is almost-continuous at x.

Thorem 2.10. *Let $f_\alpha : X_\alpha \to X^*_\alpha$ be almost-continuous for each $\alpha\epsilon I$ and let $f : \prod_{\alpha\epsilon I} X_\alpha \to \prod_{\alpha\epsilon I} X^*_\alpha$ be defined by setting $f((x_\alpha))=(f_\alpha(x_\alpha))$ for each point $(x_\alpha)\epsilon\prod_{\alpha\epsilon I} X_\alpha$. Then f is almost-continuous.*

Proof. Let $(x_\alpha)\epsilon\prod_{\alpha\epsilon I} X_\alpha$ and let O^* be a regularly-open subset of $\prod X^*_\alpha$ containing $f((x_\alpha))$. Then there is a member $\prod_{\alpha\epsilon I} O^*_\alpha$ of the defining base of the product topology on $\prod_{\alpha\epsilon I} X^*_\alpha$ such that $f((x_\alpha))\epsilon\prod_{\alpha\epsilon I} O^*_\alpha \subset O^*$ and $O^*_\alpha = X^*_\alpha$ for all $\alpha\epsilon I$ except for a finite number of indices α_i, $i=1, 2, \cdots, n$ (*say*) and $O^*_{\alpha_i}$ is an open subset of $X^*_{\alpha_i}$, $i=1, 2, \cdots, n$. Now, since O^* is regularly-open, therefore $\overline{\prod O^*_\alpha}^0 \subset O^*$. Thus, each α_i, $f_{\alpha_i}(x_{\alpha_i})\epsilon O^*_{\alpha_i}\subset\overline{O^*_{\alpha_i}}$ and f_{α_i} being almost-continuous, there is an open subset U_{α_i} of X_{α_i} such that $x_{\alpha_i}\epsilon U_{\alpha_i}$ and $f_{\alpha_i}(x_{\alpha_i})\epsilon f_{\alpha_i}(U_{\alpha_i})\subset\overline{O^*_{\alpha_i}}^0$. Thus, $\prod_{\alpha\epsilon I} U_\alpha$ where $U_\alpha = X_\alpha$ when $\alpha\neq\alpha_i$, $i=1, 2, \cdots n$, is an open set containing (x_α) such that $f(\prod_{\alpha\epsilon I} U_\alpha)\subset O^*$. Hence f is almost-continuous.

Theorem 2.11. *Let $h : X \to \prod_{\alpha\epsilon I} X_\alpha$ be almost-continuous. For each $\alpha\epsilon I$, define $f_\alpha : X \to X_\alpha$ by setting $f_\alpha(x)=(h(x))_\alpha$. Then f_α is almost-continuous for all $\alpha\epsilon I$.*

Proof. Let P_α denote the projection of X into X_α. Then $P_\alpha\circ h=f_\alpha$ for each α. Now P_α is open and continuous for each α and h is almost-continuous. Therefore by theorem 2.5, $P_\alpha\circ h$ is almost-continuous, i. e., f_α is almost-continuous for each α.

Definition 2.5. *A point x of a subset A of a space is called a boundary point of A if it is not an interior point of A* [4].

Theorem 2.12. *The set of all points of X at which $f: X\to Y$ is not almost-continuous is identical with the union of the boundaries of the inverse images of regularly-open subsets of Y.*

Proof. Suppose f is not almost-continuous at a point $x\epsilon X$. Then there exists a regularly-open set V such that $f(x)\epsilon V$ and for every open set U containing x, we have $f(U)\cap(Y\sim V)\neq\phi$. Thus, for every open set U containing x, we must have $U\cap[X\sim f^{-1}(V)]\neq\phi$. Therefore x cannot be an interior point of $f^{-1}(V)$. But x belongs to $f^{-1}(V)$. Hence x is a point of the boundary of $f^{-1}(V)$.

Now, let x belong to the boundary of $f^{-1}(G)$ for some regularly-open subset G of Y. Then $f(x)$ belongs to G. If f is almost-continuous at x, then there is an open set U such that x belongs to U and $f(U)\subset G$. Thus $x\epsilon U\subset f^{-1}(f(U))\subset f^{-1}(G)$. Therefore x is an interior point of $f^{-1}(G)$, which is a contradiction. H ence f is not almost-continuous at x.

Definition 2.6. *A space is called a Urysohn space if for every pair of distinct points x and y, there exist open sets U and V such that $x\epsilon U$, $y\epsilon V$ and $\bar{U}\cap\bar{V}=\phi$* [5].

Theorem 2.13. *If f is a weakly-continuous, one-to-one mapping of X onto Y*

and if X is compact and Y is Urysohn, then f is open.

Proof. Let A be an open subset of X. Then $X \sim A$, being a closed subset of the compact space X, is compact. Since every weakly-continuous image of a compact space is almost-compact, therefore $f(X \sim A)$, is almost-compact [6]. Since f is one-to-one, therefore, $f(X \sim A) = Y \sim f(A)$, whence $Y \sim f(A)$ is almost-compact. Since Y is a Urysohn space, therefore $Y \sim f(A)$ is closed and hence $f(A)$ is open.

Corollary 2.2. *If f is an almost-continnous, one-to-one mapping of X onto Y and if X is compact and Y is Urysohn, then f is open.*

Proof. Every almost-continuous mapping is weakly-continuous.

Defiuition 2.7. *A space X is said to be almost-regular if for each regularly-closed set A and each point $x \notin A$, there are disjoint open sets U and V such that $x \in U, A \subset V$* [7].

Theorem 2.14. *If f is an almost-continuous, closed mapping of a regular space X onto a space Y such that $f^{-1}(y)$ is compact for each point $y \in Y$, then Y is almost-regular.*

Proof. Let A be a regularly-closed subset of Y and suppose that $y \notin A$. Then, $f^{-1}(y) \cap f^{-1}(A) = \phi, f^{-1}(A)$ is closed by the almost continuity of f and $f^{-1}(y)$ is compact. Since X is regular, there exist disjoint open sets G and H such that $f^{-1}(A) \subset G, f^{-1}(y) \subset H$. Now, let $P = \{z : f^{-1}(z) \subset G\}$ and $Q_y = \{z : f^{-1}(z) \subset H\}$. Then, $y \in P, A \subset Q, P \cap Q = \phi$. Also since f is closed, therefore P and Q are open. Hence Y is almost-regular.

Theorem 2.15. *If f is an almost-continuous, closed mapping of a normal space X onto a space Y, then any two disjoint regularly-closed subsets of Y can be strongly separated.*

Proof. Let A and B be two disjoint regularly-closed subsets of Y. Then $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint, closed subsets of the normal space X and therefore there exist open sets G and H such that $G \cap H = \phi, f^{-1}(A) \subset G, f^{-1}(B) \subset H$. Let $P = \{y : f^{-1}(y) \subset G\}$ and $Q = \{y : f^{-1}(y) \subset H\}$. Then, since f is closed, therefore P and Q are open sets. Also, $A \subset P, B \subset Q$ and $P \cap Q = \phi$. Hence the result.

3. ALMOST-OPEN AND ALMOST-CLOSED MAPPINGS.

Definition 3.1. *A mapping $f : X \to Y$ is said to be almost-open if the image of every regularly-open subset of X is an open subset of Y.*

Definition 3.2. *A mapping $f : X \to Y$ is said to be almost-closed if the image of every regularly-closed subset of X is a closed subset of Y.*

Remark 3.1. A one-to-one mapping is almost-open iff it is almost-closed.

Remark 3.2. Obviously, every open (closed) mapping is almost-open (almost-closed). But the converse of this statement is not necessarily true as is shown by the following example.

Example 3.1. Let $(R, \mathfrak{T})$ and $(R, \mathfrak{U})$, be the spaces of example 2.2. Let i be the identity mapping of $(R, \mathfrak{T})$ onto $(R, \mathfrak{U})$. Then, i is almost-open and almost-closed but it is neither open nor closed.

Definition 3.3. *A mapping $f: X \to Y$ is said to be θ-continuous if for each point $x \epsilon X$ and each neighbourhood U of $f(x)$, there is a neighbourhood V of x such that $f(\bar{V}) \subset \bar{U}$* [8].

Remark 3.3. It is clear that every θ-continuous mapping is weakly-continuous. A θ-continuous mapping may fail to be almost continuous. In fact, the mapping f defined in example 2.3 is θ-continuous but not almost-continuous. We do not know, however, whether every almost-continuous mapping is θ-continuous or not.

Theorem 3.1. *If f is a one-to-one, θ-continuous mapping of X onto Y and if X is almost-compact, Y is Urysohn, then f is almost-open.*

Proof. Let A be a regularly-open subset of x. Then $X \sim A$, being a regularly-closed subset of the almost-compact space X is itself almost-compact. Also, it is known that the θ-continuous image of an almost-compact space is almost-compact [8]. Therefore $f(X \sim A)$ is an almost-compact subset of the Urysohn space Y and is therefore closed. Thus $f(X \sim A) = Y \sim f(A)$ is closed, whence $f(A)$ is open and consequently f is an almost-open mapping.

Theorem 3.2. *If $f: X \to Y$ is an almost-closed mapping of X onto Y, then for every regularly-open subset G of X and for every point $y \epsilon Y$ such that $f^{-1}(y) \subset G$, we have, $y \epsilon [f(G)]^0$.*

Proof. Since G is regularly-open, therefore $X \sim G$ is regularly-closed. Since f is almost-continuous, therefore $f(X \sim G)$ is closed. Since $f^{-1}(y) \subset G$, therefore $y \notin f(X \sim G)$. Hence there must exist an open set U containing y such that $U \cap f(X \sim G) = \phi$. Then $y \epsilon U \subset f(G)$ and consequently y is an interior point of $f(G)$.

Corollary 3.1. *If $f: X \to Y$ is an almost-closed mapping of X onto Y, then for each set $S \subset X$ and for each point $x \epsilon X$, such that $f^{-1}(f(x)) \subset \bar{S}^0$, we have, $f(x) \epsilon [f(\bar{S}^0)]^0$.*

The following two theorems give sufficient conditions for an almost-closed mapping to be continuous.

Theorem 3.3. *If f is an almost-closed mapping of an almost-regular space X onto a compact space Y with regularly-closed point inverses, then f is continuous.*

Proof. Suppose f is not continuous at a point $x \epsilon X$. Then there exists an open set M containing $f(x)$ such that $f(N) \cap Y \sim M \neq \phi$ for every open set N containing x. Since f is almost-closed and N is regularly-closed therefore $f(\bar{N})$ is closed. Also $Y \sim M$ is closed. Thus $\{f(\bar{N}) \cap (Y \sim M): N$ is open and $x \epsilon N\}$ is a family of closed subsets of Y. Also this family must have finite intersection property, for if there exists a finite number of open sets $N_1, \cdots, N_n$ such that $x \epsilon N_i$ for each $i=1, \cdots, n$ and if $\bigcap_{i=1}^{n} [f(\bar{N}_i) \cap (Y \sim M)] = \phi$ then $\bigcap_{i=1}^{n} N_i$ is an open set containing x and $(Y \sim M) \cap f(\bigcap_{i=1}^{n} N_i) \subset (Y \sim M) \cap f(\bigcap_{i=1}^{n} N_i^-) \subset (Y \sim M) \cap [\bigcap_{i=1}^{n} f(\bar{N}_i)] = \bigcap_{i=1}^{n} (Y \sim M) \cap (f(\bar{N}_i)) = \phi$, which is a contradiction. Therefore $\{f(\bar{N}) \cap (Y \sim M): N$ is open and $x \epsilon N\}$ is a family of closed subsets of Y with finite intersection property. Since Y is compact, $\cap \{f(\bar{N}) \cap (Y \sim M): N$ is open and $x \epsilon N\} \neq \phi$. Let y' belong to this intersection. Then $y' \neq f(x)$. Therefore $x \notin f^{-1}(y')$. But $f^{-1}(y')$ is regularly-closed and X is almost-regular. Therefore there exist disjoint open sets U and V such that $x \epsilon U, f^{-1}(y') \subset V$. But $V \cap U = \phi \Rightarrow V \cap \bar{U} = \phi$. Therefore $y' \notin f(\bar{U})$. But this is a contradiction to the fact that y' belongs to $f(\bar{N}) \cap Y \sim M$ for every open set N containing x. Hence f must be continuous at x. But x is an arbitrary point of X. Therefore f is continuous.

Definition 3.4. *A mapping f of X into Y has at worst a removable discontinuity at a point $x \epsilon X$ if there is a point $y \epsilon Y$ such that for each neighbourhood V of y, there is a neighbourhood U of x such that $f(U - \{x\}) \subset V$.*

Theorem 3.4. *If f is an almost-closed mapping of an almost-regular space X onto a space Y with regularly-closed point inverses, then if f has at worst a removable discontinuity at $x_0 \epsilon X$ then f is continuous at x_0.*

Proof. If x_0 is isolated in X, the result is obviously true. Assume that x_0 is non-isolated and that f is not continuous at x_0. Let y be the point of Y determined by the hypothesis. Since $y \neq f(x_0)$ and $f^{-1}(y)$ is regularly closed, an open neighbourhood U of x_0 exists such that $f^{-1}(y) \cap \bar{U} = \phi$. Then, because $\bar{U}$ is regularly-closed, therefore $f(\bar{U})$ is closed and hence a neigbourhood V of y exists for which $V \cap f(\bar{U}) = \phi$. There is a neighbourhood W of x_0 such that $f(W - \{x_0\}) \subset V$. Since x_0 is non-isolated, $U \cap (W - \{x_0\}) \neq \phi$. Hence $\phi \neq f(W - \{x_0\}) \cap f(\bar{U}) \subset V \cap f(\bar{U})$, which is a contradiction.

4. ALMOST-QUASI-COMPACT MAPPINGS.

Definition 4.1. *A mapping $f: X \to Y$ is said to be almost-quasi-compact if it is onto and if A is open whenever $f^{-1}(A)$ is regularly-open.*

Remark 4.1. An onto mapping f is almost-quasi-compact iff S is closed whenever $f^{-1}(S)$ is regularly-closed.

Remark 4.2. Clearly, every quasi-compact map is almost-quasi-compact. But an almost-quasi-compact mapping may fail to be quasi-compact. The following is an example.

Example 4.1. Let $(R, \mathfrak{T})$ be the space of example 2.1. Let $X=\{a, b\}$ and let $\mathfrak{T}^*=\{X, \phi, \{a\}\}$. Let f be a mapping of $(R, \mathfrak{T})$ onto $(X, \mathfrak{T}^*)$ defined as follows:

$$f(x)=\begin{cases} a & \text{if } x \text{ is rational,} \\ b & \text{if } x \text{ is irrational.} \end{cases}$$

Then f is almost-quasi-compact but not quasi-compact.

Theorem 4.1. *A mapping f of X onto Y is almost-quasi-compact iff the image of every regularly-open inverse set is open.*

Proof. First, let f be almost-quasi-compact and let A be any regularly-open inverse set. Then since $f^{-1}[f(A)]=A$ is regularly-open, therefore $f(A)$ is open. Conversely, if $f^{-1}(S)$ be regularly-open, then because $f^{-1}(S)$ is a regularly-open inverse set, therefore $f[f^{-1}(S)]$ is open, that is S is open and hence f is almost-quasi-compact.

Corollary 4.1. *A mapping f of X onto Y is almost-quasi-compact iff the image of every regularly-closed inverse set is closed.*

Theorem 4.2. *If f is a one-to-one mapping of X onto Y, then the following properties are equivalent:*

(*a*) *f is almost-open.*

(*b*) *f is almost-closed.*

(*c*) *f is almost-quasi-compact.*

(*d*) *f^{-1} is almost-continuous.*

Proof. $(a) \Rightarrow (b)$. Let A be a regularly-closed subset of X. Then $X \sim A$ is regularly-open. Therefore $f(X \sim A)$ is open, that is, $Y \sim f(A)$ is open. Thus $f(A)$ is closed and consequently f almost-closed.

$(b) \Rightarrow (c)$. Let $f^{-1}(S)$ be regularly-closed. Then, $f(f^{-1}(S))$ is closed, that is, S is closed and hence f is almost-quasi-compact.

$(c) \Rightarrow (d)$. Let U be a regularly-open subset of X. Then $f^{-1}(f(U))=U$ is regularly-open. Hence $f(U)$ is open, that is, $(f^{-1})^{-1}(U)$ is open and therefore f^{-1} is almost-continuous.

$(d) \Rightarrow (a)$. If A is a regularly-open subset of X. Then $(f^{-1})^{-1}(A)$ is an open subset of Y and thus f is almost-open.

Theorem 4.3. *Suppose that f maps X onto Y and g maps Y onto Z. Then if f is almost-continuous and if $g \circ f$ is open (resp. closed, quasi-compact) then g is almost-open (resp. almost-closed, almost-quasi-compact).*

Proof. Suppose first that f is almost-continuous and $g \circ f$ is open (closed). Let S

be any regularly-open (regularly-closed) subset of Y. Since f is almost-continuous, therefore $f^{-1}(S)$ is an open (closed) subset of X. Now, $g \circ f$ is open (closed). Therefore $(g \circ f)(f^{-1}(S))$ is open (closed). But $(g \circ f)(f^{-1}(S)) = g(S)$. Thus $g(S)$ is open (closed). Hence g is almost-open (almost-closed).

Now, let f be almost-continuous and let $g \circ f$ be quasi-compact. Let $g^{-1}(S)$ be a regularly-open subset of Y. Then, by almost-continuity of f, $f^{-1}(g^{-1}(S))$ is open. But $f^{-1}(g^{-1}(S))$ is $(g \circ f)^{-1}(S)$. Therefore, since $g \circ f$ is quasi-compact, S must be open whence g is almost-quasi-compact.

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