# VARIOUS HAMILTON'S CANONICAL FORMALISMS AS NON-CONNECTION METHODS FOR VARIOUS CONNECTION GEOMETRIES IN THE LARGE.

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In a paper T. Takasu, [1], the present author has introduced non-connection methods for some connection geometries *in th large* based on canonical equations of Hamiltonian types of II-geodesic curves in the present author's sense. It was a six pages extraction as a preliminary report. The present paper is a *detailed exposition* of it. The main resuls are as follows. § 1. "An extension of the *duality* exposed in the excellent book H. Hund, [12] to the case depending on higher order derivatives" adding new formulas to the Hamilton's canonical formalism so that the Hamiltonian $H$ and the Lagrangian $L$ for $(x^\lambda, y_\lambda)$ are the Lagrangian and the Hamiltonian for $(x_\lambda, y^\lambda)$ respectively owing to the new relations (1.28), p. 22. The case $M=1$ is particularly note-worthy. § 2. "Another Duality in the Hamilton's Canonical Formalism" is added introducing *another Hamiltonian* $\mathfrak{H}$ and *another Lagrangian* $\mathfrak{L}$ (corresponding to $\mathfrak{H}$) given by (2.1), p. 25, (2.14), (2.15) and (2.16), p. 27 being led to *new canonical eqnations* (2.34), (2.40), p. 31, 32 *of Hamiltonian types*. The case of $M=1$ is particularly note-worthy. § 3. "Hamilton's Canonical Formalism in terms oJ Global Coordinates of the Present Author", making a sequel to T. Takasu, [1]. § 4. "The Hamiltonian Equations in the Large." It is remarkable that the extremals cover several paths (e. g. II-geodesic curves covering II-geodesic affine paths). The case $M=1$ of § 4 is particularly note-worthy. § 5. "The Groups of Extended Hamilton's Canonical Transformations." The case $M=1$ is particularly note-worthy. § 6 "The Relation between the Hamilton's Formalism of § 4 and the Present Author's Non-Connection Methods for Some Connection Geometries in the Large", each forming another method for the other. The case $M=1$ is particularly note-worthy.

A paper T. Takasu, [9], which is a unification of [3] and [4] by the present principle, is under preparation and will be read in the Autumn Meeting of the Japan. Math. Soc. in the middle of Oct. 1968 in Tokyo.

## § 1. Hamilton's Canonical Formalism depending on Higher Order Derivatives.

We will start with the duality exposed in the excellent book [12] of H. Rund adding new dualities spoken of under § 1 of the Introduction above.

We shall now base our theory on the $n$-dimensional differentiable manifold $\bigcup_{\alpha} U_{\alpha}(x_{(\alpha)})$ (in current notation) provided with the local coordinates

(1.0) $x^{\lambda}\ (\lambda, \mu, \cdots = 1, 2, \cdots, n),$ | $x_{\lambda}\ (\lambda, \mu, \cdots = 1, 2, n),$

which is denoted by $X_n$. *A relation of the two sides* will be established later by (1.16) and (1.28). A set of $n$ equations of the type

(1.1) $x^{\lambda} = x^{\lambda}(\tau),$ | $x_{\lambda} = x_{\lambda}(\tau),$

where $\tau$ is an *arbitrary parameter,* represents a curve $C$ in $X_n$. If the functions (1.1) are of Class $C^1$, we can form the derivatives

(1.2) $\dot{x}^{\lambda} = dx^{\lambda}/d\tau,$ | $\dot{x}_{\lambda} = dx_{\lambda}/d\tau,$

which we shall regard as the components of the tangent vector to $C$.

Suppose now that we are given a function

$L(x^{\lambda}, \dot{x}^{\lambda}, \cdots, \overset{(M)}{x^{\lambda}}),$ | $H(x_{\lambda}, \dot{x}_{\lambda}, \cdots, \overset{(M)}{x_{\lambda}}),$ (T. Takasu, [9,10] )

of which we shall now assume that it is of class $C^3$ in its $(M+1)n$ arguments (in fact, all our final conclusions can be established for $C^2$ functions)

$(x^{\lambda}, \dot{x}^{\lambda}, \cdots, \overset{(M)}{x^{\lambda}}).$ | $(x_{\lambda}, \dot{x}_{\lambda}, \cdots, \overset{(M)}{x_{\lambda}}).$

For two arbitrary points $P_0, P_1$ on $C$ corresponding to parameter values $\tau_0$ and $\tau_1$ respectively, we can then form the integral

(1.3) $I = \int_{\tau_0}^{\tau_1} L(x^{\lambda}, \dot{x}^{\lambda}, \cdots, \overset{(M)}{x^{\lambda}})\, d\tau.$ | $I = \int_{\tau_0}^{\tau_1} H(x_{\lambda}, \dot{x}_{\lambda}, \cdots, \overset{(M)}{x_{\lambda}})\, d\tau.$

It will be *assumed* that *the values of I be invariant* (cf. A. Kawaguchi, [1] ; M. Kawaguchi, [1], p. 724. *under an arbitrary parameter transformation of the type*

(1.4) $\sigma = \sigma(\tau),$

the function $\sigma$ being of class $C^1$ such that

(1.5) $\dot{\sigma} = d\sigma/d\tau > 0.$

This assumption implies that *our theory is to be invariant under transformations of the local coordinates*

$x^{\lambda}$ | $x_{\lambda}$

*as well as under trasformations of the parameter* $\tau$ subject to the condition (1.5). It is assumed [1] that the

*Lagrangian* $L(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})$, | *Hamiltonian* $H(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})$,

say, *is positively homogeneous of the fiirst degree in*

$$(1.6)\quad \dot{x}^\lambda:\ L(x^\lambda, \lambda\dot{x}^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) = \lambda L(x^\lambda, \dot{x}^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}),\ \lambda > 0,$$

$$\dot{x}_\lambda:\ H(x_\lambda, \lambda\dot{x}_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}) = \lambda H(x_\lambda, \dot{x}_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}),\ \lambda>0,$$

so that

$$(1.7)\quad \frac{\partial L(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})}{\partial \dot{x}^\mu}\dot{x}^\mu = L(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}),$$

$$\frac{\partial H(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})}{\partial \dot{x}_\mu}\dot{x}_\mu = H(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}),$$

*where the partial derivatives themselves are now positively homogeneous of degree zero in*

$\dot{x}^\lambda$. | $\dot{x}_\lambda$.

Hence

$$(1.8)\quad \frac{\partial^2 L(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})}{\partial \dot{x}^\mu\, \partial \dot{x}^\nu}\dot{x}^\nu = 0, \qquad \frac{\partial^2 H(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})}{\partial \dot{x}_\mu\, \partial \dot{x}_\nu}\dot{x}_\nu = 0,$$

which entails

$$(1.9)\quad \det\left(\frac{\partial^2 L}{\partial \dot{x}^\mu\, \partial \dot{x}^\nu}\right) \equiv 0. \qquad \det\left(\frac{\partial^2 H}{\partial \dot{x}_\mu\, \partial \dot{x}_\nu}\right) \equiv 0.$$

If we write

$$(1.10)\quad p_\mu \overset{\text{def}}{=} \partial L(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) \,/\, \partial \dot{x}^\mu, \qquad p^\mu \overset{\text{def}}{=} \partial H(x_\lambda, \dot{x}_\lambda, \cdots \overset{(M)}{x_\lambda}) \,/\, \partial \dot{x}_\mu,$$

we can not express

$\dot{x}^\lambda$ | $\dot{x}_\lambda$

as functions of

$(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, p_\lambda)$. | $(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}, p^\lambda)$.

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1) This restriction on $L(H)$ is not as serve as may appear to be the case at first sight; a homogeneous Lagrangian (Hamiltonian) can always be obtained from an arbitrary problem in the calculus of variations. (Cf. Section 7 of Chap. 2 of H. Rund, [12]). Other cases shall be studied later.

(Indeed the right–hand side of (1.10) is homogeneous of *degree zero* in

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| $\dot{x}^\lambda$.) | $\dot{x}_\lambda$.) |

Substituting (1.10) in (1.7), we obtain

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| (1.11) $-L(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) + p_\mu \dot{x}^\mu \equiv 0.$ | $-H(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}) + p^\mu \dot{x}_\mu \equiv 0.$ |

Thus *the left–hand side of* (1.11) *can not serve as Hamiltonian* (*Lagrangian*) *function.*

*We have,* then, *to seek for suitable canonical variables and a corresponding*

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| *Hamiltonian.* | *Lagrangian.* |

This is achieved by means of the so–called *fundamental tensor,* whose components are

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| (1.12) $g_{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) \overset{\text{def}}{=} \frac{1}{2} \frac{\partial^2 L(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})}{\partial \dot{x}^\mu \partial \dot{x}^\nu}.$ | $g^{\mu\nu}(x_\lambda, \dot{x}_\lambda \cdots, \overset{(M)}{x_\lambda}) \overset{\text{def}}{=} \frac{1}{2} \frac{\partial^2 H(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})}{\partial \dot{x}_\mu \partial \dot{x}_\nu}.$ |

It is easily verified that the

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| $g_{\mu\nu}$ | $g^{\mu\nu}$ |

form a

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| covariant | contravariant |

tensor of rank 2 as a result of the invariance of

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| $L(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})$ | $H(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})$ |

under arbitrary coordinate transformation.

We note that

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| $g_{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})$ | $g^{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x^\lambda})$ |

*are positively homogeneous of degree zero in*

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| $\dot{x}^\lambda$, | $\dot{x}_\lambda$, |

so that

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| (1.13) $\frac{\partial g_{\sigma\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})}{\partial \dot{x}^\mu} \dot{x}^\mu \equiv 0,$ $\frac{\partial g_{\eta\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})}{\partial \dot{x}^\sigma} \ddot{x}^\mu = 0,$ | $\frac{\partial g^{\sigma\mu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})}{\partial \dot{x}_\mu} \dot{x}_\mu \equiv 0,$ $\frac{\partial g^{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})}{\partial \dot{x}_\sigma} \dot{x}_\mu = 0,$ |

where we have made use of the fact that according to definition (1.12), the derivative

$$\frac{\partial g_{\mu\nu}}{\partial \dot{x}^{\sigma}} \qquad \Big| \qquad \frac{\partial g^{\mu\nu}}{\partial \dot{x}_{\sigma}}$$

are symmetric in the three indices $\mu, \nu, \sigma$.

*Let us consider the case*

$$\text{(1.14)} \quad \det\{g_{\mu\nu}(x^{\lambda}, \dot{x}^{\lambda}, \cdots, \overset{(M)}{x^{\lambda}})\} \neq 0 \qquad \Big| \qquad \det\{g^{\mu\nu}(x_{\lambda}, \dot{x}_{\lambda}, \cdots, \overset{(M)}{x_{\lambda}})\} \neq 0$$

*only.*

Owing to (1.14), the matrix

$$(g_{\mu\nu}(x^{\lambda}, \dot{x}^{\lambda}, \cdots, \overset{(M)}{x^{\lambda}})) \qquad \Big| \qquad (g_{\mu\nu}(x_{\lambda}, \dot{x}_{\lambda}, \cdots, \overset{(M)}{x_{\lambda}}))$$

has the *inverse*

$$(g^{\mu\nu}(x^{\lambda}, \dot{x}^{\lambda}, \cdots, \overset{(M)}{x^{\lambda}})): \qquad \Big| \qquad (g_{\mu\nu}(x_{\lambda}, \dot{x}_{\lambda}, \cdots, \overset{(M)}{x_{\lambda}})):$$

$$\text{(1.15)} \quad g^{\mu\nu}(x^{\lambda}, \dot{x}^{\lambda}, \cdots, \overset{(M)}{x^{\lambda}})\, g_{\mu\sigma}(x^{\lambda}, \dot{x}^{\lambda}, \cdots, \overset{(M)}{x^{\lambda}}) = \delta^{\nu}_{\sigma}. \qquad \Big| \qquad g_{\mu\nu}(x_{\lambda}, \dot{x}_{\lambda}, \cdots, \overset{(M)}{x_{\lambda}})\, g^{\mu\sigma}(x_{\lambda}, \dot{x}_{\lambda}, \cdots, \overset{(M)}{x_{\lambda}}) = \delta^{\sigma}_{\nu}.$$

Since

$$g_{\mu\sigma}(x^{\lambda}, \dot{x}^{\lambda}, \cdots, \overset{(M)}{x^{\lambda}}) \qquad \Big| \qquad g^{\mu\sigma}(x_{\lambda}, \dot{x}_{\lambda}, \cdots, \overset{(M)}{x_{\lambda}})$$

are homogeneous of *degree zero* in $(\dot{x})$,

$$g^{\mu\sigma}(x^{\lambda}, \dot{x}^{\lambda}, \cdots, \overset{(M)}{x^{\lambda}}) \qquad \Big| \qquad g_{\mu\nu}(x_{\lambda}, \dot{x}_{\lambda}, \cdots, \overset{(M)}{x_{\lambda}})$$

*must also be homogeneous of degree zero in* $(\dot{x})$.

We shall define the components of an analogue of the "*canonical momentum*" by

$$\text{(1.16)}^{2)} \quad y_{\mu} \overset{\text{def}}{=} \dot{x}_{\mu} \overset{\text{def}}{=} g_{\mu\nu}(x^{\lambda}, \dot{x}^{\lambda}, \cdots, \overset{(M)}{x^{\lambda}})\, \dot{x}^{\nu}. \qquad \Big| \qquad y^{\mu} \overset{\text{def}}{=} \dot{x}^{\mu} \overset{\text{def}}{=} g^{\mu\nu}(x_{\lambda}, \dot{x}_{\lambda}, \cdots, \overset{(M)}{x_{\lambda}})\, \dot{x}_{\nu}.$$

Since (1.12) can be written out as

$$\text{(1.17)} \quad g_{\mu\nu} = \frac{\partial L}{\partial \dot{x}^{\mu}} \frac{\partial L}{\partial \dot{x}^{\nu}} + L \frac{\partial^2 L}{\partial \dot{x}^{\mu} \partial \dot{x}^{\nu}}, \qquad \Big| \qquad g^{\mu\nu} = \frac{\partial H}{\partial \dot{x}_{\mu}} \frac{\partial H}{\partial \dot{x}_{\nu}} + H \frac{\partial^2 H}{\partial \dot{x}_{\mu} \partial \dot{x}_{\nu}},$$

if follows from (1.7) and (1.8) that the definition (1.16) *is equivalent to*

$$\text{(1.18)}^{3)} \quad y_{\mu} = L \frac{\partial L}{\partial \dot{x}^{\mu}}. \qquad \Big| \qquad y^{\mu} = H \frac{\partial H}{\partial \dot{x}_{\mu}}.$$

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2), 3) will be seen later in (1.27).

Thus

$$y_\lambda \text{ and } p_\lambda \qquad\Big|\qquad y^\lambda \text{ and } p^\lambda$$

*differ only by the multiplication factor*

$$L. \qquad\Big|\qquad H.$$

The equation (1.16) may be solved uniquely for the

$$\dot{x}^\lambda \qquad\Big|\qquad \dot{x}_\lambda$$

as functions of

$$(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda, y_\lambda) \qquad\Big|\qquad (x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda, y^\lambda)$$

as a consequence of (1.14). For, it follows from (1.16) by differentiation with respect to

$$\dot{x}^\nu \qquad\Big|\qquad \dot{x}_\nu$$

that

$$(1.19) \qquad \frac{\partial y_\lambda}{\partial \dot{x}^\nu} = \frac{\partial g_{\lambda\mu}}{\partial \dot{x}^\nu}\dot{x}^\mu + g_{\mu\nu}\,\delta^\mu_\nu = g_{\lambda\nu} \qquad\Big|\qquad \frac{\partial y^\lambda}{\partial \dot{x}_\nu} = \frac{\partial g^{\lambda\mu}}{\partial \dot{x}_\nu}\dot{x}_\mu + g^{\lambda\mu}\,\delta^\nu_\mu = g^{\lambda\nu}$$

by virtue of (1.13). Thus the Jacobian of (1.16) is simply the matrix:

$$(1.20) \qquad \left|\frac{\partial y_\mu}{\partial \dot{x}^\nu}\right| = |g_{\mu\nu}|; \qquad\Big|\qquad \left|\frac{\partial y^\mu}{\partial \dot{x}_\nu}\right| = |g^{\mu\nu}|;$$

and hence, by virtue of (1.13) and (1.16) we may write

$$(1.21) \qquad \dot{x}^\lambda = \psi^\lambda(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda, y_\lambda). \qquad\Big|\qquad \dot{x}_\lambda = \psi_\lambda(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda, y^\lambda).$$

For, owing to (1.14), the matrix

$$(g_{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda)) \qquad\Big|\qquad (g^{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda))$$

has the inverse

$$(g^{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda)): \qquad\Big|\qquad (g_{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda)):$$

$$(1.22) \quad g^{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda)\, g_{\mu\sigma}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda) = \delta^\nu_\sigma; \quad\Big|\quad g_{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda)\, g^{\mu\sigma}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda) = \delta^\sigma_\nu;$$

thus, if we multiply (1.16) by

$$g^{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda), \qquad\Big|\qquad g_{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda),$$

it results that

$$\begin{aligned} & g^{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda)\, y_\mu \\ &= g^{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda)\, g_{\mu\kappa}(x^\lambda, \dot{x}^\lambda, \cdots \overset{(M)}{x}{}^\lambda)\, \dot{x}^\kappa \\ &= \delta^\nu_\kappa\, \dot{x}^\kappa = \dot{x}^\nu, \end{aligned} \qquad \Bigg| \qquad \begin{aligned} & g_{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda)\, y^\mu \\ &= g_{\mu\nu}(x_\lambda, x_\lambda, \cdots, \overset{(M)}{x}_\lambda)\, g^{\mu\kappa}(x_\lambda\, \dot{x}_\lambda, \cdots \overset{(M)}{x}_\lambda)\, \dot{x}_\kappa \\ &= \delta^\kappa_\nu\, \dot{x}_\kappa = \dot{x}_\nu, \end{aligned}$$

and by virtue of (1.20) and (1.13), we have

$$\dot{x}^\nu = \dot{x}^\nu(y_\mu, x^\mu, \ddot{x}^\mu, \cdots, \overset{(M)}{x}{}^\mu), \qquad \Big| \qquad \dot{x}_\nu = \dot{x}_\nu(y^\mu, x_\mu, \ddot{x}_\mu, \cdots, \overset{(M)}{x}_\mu), \tag{1.23}$$

so that

$$\begin{aligned} \dot{x}^\nu &= g^{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda)\, y_\mu \\ &= g^{\mu\nu}(x^\lambda, \dot{x}^\lambda(y, x^\mu, \ddot{x}^\mu, \cdots, \overset{(M)}{x}{}^\mu), \\ &\qquad \ddot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda)\, y_\mu, \end{aligned} \qquad \Bigg| \qquad \begin{aligned} \dot{x}_\nu &= g_{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda)\, y^\mu \\ &= g_{\mu\nu}(x_\lambda, \dot{x}_\lambda(y^\mu, x_\mu, \ddot{x}_\mu, \cdots, \overset{(M)}{x}_\mu), \\ &\qquad \ddot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda)\, y^\mu, \end{aligned} \tag{1.24}$$

which is of the form

$$\dot{x}^\nu = \phi^\nu(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda, y_\lambda). \qquad \Big| \qquad \dot{x}_\nu = \phi_\nu(x_\lambda, \ddot{x}_\lambda \cdots, \overset{(M)}{x}_\lambda, y^\lambda). \tag{1.25}$$

Hence *the functions*

$$\phi^\nu(x^\lambda, \ddot{x}^\lambda \cdots, \overset{(M)}{x}{}^\lambda, y_\lambda) \qquad \Big| \qquad \phi_\nu(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda, y^\lambda)$$

*are of class* $C^2$.

By virtue of (1.25), the (1.22) *may be written as*

$$\begin{aligned} & g^{\mu\nu}(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda, y_\lambda)\, g_{\mu\sigma}(x^\lambda, \dot{x}^\lambda, \\ &\qquad \cdots \overset{(M)}{x}{}^\lambda) = \delta^\nu_\sigma. \end{aligned} \qquad \Bigg| \qquad \begin{aligned} & g_{\mu\nu}(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda, y^\lambda)\, g^{\mu\sigma}(x_\lambda, \dot{x}_\lambda, \\ &\qquad \cdots, \overset{(M)}{x}_\lambda) = \delta^\sigma_\nu. \end{aligned} \tag{1.26}$$

From (1.16) it follows that

$$y_\lambda \qquad \Big| \qquad y^\lambda$$

*are positively homogeneous of the first degree in*

$$\dot{x}^\lambda. \qquad \Big| \qquad \dot{x}_\lambda.$$

The (1.21) becomes now by (1.16) to

$$\dot{x}^\nu = g^{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda)\, y_\mu. \qquad \Big| \qquad \dot{x}_\nu = g_{\mu\nu}(x^\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda)\, y^\mu. \tag{1.27}$$

From (1.27), we infer that *the*

$$g^{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda, y_\lambda) \qquad \Big| \qquad g_{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda, y^\lambda)$$

*are positively homogeneous of degree zero in the*

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| $y_\mu.$ | $y^\mu.$ |

Now let us prove that

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| (1.28) $g^{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) = g^{\nu\mu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}).$ | $g_{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}) = g_{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}).$ |

From (1.26), we habe namely

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| $\dot{x}_\mu \dot{x}^\mu = \delta^\nu_\sigma \dot{x}_\nu \dot{x}^\sigma$ <br> $= g^{\mu\nu}(x^\lambda, \dot{x}_\lambda, \cdots, x_\lambda) \dot{x}^\nu g_{\mu\sigma}(x^\lambda, \dot{x}^\lambda, \cdots, x^\lambda) \dot{x}_\sigma,$ | $\dot{x}^\mu \dot{x}_\mu = \delta^\sigma_\nu \dot{x}^\nu \dot{x}_\sigma$ <br> $= g_{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) \dot{x}^\nu g^{\mu\sigma}(y_\lambda, \dot{x}_\lambda, \cdots, x_\lambda) \dot{x}_\sigma,$ |

so that

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| $g^{\mu\nu}(x^\lambda, \dot{x}_\lambda, \cdots \overset{(M)}{x_\lambda}) g_{\mu\sigma}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) = \delta^\nu_\sigma.$ | $g_{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) g^{\mu\sigma}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x_\lambda}) = \delta^\sigma_\nu.$ |

Comparing this with (1.15), we see (1.28).

**N. B,** By virtue of (1.28), the (1.16) gives

$$\dot{x}_\mu = g_{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) \dot{x}^\nu = g_{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}) \dot{x}^\nu,$$

whose inverse is

$$\dot{x}^\mu = g^{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, x_\lambda) x_\nu,$$

which is nothing other than the right–hand side of (1.16)). Thus the two sides of (1.16) are consistently related by

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| (1.16)′ $y_\mu = \dot{x}_\mu = g_{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) \dot{x}^\nu$ <br> $= g_{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}) \dot{x}^\nu.$ | $y^\mu = \dot{x}^\mu = g^{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}) \dot{x}_\nu$ <br> $= g^{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) \dot{x}_\nu.$ |

Again it is easily verified that *the*

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| $y_\lambda$ | $y^\lambda$ |

*are components of a*

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| *covariant* | *contravariant* |

*vector, its*

| | |
|---|---|
| *contravariant* | *covariant* |

*counter–part possessing the components*

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| $y^\lambda = \dot{x}^\lambda,$ | $y_\lambda = \dot{x}_\lambda,$ |

*while the*

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| $g^{\mu\nu}(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y_\lambda)$ | $g_{\mu\nu}(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}, y^\lambda)$ |

*represents a*

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| *contravariant* | *covariant* |

*tensor of rank* 2.

Also, in view of the homogeneiety relations (1.7) and (1.8), we find by multiplication of (1.7) by

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| $\dot{x}^\mu$ and $\dot{x}^\nu$ | $\dot{x}_\mu$ and $\dot{x}_\nu$ |

that

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| (1.29) $L^2(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) = g_{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})\dot{x}^\mu \dot{x}^\nu.$ | $H^2(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}) = g^{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})\dot{x}_\mu \dot{x}_\nu.$ |

Thus, for

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| $L$ positive, | $H$ positive, |

we can write fundamental integral (1.3) in the form

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| (1.30) $I = \int_{\tau_0}^{\tau_1} \{g_{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})\, dx^\mu dx^\nu\}^{\frac{1}{2}}.$ | $I = \int_{\tau_0}^{\tau_1} \{g^{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})\, dx_\mu dx_\nu\}^{\frac{1}{2}}.$ |

The identity (1.29) suggests that we should define *the*

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|---|---|
| *Hamiltonian* | *Lagrangian* |

*function*

| | |
|---|---|
| $H(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y_\lambda)$ | $L(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}, y^\lambda)$ |

*by the relation*

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| (1.31)[4] $H^2(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x}, y_\lambda) \overset{\text{def}}{=} g^{\mu\nu}(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y_\lambda)\, y_\mu y_\nu.$ | $L^2(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}, y^\lambda) \overset{\text{def}}{=} g_{\mu\nu}(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}, y^\lambda)\, y^\mu y^\nu.$ |

We can prove the equality

$$(1.32)\quad H^2(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) = H^2(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y_\lambda) = g^{\mu\nu} y_\mu y_\nu = y_\lambda \dot{x}^\lambda$$
$$= y_\lambda y^\lambda = \dot{x}_\lambda y^\lambda = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = g_{\mu\nu} y^\mu y^\nu = L^2(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}, y^\lambda) = L^2(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}).$$

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4) Cf. (1.26) of H. Rund, [12].

Indeed (1.16) gives

$$(1.33)\qquad \dot{x}^{\nu}=g^{\mu\nu}y_{\mu}, \qquad\Big|\qquad \dot{x}_{\nu}=g_{\mu\nu}y^{\mu},$$

so that we have

$$\begin{aligned}L^2&=g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}=y_{\lambda}\dot{x}^{\lambda}\\&=g_{\mu\nu}(g^{\sigma\mu}y_{\sigma})(g^{\tau\nu}y_{\tau})=g^{\sigma\tau}y_{\sigma}y_{\tau}.\end{aligned}\qquad\Bigg|\qquad\begin{aligned}H^2&=g^{\mu\nu}\dot{x}_{\mu}\dot{x}_{\nu}=y^{\lambda}\dot{x}_{\lambda}\\&=g^{\mu\nu}(g_{\sigma\mu}y^{\sigma})(g_{\tau\nu}y^{\tau})=g_{\sigma\tau}y^{\sigma}y^{\tau}.\end{aligned}$$

For the signs of $H$ and $L$ in (1.31), *we stipulate that the signs of $L$ and $H$ must coincide*; thus

$$(1.34)\qquad\begin{aligned}&H(x^{\lambda},\ddot{x}^{\lambda},\cdots,\overset{(M)}{x}{}^{\lambda},y_{\lambda})\\&=L(x^{\lambda},\dot{x}^{\lambda},\cdots,\overset{(M)}{x}{}^{\lambda})\\&=L(x^{\lambda},\dot{x}^{\lambda},\cdots,\overset{(M)}{x}{}^{\lambda},\psi_{\lambda}(x^{\lambda},\ddot{x}^{\lambda},\cdots,\overset{(M)}{x}{}^{\lambda},y_{\lambda}))\end{aligned}\qquad\Bigg|\qquad\begin{aligned}&L(x_{\lambda},\ddot{x}_{\lambda},\cdots,\overset{(M)}{x}_{\lambda},y^{\lambda})\\&=H(x_{\lambda},\dot{x}_{\lambda},\cdots,\overset{(M)}{x}_{\lambda})\\&=H(x_{\lambda},\dot{x}_{\lambda},\cdots,\overset{(M)}{x}_{\lambda},\psi^{\lambda}(x_{\lambda},\ddot{x}_{\lambda},\cdots\overset{(M)}{x}_{\lambda},y^{\lambda}))\end{aligned}$$

by (1.21).

Next let us prove the relation

(1.35)

$$\frac{\partial H(x^{\lambda},\ddot{x}^{\lambda},\cdots\overset{(M)}{x}{}^{\lambda},y_{\lambda})}{\partial x^{\mu}}=-\frac{\partial L(x_{\lambda},\ddot{x}_{\lambda},\cdots,\overset{(M)}{x}_{\lambda},y^{\lambda})}{\partial x^{\mu}}.\qquad\Bigg|\qquad\frac{\partial L(x_{\lambda},\ddot{x}_{\lambda},\cdots,\overset{(M)}{x}_{\lambda},y^{\lambda})}{\partial x_{\mu}}=-\frac{\partial H(x^{\lambda},\ddot{x}^{\lambda},\cdots,\overset{(M)}{x}{}^{\lambda},y_{\lambda})}{\partial x_{\mu}}.$$

For this, we differentiate (1.34) with respect to

$x^{\mu}$, the $y_{\lambda}$ | $x_{\mu}$, the $y^{\lambda}$

being held constant for the purpose of partial differentiation, and we obtain

$$(1.36)\qquad\begin{aligned}\frac{\partial H}{\partial x^{\mu}}&=\frac{\partial L}{\partial x^{\mu}}+\frac{\partial L}{\partial \dot{x}^{\nu}}\frac{\partial \psi^{\nu}}{\partial x^{\mu}}\\&=\frac{\partial L}{\partial x^{\mu}}+\frac{y_{\lambda}}{L}\frac{\partial}{\partial x^{\mu}}\left(H\frac{\partial H}{\partial y_{\lambda}}\right),\end{aligned}\qquad\Bigg|\qquad\begin{aligned}\frac{\partial L}{\partial x_{\mu}}&=\frac{\partial H}{\partial x_{\mu}}+\frac{\partial H}{\partial \dot{x}_{\nu}}\frac{\partial \psi_{\nu}}{\partial x_{\mu}}\\&=\frac{\partial H}{\partial x_{\mu}}+\frac{y^{\lambda}}{L}\frac{\partial}{\partial x_{\mu}}\left(L\frac{\partial L}{\partial y^{\lambda}}\right),\end{aligned}$$

where we have used (1.18) and (1.21). But, since $H(L)$ is positively homogeneous of the *first degree* in $y_{\lambda}(y^{\lambda})$, we have

$$(1.37)\qquad y_{\lambda}\frac{\partial H}{\partial y_{\lambda}}=H,\qquad\Big|\qquad y^{\lambda}\frac{\partial L}{\partial y^{\lambda}}=L,$$

$$(1.38)\qquad y_{\lambda}\frac{\partial^2 H}{\partial x^{\mu}y_{\lambda}}=\frac{\partial H}{\partial x^{\mu}},\qquad\Big|\qquad y^{\lambda}\frac{\partial^2 L}{\partial x_{\mu}\partial y^{\lambda}}=\frac{\partial L}{\partial x_{\mu}},$$

so that (1.36) becomes

$$(1.39)\qquad \frac{\partial H}{\partial x^{\mu}}=\frac{\partial L}{\partial x^{\mu}}+\frac{2}{L}\left(\frac{\partial H}{\partial x^{\mu}}H\right),\qquad\Big|\qquad \frac{\partial L}{\partial x_{\mu}}=\frac{\partial H}{\partial x_{\mu}}+\frac{2}{H}\left(\frac{\partial L}{\partial x_{\mu}}L\right),$$

or, if we use (1.34) once more, the relation (1.39) follows.

Next, let us prove the relation

(1.35)′

$$\frac{\partial H(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y_\lambda)}{\partial y_\mu} = -\frac{\partial L(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}, y^\lambda)}{\partial y_\mu}. \quad \Bigg| \quad \frac{\partial L(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}, y^\lambda)}{\partial y^\mu} = -\frac{\partial H(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y_\lambda)}{\partial y^\mu}.$$

For this, we differentiate (1.34) with respect to

$y_\mu$, the $x_\lambda$ | $y^\mu$, the $x^\lambda$

being held constant for the purpose of partial differentiation, and obtain

(1.36)′

$$\frac{\partial H(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y_\lambda)}{\partial y_\mu} = \frac{\partial L}{\partial y_\mu} + \frac{\partial L}{\partial \dot{x}^\nu}\frac{\partial \psi^\nu}{\partial y_\mu} = \frac{\partial L}{\partial y_\mu} + \frac{y_\lambda}{L}\frac{\partial}{\partial y_\mu}\left(H\frac{\partial H}{\partial y_\lambda}\right),$$

$$\frac{\partial L(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda} y^\lambda)}{\partial y^\mu} = \frac{\partial H}{\partial y^\mu} + \frac{\partial H}{\partial \dot{x}_\nu}\frac{\partial \psi_\nu}{\partial y^\mu} = \frac{\partial H}{\partial y^\mu} + \frac{y^\lambda}{H}\frac{\partial}{\partial y^\mu}\left(L\frac{\partial L}{\partial y^\lambda}\right),$$

where we have used (1.18) and (1.21). But, since $H(L)$ is positively homogeneous of the *first degree in* $y_\lambda$ $(y^\lambda)$, we have

$$y_\lambda\frac{\partial H}{\partial y_\lambda} = H, \; y_\lambda\frac{\partial^2 H}{\partial y_\mu \partial y_\lambda} = \frac{\partial H}{\partial y_\mu}, \quad \Bigg| \quad y^\lambda\frac{\partial L}{\partial y^\lambda} = L, \; y^\lambda\frac{\partial^2 L}{\partial y^\mu \partial y^\lambda} = \frac{\partial L}{\partial y^\mu},$$

so that (1.36)′ becomes

$$\frac{\partial H}{\partial y_\mu} = \frac{\partial L}{\partial y_\mu} + \frac{2}{L}\left(\frac{\partial H}{\partial y_\mu}H\right), \quad \Bigg| \quad \frac{\partial L}{\partial y^\mu} = \frac{\partial H}{\partial y^\mu} + \frac{2}{H}\left(\frac{\partial L}{\partial y^\mu}L\right),$$

or, if we use (1.34) once more, being put $2/L = 2/H$, the relation (1.35)′ follows.

The case $M=1$ of § 1 is particularly note-worthy.

As for $L$ resp. $H$, we shall find later the *canonical equations of*

*Hamiltonian types* (2.25). | *Lagrangian types* (2.26).

## § 2 Another Duality in the Hamilton's Canonical Formalism.

In (1.32), for (1.11) we have found

(2.1)

$$\begin{aligned} \dot{s}^2 &= H^2(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}) \equiv H^2(x, \ddot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}, y_\lambda) \\ &= g^{\mu\nu}(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y_\lambda)\, y_\mu y_\nu = g_{\mu\nu}(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y^\lambda)\, y^\mu y^\nu \\ &= g^{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})\, \dot{x}_\mu \dot{x}_\nu = g_{\mu\nu}(x^\lambda, \cdots, \overset{(M)}{x^\lambda})\, \dot{x}^\mu \dot{x}^\nu = L^2(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y^\lambda) \equiv L^2(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}). \end{aligned}$$

Now, quadratic differential forms are always expressible in the forms of the sums of (generalized) Pfaffians:

$$(2.2)\qquad \dot{s}^2 (d\tau)^2 = g^{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})\, \dot{x}_\mu \dot{x}_\nu (d\tau)^2 = \omega_l \omega_l = \omega^l \omega^l$$
$$= g_{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})\, \dot{x}_\mu \dot{x}_\nu (d\tau)^2$$

*except undergoing*[5] *extended as well as doubly extended orthogonal transformations,* where

$$(2.3)\qquad \omega_l \overset{\text{def}}{=} \omega^\mu_l (x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})\, \dot{x}_\mu\, d\tau. \qquad\Big|\qquad \omega^l \overset{\text{def}}{=} \omega^l_\mu (x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})\, \dot{x}^\mu\, d\tau.$$

Thereby (1.28) and (1.30) (for $H=\dot{s}=L$) are taken into consideration.

Set (cf. Art. 3 and T. Takasu, [1])

$$(2.4)\qquad H^l = c^l H = c^l L = L^l, \qquad\Big|\qquad L_l = c_l L = c_l H = H_l,$$

where $c^l$ and $c_l$ are arbitrary constants such that

$$(2.5)\qquad c^l c^l = 1, \qquad\Big|\qquad c_l c_l = 1,$$

so that

$$(2.6)\qquad H^l H^l = H^2 = H_l H_l = L_l L_l = L^2 = L^l L^l.$$

Thus we have

$$(2.7)\qquad y_\mu = \dot{x}_\mu = \omega^l_\mu H_l = \omega^l_\mu L^l = g_{\mu\nu} \dot{x}^\nu, \qquad\Big|\qquad y^\mu = \dot{x}^\mu = \omega^\mu_l L_l = \omega^\mu_l H_l = g^{\mu\nu} \dot{x}_\nu,$$

where

$$(2.8)\qquad g_{\mu\nu} = \omega^l_\mu \omega^l_\nu, \ (\omega_\lambda = (\omega^l_\lambda)), \qquad\Big|\qquad g^{\mu\nu} = \omega^\mu_l \omega^\nu_l, \ (\omega^\lambda = (\omega^\lambda_l)),$$

$$(2.9)\qquad \omega^l_\mu \omega^\nu_l = \delta^\nu_\mu.$$

From (2.9), we have

$$(2.10)\qquad y_\lambda = g_{\lambda\nu} \dot{x}^\nu = \omega_\lambda \omega_\nu \dot{x}^\nu = \omega_\lambda H, \ (\omega_\lambda = (\omega^l_\lambda)), \qquad\Big|\qquad y^\lambda = g^{\lambda\nu} \dot{x}_\nu = \omega^\lambda \omega^\nu \dot{x}_\nu = \omega^\lambda L, \ (\omega^\lambda = (\omega^\lambda_l)).$$

Since

$$(2.11)\qquad H(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}, y_\lambda) = H(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}, H\omega_\lambda) \qquad\Big|\qquad L(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y^\lambda) = L(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, L\omega^\lambda)$$

*is of the first degree in*

---

5) T. Takasu, [1].

| | |
|---|---|
| $y_\lambda$, *the* $\omega_\lambda$ | $y^\lambda$, *the* $\omega^\lambda$ |

*must be of zero degree in*

| | |
|---|---|
| $y_\lambda$, | $y^\lambda$, |

so that

| | |
|---|---|
| (2.12) $\dfrac{\partial y_\lambda}{\partial \omega_\lambda} = H + \omega_\lambda \dfrac{\partial H}{\partial \omega_\lambda} = H.$ | $\dfrac{\partial y^\lambda}{\partial \omega^\lambda} = L + \omega^\lambda \dfrac{\partial L}{\partial \omega^\lambda} = L.$ |

Next, let us prove the relation

| | |
|---|---|
| (2.13) $\dfrac{\partial H(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}, \omega_\lambda H)}{d\omega_\mu} = -\dfrac{\partial L(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}\, \omega^\lambda L)}{\partial \omega_\mu}.$ | $\dfrac{\partial L(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, \omega^\lambda L)}{\partial \omega^\mu} = -\dfrac{\partial H(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}, \omega_\lambda H)}{\partial \omega^\mu}.$ |

Multiplying namely (1.35)′ with (2.12)

| | |
|---|---|
| $\dfrac{\partial y_\lambda}{\partial \omega_\lambda} = H,$ | $\dfrac{\partial y^\lambda}{\partial \omega^\lambda} = L,$ |

we obtain (2.13).

Set further

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|---|---|
| (2.14) $\mathfrak{H} \overset{\text{def}}{=} y_\mu \dot{x}^\mu$ | $\mathfrak{L} \overset{\text{def}}{=} y^\mu \dot{x}_\mu$ |

$$= y_\mu y^\mu,$$

then we have also

| | |
|---|---|
| (2.15) $\mathfrak{H} = \omega_\mu^l H^l \dot{x}^\mu = \omega_\mu^l L^l \dot{x}_\mu$ | $\mathfrak{L} = \omega^\mu_l L^l \dot{x}_\mu = \omega^\mu_l H_l \dot{x}^\mu$ |

$$= H^2 = \dot{s}^2 = L^2.$$

From (2.14) and (2.15), we have

$$\text{(2.16)} \qquad \mathfrak{H} = y_\mu \dot{x}^\mu = \mathfrak{L} = y^\mu \dot{x}_\mu$$

| | |
|---|---|
| $= g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = g_{\mu\nu}, y^\mu y^\nu$ | $= g^{\mu\nu} \dot{x}_\mu \dot{x}_\nu = g^{\mu\nu} y_\mu y_\nu$ |

$$= y_\mu y^\mu.$$

Subsequently it will become convenient to introduce $\omega_\mu (x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})$ and $\omega^\mu (x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})$ such that

| | |
|---|---|
| (2.17) $c^l \omega_\mu (x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}) = \omega_\mu^l (x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}),$ | $c_l\, \omega^\mu (x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}) = \omega^\mu_l (x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}),$ |

so that

(2.18) $$H=\omega_\mu \dot{x}^\mu=\omega_\mu y^\mu=L=\omega^\mu \dot{x}_\mu=\omega^\mu y_\mu$$

$$=(H_\iota H_\iota)^{\frac{1}{2}}=(H^\iota H^\iota)^{\frac{1}{2}}. \qquad\qquad =(L^\iota L^\iota)^{\frac{1}{2}}=(L_\iota L_\iota)^{\frac{1}{2}}.$$

From (2.16), we have

$$\delta\mathfrak{H}=\delta y_\mu \dot{x}^\mu+y_\mu \delta\dot{x}^\mu, \qquad\qquad \delta\mathfrak{L}=\delta y^\mu \dot{x}_\mu+y^\mu \delta\dot{x}_\mu,$$

(2.19)
$$0=\delta\int_{\tau_0}^{\tau_1}\mathfrak{H}d\tau=\int_{\tau_0}^{\tau_1}\delta\mathfrak{H}\,d\tau$$
$$=\int_{\tau_0}^{\tau_1}\delta y_\mu \dot{x}^\mu\,d\tau+\int_{\tau_0}^{\tau_1}y_\mu \delta\dot{x}^\mu\,d\tau$$
$$=\int_{\tau_0}^{\tau_1}\delta y_\mu \dot{x}^\mu\,d\tau+\int_{\tau_0}^{\tau_1}y_\mu \frac{d}{d\tau}\delta x^\mu\,d\tau$$
$$=\int_{\tau_0}^{\tau_1}\delta y_\mu \dot{x}^\mu\,d\tau+[y_\mu \delta x^\mu]_{\tau_0}^{\tau_1}-\int_{\tau_0}^{\tau_1}\delta x^\mu \dot{y}_\mu\,d\tau,$$

$$0=\delta\int_{\tau_0}^{\tau_1}\mathfrak{L}d\tau=\int_{\tau_0}^{\tau_1}\delta\mathfrak{L}\,d\tau$$
$$=\int_{\tau_0}^{\tau_1}\delta y^\mu \dot{x}_\mu\,d\tau+\int_{\tau_0}^{\tau_1}y^\mu \delta\dot{x}_\mu\,d\tau$$
$$=\int_{\tau_0}^{\tau_1}\delta y^\mu \dot{x}_\mu\,d\tau+\int_{\tau_0}^{\tau_1}y^\mu \frac{d}{d\tau}\delta x_\mu\,d\tau$$
$$=\int_{\tau_0}^{\tau_1}\delta y^\mu \dot{x}_\mu\,d\tau+[y^\mu \delta x_\mu]_{\tau_0}^{\tau_1}-\int_{\tau_0}^{\tau_1}\delta x_\mu \dot{y}^\mu\,d\tau,$$

(2.20 $$0=\int_{\tau_0}^{\tau_1}[\delta y_\mu \dot{x}^\mu-\delta x^\mu \dot{y}_\mu]\,d\tau. \qquad\qquad 0=\int_{\tau_0}^{\tau_1}[\delta y^\mu \dot{x}_\mu-\delta x_\mu \dot{y}^\mu]\,d\tau.$$

Hence

(2.21) $$\delta\mathfrak{H}=\delta y_\mu \dot{x}^\mu-\delta x^\mu \dot{y}_\mu, \qquad\qquad \delta\mathfrak{L}=\delta y^\mu \dot{x}_\mu-\delta x_\mu \dot{y}^\mu,$$

$y_\mu$ and $x^\mu$ | $y^\mu$ and $x_\mu$

being considered as canonical conjugates. Thus we obtain the *Hamilton's global canonical equations*:

(2.22) $$\frac{\partial\mathfrak{H}}{\partial y_\mu}=\dot{x}^\mu,\ \frac{\partial\mathfrak{H}}{\partial x^\mu}=-\dot{y}_\mu \qquad\qquad \frac{\partial\mathfrak{L}}{\partial y_\mu}=\dot{x}_\mu,\ \frac{\partial\mathfrak{L}}{\partial x_\mu}=-\dot{y}^\mu$$

of the extremals

$$\delta\int_{\tau_0}^{\tau_1}(\dot{s})^2\,d\tau=\int_{\tau_0}^{\tau_1}\delta(\dot{s})^2\,d\tau=2\int_{\tau_0}^{\tau_1}\dot{s}\,\ddot{s}\,\delta\tau\,d\tau=0$$

for all values of $\delta\tau$, i.e. of the extremals

$$s=a\tau+c. \qquad (a, c:\ \text{const.}),$$

From (2.18), we have

(2.23) $$H=\omega_\mu \dot{x}^\mu, \qquad\qquad L=\omega^\mu \dot{x}_\mu.$$

Hence

$$\delta\int_{\tau_0}^{\tau_1} H d\tau=\int_{\tau_0}^{\tau_1}\delta H\, d\tau$$
$$=\int_{\tau_0}^{\tau_1}[\delta\omega_\mu\,\dot{x}^\mu+\omega_\mu\,\delta\dot{x}]\,d\tau$$
$$=\int_{\tau_0}^{\tau_1}[\delta\omega_\mu\,\dot{x}^\mu\,d\tau+\int_{\tau_0}^{\tau_1}\omega_\mu\frac{d}{d\tau}\delta x^\mu\,d\tau$$
$$=\int_{\tau_0}^{\tau_1}\delta\omega_\mu\,\dot{x}^\mu\,d\tau+[\omega_\mu\,\delta x^\mu]_{\tau_0}^{\tau_1}-\int_{\tau_0}^{\tau_1}\delta x^\mu\,\dot{\omega}_\mu\,d\tau$$
$$=\int_{\tau_0}^{\tau_1}[\delta\omega_\mu\,\dot{x}^\mu-\delta x^\mu\,\dot{\omega}_\mu]\,d\tau=0,$$

$$\delta\int_{\tau_0}^{\tau_1} L d\tau=\int_{\tau_0}^{\tau_1}\delta L d\tau$$
$$=\int_{\tau_0}^{\tau_1}[\delta\omega^\mu\,\dot{x}_\mu+\omega^\mu\,\delta\dot{x}_\mu]\,d\tau$$
$$=\int_{\tau_0}^{\tau_1}\delta\omega^\mu\,\dot{x}_\mu\,d\tau+\int_{\tau_0}^{\tau_1}\omega^\mu\frac{d}{d\tau}\delta x^\mu\,d\tau$$
$$=\int_{\tau_0}^{\tau_1}\delta\omega^\mu\,\dot{x}_\mu\,d\tau+[\omega^\mu\,\delta x_\mu]_{\tau_0}^{\tau_1}-\int_{\tau_0}^{\tau_1}\delta x_\mu\,\dot{\omega}^\mu\,d\tau$$
$$=\int_{\tau_0}^{\tau_1}[\delta\omega^\mu\,\dot{x}_\mu-\delta x_\mu\,\dot{\omega}^\mu]\,d\tau=0,$$

so that we may put

(2.24) $$\delta H=\delta\omega_\mu\,\dot{x}^\mu-\delta x^\mu\,\dot{\omega}_\mu \qquad\Big|\qquad \delta L=\delta\omega^\mu\,\dot{x}_\mu-\delta x_\mu\,\dot{\omega}^\mu,$$

whence follows the *Hamilton's canonicat equations*:

(2.25) $$\frac{\partial H}{\partial\omega_\mu}=\dot{x}^\mu,\ \frac{\partial H}{\partial x^\mu}=-\dot{\omega}_\mu \qquad\Big|\qquad \frac{\partial L}{\partial\omega^\mu}=\dot{x}_\mu,\ \frac{\partial L}{\partial x_\mu}=-\dot{\omega}^\mu$$

of the extremals (*II-geodesic curves* in the present author's sense, T. Takasu, [1])

$$\delta s=\delta\int_{\tau_0}^{\tau_1}\dot{s}d\tau=\int_{\tau_0}^{\tau_1}\delta\dot{s}d\tau=\int_{\tau_0}^{\tau_1}\frac{d}{d\tau}\delta s\,d\tau=[\delta s]_{\tau_0}^{\tau_1}=0.$$

The left- | The right-

hand side becomes to the *Lagrangian canonical equations*:

(2.26) $$\frac{\partial L}{\partial\omega_\mu}=-\dot{x}^\mu,\ \frac{\partial L}{\partial x^\mu}=\dot{\omega}_\mu \qquad\Big|\qquad \frac{\partial H}{\partial\omega^\mu}=-\dot{x}_\mu,\ \frac{\partial H}{\partial x_\mu}=\dot{\omega}^\mu$$

owing to (1.35).

**N. B.** *Along the* II-*geodesic curves* $\delta s=0$ *there holds*:

(2.27) $$\dot{H}=0. \qquad\Big|\qquad \dot{L}=0.$$

The (1.34) and (2.12) give

(2.28) $$\mathfrak{H}(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda, y_\lambda) = y_\mu \dot{x}^\mu = y_\mu y^\mu = \mathfrak{L}(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y^\lambda) = y^\mu \dot{x}_\mu$$

| | |
|---|---|
| $=\mathfrak{L}(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, \phi_\nu(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y_\lambda))$ | $=\mathfrak{H}(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, \phi^\nu(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda, y^\lambda))$ |
| $=g_{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda)\, \dot{x}^\mu \dot{x}^\nu$ | $=g^{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})\, \dot{x}_\mu \dot{x}_\nu$ |
| $=g_{\mu\nu}(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda, y_\lambda)\, y^\mu y^\nu.$ | $=g^{\mu\nu}(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y^\lambda)\, y_\mu y_\nu.$ |
| (2.29) $\dfrac{\partial \mathfrak{H}}{\partial \dot{x}_\mu} = g^{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda)\, \dot{x}_\nu,$ | $\dfrac{\partial \mathfrak{L}}{\partial \dot{x}^\mu} = g_{\nu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})\, \dot{x}^\nu,$ |

since

| | |
|---|---|
| $g^{\mu\nu}$ | $g_{\eta\nu}$ |

are homogeneous of *zero degree in*

| | |
|---|---|
| $\dot{x}_\mu$ | $\dot{x}^\mu$ |

and

| | |
|---|---|
| (2.30) $\dfrac{\partial^2 \mathfrak{H}}{\partial \dot{x}_\mu \partial \dot{x}_\nu} = g^{\mu\nu}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda).$ | $\dfrac{\partial^2 \mathfrak{L}}{\partial \dot{x}^\mu \partial \dot{x}^\nu} = g_{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}).$ |

Now, let us prove the relation

(2.31)

| | |
|---|---|
| $\dfrac{\partial \mathfrak{H}(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda, y_\lambda)}{\partial x^\mu} = -\dfrac{\partial \mathfrak{L}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y^\lambda)}{\partial x^\mu}.$ | $\dfrac{\partial \mathfrak{L}(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}, y^\lambda)}{\partial x_\mu} = -\dfrac{\partial \mathfrak{H}(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda, y_\lambda)}{\partial x_\mu}.$ |

For this, we differentiate (2.27) with respect to

| | |
|---|---|
| $x^\mu$, the $y_\lambda$ | $x_\mu$, the $y^\lambda$ |

being held constant for the purpose of partial differentiation, and obtain

| | |
|---|---|
| (2.32) $\dfrac{\partial \mathfrak{H}}{\partial x^\mu} = \dfrac{\partial \mathfrak{L}}{\partial x^\mu} + \dfrac{\partial \mathfrak{L}}{\partial \dot{x}^\nu} \dfrac{\partial \phi^\nu}{\partial x^\mu}$ | $\dfrac{\partial \mathfrak{L}}{\partial x_\mu} = \dfrac{\partial \mathfrak{H}}{\partial x_\mu} + \dfrac{\partial \mathfrak{H}}{\partial \dot{x}_\nu} \dfrac{\partial \phi_\nu}{\partial x_\mu}$ |
| $= \dfrac{\partial \mathfrak{L}}{\partial x^\mu} + \dfrac{y_\lambda}{L} \dfrac{\partial}{\partial x^\mu}\left(H \dfrac{\partial H}{\partial y_\lambda}\right)$ | $= \dfrac{\partial \mathfrak{H}}{\partial x_\mu} + \dfrac{y^\lambda}{H} \dfrac{\partial}{\partial x_\mu}\left(L \dfrac{\partial L}{\partial y^\lambda}\right)$ |

by (2.29), where we have used (1.18) and (8.21). But, since $H(L)$ is positively homogeneous of the first degree in $y_\lambda(y^\lambda)$, we have

| | |
|---|---|
| $y_\lambda \dfrac{\partial H}{\partial y_\lambda} = H,\ y_\lambda \dfrac{\partial^2 H}{\partial y_\mu \partial y_\nu} = \dfrac{\partial H}{\partial y_\mu},$ | $y^\lambda \dfrac{\partial L}{\partial y^\lambda} = L,\ y^\lambda \dfrac{\partial^2 L}{\partial y^\mu \partial y^\lambda} = \dfrac{\partial L}{\partial y^\mu},$ |

so that (2.32) becomes

$$\frac{\partial\mathfrak{H}}{\partial x^\mu}=\frac{\partial\mathfrak{L}}{\partial x^\mu}+\frac{2}{L}\left(\frac{\partial H}{\partial y_\mu}\right), \qquad \Bigg| \qquad \frac{\partial\mathfrak{L}}{\partial x_\mu}=\frac{\partial\mathfrak{H}}{\partial x_\mu}+\frac{2}{H}\left(\frac{\partial L}{\partial y^\mu}L\right),$$

or, if we use (1.34) once more, being put $2/H=2/L$, the relation (2.31) follows.

Now, let us prove

$$\text{(2.33)} \quad \frac{\partial\mathfrak{H}(x_\lambda,\dot{x}_\lambda,\cdots,\overset{(M)}{x}_\lambda y_\lambda)}{\partial y_\mu}=\frac{\partial\mathfrak{L}(x^\lambda,\ddot{x}^\lambda,\cdots\overset{(M)}{x^\lambda},y^\lambda)}{\partial y_\mu}. \quad \Bigg| \quad \frac{\partial\mathfrak{L}(x^\lambda,\ddot{x}^\lambda,\cdots,\overset{(M)}{x^\lambda},y^\lambda)}{\partial y^\mu}=-\frac{\mathfrak{H}(x_\lambda,\ddot{x}_\lambda,\cdots,\overset{(M)}{x}_\lambda,y_\lambda)}{\partial y^\mu}.$$

$$\mathfrak{H}=H^2=L^2=\mathfrak{L}.$$

$$\frac{\partial\mathfrak{H}(x_\lambda,\ddot{x}_\lambda,\cdots\overset{(M)}{x}_\lambda,y_\lambda)}{\partial y_\mu}=-2H\frac{\partial H}{\partial y_\mu}$$
$$=(-2H)\left(-\frac{\partial L(x^\lambda,\ddot{x}^\lambda,\cdots\overset{(M)}{x^\lambda},y^\lambda)}{\partial y_\mu}\right)$$
$$=2H\frac{\partial L(x^\lambda,\ddot{x}^\lambda,\cdots,\overset{(M)}{x^\lambda},y^\lambda)}{\partial y_\mu}$$

$$\frac{\partial\mathfrak{L}(x^\lambda,\ddot{x}^\lambda,\cdots,\overset{(M)}{x^\lambda},y^\lambda)}{\partial y^\mu}=-2L\frac{\partial L}{\partial y^\mu}$$
$$=(-2L)\left(-\frac{\partial H(x_\lambda,\ddot{x}_\lambda,\cdots,\overset{(M)}{x}_\lambda,y_\lambda}{\partial y^\mu}\right)$$
$$=2L\frac{\partial H(x_\lambda,\ddot{x}_\lambda,\cdots,\overset{(M)}{x}_\lambda,y_\lambda)}{\partial y^\mu}$$

and

$$\frac{\partial\mathfrak{L}(x^\lambda,\ddot{x}^\lambda,\cdots,\overset{(M)}{x^\lambda},y^\lambda)}{\partial y_\mu}=-2L\frac{\partial L}{\partial y_\mu}$$
$$=(-2L)\left(-\frac{\partial H}{\partial y_\mu}\right)$$
$$=2L\frac{\partial H(x^\lambda,\ddot{x}^\lambda,\cdots,\overset{(M)}{x^\lambda},y^\lambda)}{\partial y_\mu}$$

$$\frac{\partial\mathfrak{H}(x_\lambda,\ddot{x}_\lambda\cdots,\overset{(M)}{x}_\lambda,y_\lambda)}{\partial y^\mu}=-2H\frac{\partial H}{\partial y^\mu}$$
$$=(-2H)\left(-\frac{\partial L}{\partial y^\mu}\right)$$
$$=2H\frac{\partial L(x^\lambda,\ddot{x}^\lambda,\cdots,\overset{(M)}{x^\lambda},y^\lambda)}{\partial y^\mu}$$

by (1.35)′. Eliminating

$$\frac{\partial H}{\partial y_\mu} \text{ resp. } \frac{\partial L}{\partial y_\mu}, \qquad \Bigg| \qquad \frac{\partial L}{\partial y^\mu} \text{ resp. } \frac{\partial H}{\partial y^\mu},$$

we obtain (2.33).

By virtue of (2.31) and (2.33), the (2.22) becomes to the *global Lagrangian equations*

$$\text{(2.34)} \quad \frac{\partial\mathfrak{L}}{\partial y_\mu}=\dot{x}^\mu,\ \frac{\partial\mathfrak{L}}{\partial x^\mu}=\dot{y}_\mu, \qquad \Bigg| \qquad \frac{\partial\mathfrak{H}}{\partial y^\mu}=\dot{x}_\mu,\ \frac{\partial\mathfrak{H}}{\partial x_\mu}=\dot{y}^\mu,$$

the first of which follows from (2.14) at once and the second are nothing other than the *Euler-Lagrange equations* of (2.19), where

$$\text{(2.35)} \quad y_\mu\overset{\text{def}}{=}\frac{\partial\mathfrak{L}}{\partial\dot{x}^\mu}-\frac{d}{d\tau}\frac{\partial\mathfrak{L}}{\partial\ddot{x}^\mu}+\cdots+(-1)^{M-1}\frac{d^{M-1}}{d\tau^{M-1}}\frac{\partial\mathfrak{L}}{\partial\overset{(M)}{x^\mu}}. \qquad \Bigg| \qquad y^\mu\overset{\text{def}}{=}\frac{\partial\mathfrak{H}}{\partial\dot{x}_\mu}-\frac{d}{d\tau}\frac{\partial\mathfrak{H}}{\partial\ddot{x}_\mu}+\cdots+(-1)^{M-1}\frac{d^{M-\lambda}}{d\tau^{M-1}}\frac{\partial\mathfrak{H}}{\partial\overset{(M)}{x}_\mu}.$$

Since

$$H=\omega_\mu \dot{x}^\mu, \qquad\qquad L=\omega^\mu \dot{x}_\mu,$$

$$\mathfrak{H}(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda, y_\lambda)=H^2=\dot{s}^2=K^2=\mathfrak{L}(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda, y^\lambda),$$

we have

$$\tfrac{1}{2}\,\delta\mathfrak{H}=H\delta H=H(\delta\omega_\mu\,\dot{x}^\mu-\delta x^\mu\,\dot{\omega}_\mu), \qquad\qquad \tfrac{1}{2}\,\delta\mathfrak{L}=L\delta L=L(\delta\omega^\mu\,\dot{x}_\mu-\delta x_\mu\,\dot{\omega}^\mu),$$

whence follows

$$(2.36)\qquad \frac{1}{2}\frac{\partial\mathfrak{H}}{\partial\omega_\mu}=H\dot{x}^\mu,\ \frac{1}{2}\frac{\partial\mathfrak{H}}{\partial x^\mu}=-H\dot{\omega}_\mu \qquad\qquad \frac{1}{2}\frac{\partial\mathfrak{L}}{\partial\omega^\mu}=L\dot{x}_\mu,\ \frac{1}{2}\frac{\partial\mathfrak{L}}{\partial x_\mu}=-L\dot{\omega}^\mu$$

*for the extremals* ("II-geodesic curves" T. Takasu, [1] ) $\delta s=0$:

$$(2.37)\qquad \partial\mathfrak{H}=0. \qquad\qquad \delta\mathfrak{L}=0.$$

In the special case, where

$$(2.38)\qquad \tau=s,\ H=K=1,^{6)}\quad \tilde{\mathfrak{H}}\overset{\text{def}}{=}\mathfrak{H}/2,\ \tilde{\mathfrak{L}}\overset{\text{def}}{=}\mathfrak{L}/2,$$

$$(2.39)\qquad y_\lambda=\omega_\lambda \qquad\qquad y^\lambda=\omega^\lambda$$

as will be read out from (2.10), the (2.36) becomes for (2.37) to

$$(2.40)\qquad \frac{\partial\tilde{\mathfrak{H}}}{\partial\omega_\mu}=\frac{dx^\mu}{ds},\ \frac{\partial\tilde{\mathfrak{H}}}{dx^\mu}=-\frac{d\omega_\mu}{ds}, \qquad\qquad \frac{\partial\tilde{\mathfrak{L}}}{\partial\omega^\mu}=\frac{dx_\mu}{ds},\ \frac{\partial\tilde{\mathfrak{L}}}{\partial x_\mu}=-\frac{d\omega^\mu}{ds},$$

which are *Hamilton's canonical equations of the* II-*geodesic curves.*

Next, let us prove the following relations as *counter-parts of* (2.31)

$$(2.41)\qquad \frac{\partial\tilde{\mathfrak{H}}(x_\lambda, \ddot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda, \omega_\lambda H)}{\partial\omega_\mu} =-\frac{\partial\tilde{\mathfrak{L}}(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda, \omega^\lambda L)}{\partial\omega_\mu}. \qquad\qquad \frac{\partial\tilde{\mathfrak{L}}(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda, \omega^\lambda L)}{\partial\omega^\mu} =-\frac{\partial\tilde{\mathfrak{H}}(x^\lambda, \ddot{x}^\lambda, \cdots, \overset{(M)}{x}_\lambda, \omega_\lambda H)}{\partial\omega^\mu}.$$

Utilizing namely (2.12), we obtain

$$\frac{\partial\tilde{\mathfrak{H}}}{\partial\omega_\mu}=\frac{\partial\tilde{\mathfrak{H}}}{\partial y_\mu}\frac{\partial y_\mu}{\partial\omega_\mu}=\frac{\partial\tilde{\mathfrak{L}}}{\partial y_\mu}. \qquad\qquad \frac{\partial\tilde{\mathfrak{L}}}{\partial\omega^\mu}=\frac{\partial\tilde{\mathfrak{L}}}{\partial y^\mu}\frac{\partial y^\mu}{\partial\omega^\mu}=\frac{\partial\tilde{\mathfrak{L}}}{\partial y^\mu}.$$

Thus (2.41) follows from (23.3).

---

6) Cf. (4.29) in H. Rund, [12], p. 172.

Next, let us deduce *the Hamilton's canonical equations of the* II*-geodesic curves* $\delta s=0$:

(2.42) $$\frac{\partial \tilde{\mathfrak{H}}(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x}_\lambda, \omega_\lambda H)}{\partial \omega^\mu}=\frac{dx_\mu}{ds} \qquad \Bigg| \qquad \frac{\partial \tilde{\mathfrak{L}}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda, \omega^\lambda L)}{\partial \omega_\mu}=\frac{dx^\mu}{ds}$$

*as the counter-part of*

(2.43) $$\frac{\partial \tilde{\mathfrak{H}}}{\partial x_\mu}=-\frac{d\omega^\mu}{ds}. \qquad \Bigg| \qquad \frac{\partial \tilde{\mathfrak{L}}}{\partial x^\mu}=-\frac{d\omega_\mu}{ds}.$$

This is seen e. g. for the right-hand side as follows.

$$\frac{\partial \tilde{\mathfrak{L}}}{\partial x^\mu}=-\frac{\partial \tilde{\mathfrak{H}}}{\partial x^\mu}=-H\dot{\omega}_\mu \rightarrow -\frac{d\omega_\mu}{ds}$$

by (2.31) and (2.37).

By (2.41), we bave

(2.44) $$\frac{\partial \tilde{\mathfrak{H}}}{\partial \omega^\mu}=L\dot{x}_\mu \qquad \Bigg| \qquad \frac{\partial \tilde{\mathfrak{L}}}{\partial \omega_\mu}=H\dot{x}^\mu$$

by (2.35). | by (2.39) and (2.37).

The (2.44) decomposes to

(2.45) $$\frac{\partial \tilde{\mathfrak{H}}}{\partial \omega^\mu}=\frac{dx_\mu}{ds} \qquad \Bigg| \qquad \frac{\partial \tilde{\mathfrak{L}}}{\partial \omega_\mu}=\frac{dx^\mu}{ds}$$

in the case (2.38), which is (2.42).

The case $M=1$ of § 2 is particularly note-worthy.

## § 3. Hamilton's Canonical Formalism in terms of Global Coordinates of the Present Author and Two Allied Dualities.

Set

(3.1) $$\omega \overset{\text{def}}{=} y_\lambda \dot{x}^\lambda = H\omega_\lambda \dot{x}^\lambda \qquad \Bigg| \qquad \omega \overset{\text{def}}{=} y^\lambda \dot{x}_\lambda = L\omega^\lambda \dot{x}_\lambda$$

utilizing the relation (2.10) and considering the special case (2.38), (2.39), where we may consider (by virtue of (2.7) and (2.38)):

(3.2) $$\omega_\mu \omega^\lambda = y_\mu y^\lambda = \delta^\lambda_\mu.$$

Then straight forward calculation gives us the following relations

(3.3) $$\frac{d}{ds}\frac{\omega}{ds} \equiv \omega_\lambda (\ddot{x}^\lambda + \Lambda^\lambda_{\mu\nu}\dot{x}^\mu \dot{x}^\nu) = \omega^\lambda (\ddot{x}_\lambda + \Lambda^{\mu\nu}_\lambda \dot{x}_\mu \dot{x}_\nu),$$

where e. g. $\omega^\lambda \frac{\partial \omega_\mu}{\partial \dot{x}^\nu} d\dot{x}^\nu = 0$ *owing to the zero degree homogeneiety of* $\omega_\mu$ *in* $\dot{x}^\nu$ and

(3.4) $$\Lambda^\lambda_{\mu\nu} \overset{\text{def}}{=} \omega^\lambda \frac{\partial \omega_\mu}{\partial x^\nu}. \qquad \Big| \qquad \Lambda^{\mu\nu}_\lambda \overset{\text{def}}{=} \omega_\lambda \frac{\partial \omega^\mu}{\partial x_\nu}.$$

*Since* (3.1) *is written in invariant forms,* (3.1) *and* (3.3) *are global in the differentiable manifold* $X_n = \bigcup_\alpha U_\alpha(x_{(\alpha)})$. *The factor* $\omega_\lambda$ *resp.* $\omega^\lambda$ *maps the local paths*

(3.5) $$\ddot{x}^\lambda + \Lambda^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = 0 \qquad \Big| \qquad \ddot{x}_\lambda + \Lambda^{\mu\nu}_\lambda \dot{x}_\mu \dot{x}_\nu = 0$$

*into the global paths*

(3.6) $$\frac{d}{ds}\frac{\omega}{ds} = 0, \quad \omega = cds, \quad \int \omega = cs + d,$$

where $c$ and $d$ are constants. For (3.6), we may write down

(3.7) $$d\xi \overset{\text{def}}{=} cds.$$

*The paths* (3.6) *are linear combinations of* (3.5) *by the coefficients* $\omega^\lambda(\omega^\lambda)$.
As we have assumed (1.14), the

(3.8) $$ds^2 = g_{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, x^\lambda)\, dx^\mu dx^\nu$$

is always expressible in the form (2.2), *where* (2.3) *are linearly independent.* The admissibility of $c_l$ and $c^l$ in (2.4), (2.5) and (2.6) is in concordance with the introduction of the constant vectors $c = (c_l)$ and $c = (c^l)$ in (3.7). Hence we may set

(3.9) $$\omega_l = c_l \omega, \quad y^l_\lambda \overset{\text{def}}{=} c^l y_\lambda, \quad \xi^l \overset{\text{def}}{=} c^l \xi, \qquad \Big| \qquad \omega^l \overset{\text{def}}{=} c^l \omega, \quad y^\lambda_l \overset{\text{def}}{=} c_l y^\lambda, \quad \xi_l = c_l \xi,$$

and for (3.1), (3.2), (3.3), (3.4) and (3.6) respectively we may write

(3.10) $$\omega^l = y^l_\lambda \dot{x}^\lambda d\tau. \qquad \Big| \qquad \omega_l = y^\lambda_l \dot{x}_\lambda d\tau.$$

(3.11) $$\omega^l_\mu \omega^\lambda_l = y^l_\mu y^\lambda_l = \delta^\lambda_\mu,$$

(3.12) $$\frac{d}{ds}\frac{\omega^l}{ds} = \omega^l_\lambda (\ddot{x}^\lambda + \Lambda^\lambda_{\mu\nu} \dot{x}^\mu \dot{x}^\nu) = \omega^\lambda_l (\ddot{x}_\lambda + \Lambda^{\mu\nu}_\lambda \dot{x}_\mu \dot{x}_\nu),$$

(3.13) $$\Lambda^\lambda_{\mu\nu} \overset{\text{def}}{=} \omega^\lambda_l \frac{\partial \omega^l_\mu}{\partial x^\nu}, \qquad \Big| \qquad \Lambda^{\mu\nu}_\lambda \overset{\text{def}}{=} \omega^l_\lambda \frac{\partial \omega^\mu_l}{\partial x_\nu},$$

(3.14) $$\frac{d^2 \xi^l}{ds^2} = \frac{d}{ds}\frac{\omega^l}{ds} = 0, \qquad \Big| \qquad \frac{d^2 \xi_l}{ds^2} = \frac{d}{ds}\frac{\omega_l}{ds} = 0,$$

(3.15) $$d\xi^l = \omega^l = c^l ds, \ \xi^l = c^l s + d^l, (c^l c^l = 1). \qquad \Big| \qquad d\xi_l = \omega_l = c_l\, ds, \ \xi_l = c_l s + d_l, (c_l c_l = 1).$$

The extremals represented by (3.14) has been called by the present author (T. Takasu, [1]) the II-*geodesic curves*; *they are global in* $X_n=\bigcup_\alpha U_\alpha(x_{(\alpha)})$ *and behave as for meet and join as well as for the extremal* $\delta s=0$ *like straight lines.*

*The Euler-Lagrange equation* for the extremal problem

$$(3.16)\qquad \delta\int \mathscr{H}(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi})\,ds=0 \qquad\Big|\qquad \delta\int \mathscr{L}(\dot{\xi},\cdots,\overset{(M)}{\xi})\,ds=0$$

for

$$(3.17)\qquad \mathscr{H}(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi})=\eta^l(\xi,\dot{\xi},\cdots\overset{(M)}{\xi})\,\dot{\xi}_l \qquad\Big|\qquad \mathscr{L}(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi})=\eta_l(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi})\,\dot{\xi}^l$$

become

$$(3.18)\qquad \frac{\partial\mathscr{H}}{\partial\xi_l}-\frac{d}{ds}\,\eta^l=0, \qquad\Big|\qquad \frac{\partial\mathscr{L}}{\partial\xi^l}-\frac{d}{ds}\,\eta_l=0,$$

where (cf. (2.34))

$$(3.19)\qquad \eta^l\overset{\text{def}}{=}\frac{\partial\mathscr{H}}{\partial\dot{\xi}_l}-\frac{d}{ds}\frac{\partial\mathscr{H}}{\partial\ddot{\xi}_l}+\cdots+(-1)^{M-1}\frac{d^{M-1}}{ds^{M-1}}\frac{\partial\mathscr{H}}{\partial\overset{(M)}{\xi}_l}. \qquad\Big|\qquad \eta_l\overset{\text{def}}{=}\frac{\partial\mathscr{L}}{\partial\dot{\xi}^l}-\frac{d}{ds}\frac{\partial\mathscr{L}}{\partial\ddot{\xi}^l}+\cdots+(-)^{M-1}\frac{d^{M-1}}{ds^{M-1}}\frac{\partial\mathscr{L}}{\partial\overset{(M)}{\xi^l}}.$$

The (3.18) and (3.17) give the *Lagrangian canonical equations*

$$(3.20)\qquad \frac{\partial\mathscr{H}}{\partial\xi^l}=\dot{\eta}^l,\quad \frac{\partial\mathscr{H}}{\partial\eta^l}=\dot{\xi}_l \qquad\Big|\qquad \frac{\partial\mathscr{L}}{\partial\xi^l}=\dot{\eta}_l,\quad \frac{\partial\mathscr{L}}{\partial\eta_l}=\dot{\xi}^l$$

*of the* II-*geodesic curves, which are global in* $X_n=\bigcup_\alpha U_\alpha(x_{(\alpha)})$, so that we have

$$(3.21)\qquad \delta\mathscr{H}=\frac{\partial\mathscr{H}}{\partial\xi_l}\delta\xi_l+\frac{\partial\mathscr{H}}{\partial\eta^l}\delta\eta^l \qquad\Big|\qquad \delta\mathscr{L}=\frac{\partial\mathscr{L}}{\partial\xi^l}\delta\xi^l+\frac{\partial\mathscr{L}}{\partial\eta_l}\delta\eta_l$$

$$\delta H=\dot{\eta}^l\,\delta\xi_l+\dot{\xi}_l\,\delta\eta^l, \qquad\Big|\qquad =\dot{\eta}_l\,\delta\xi^l+\dot{\xi}^l\,\delta\eta_l,$$

whence follows

$$(3.22)\qquad \delta\mathscr{H}=2\dot{\xi}_l\,\delta\eta^l+(\dot{\eta}^l\,\delta\xi_l-\dot{\xi}_l\,\delta\eta^l) \qquad\Big|\qquad \delta\mathscr{L}=2\dot{\xi}^l\,\delta\eta_l+(\dot{\eta}_l\,\delta\xi^l-\dot{\xi}^l\,\delta\eta_l)$$

by integration by parts of

$$(3.23)\qquad \delta\int_{s_0}^{s}\mathscr{H}\,ds=\int_{s_0}^{s}\delta\mathscr{H}\,ds. \qquad\Big|\qquad \delta\int_{s_0}^{s}ds=\int_{s_0}^{s}\delta\mathscr{L}\,ds.$$

If we put

$$(3.24)\qquad \delta\mathscr{L}\overset{\text{def}}{=}\dot{\xi}_l\,\delta\eta^l-\dot{\eta}^l\,\delta\xi_l, \qquad\Big|\qquad \delta\mathscr{H}\overset{\text{def}}{=}\dot{\xi}^l\,\delta\eta_l-\dot{\eta}_l\,\delta\xi^l,$$

then (3.22) becomes

$$(3.25)\qquad \delta\mathscr{H}+\delta\mathscr{L}=2\dot{\xi}_l\,\delta\eta^l. \qquad\Bigg|\qquad \delta\mathscr{L}+\delta\mathscr{H}=2\dot{\xi}^l\,\delta\eta_l.$$

From (3.22) and (3.24), we read out the *canonical equations*

$$(3.26)\qquad \frac{\partial\mathscr{L}}{\partial\eta^l}=\dot{\xi}_l,\quad \frac{\partial\mathscr{L}}{\partial\xi_l}=-\dot{\eta}^l \qquad\Bigg|\qquad \frac{\partial\mathscr{H}}{\partial\eta_l}=\dot{\xi}^l,\quad \frac{\partial\mathscr{H}}{\partial\xi^l}=-\dot{\eta}_l$$

*of the Hamilton's types of the* II*-geodesic curves.*

(3.20) and (3.26) give *two dualities.*

The case $M=1$ of § 3 is particularly note-worthy.

## § 4. The Hamilton-Jacobi Equation in the Large.

In our differentiable manifold $X_n=\bigcup_\alpha U_\alpha(x_{(\alpha)})$, let us consider a 1-parameter family of hypersurfaces

$$(4.1)\qquad S(\xi_l)=\Sigma, \qquad\Bigg|\qquad S(\xi^l)=\Sigma,$$

where the functions $S$ ore of class $C^2$, while $\Sigma$ denotes the parameter of the family.

It will be supposed that (4.1) *covers the region* $X_n(\xi)=\bigcup_\alpha U_\alpha(x_{(\alpha)})$ *simply.*

The derivatives $\dfrac{\partial S}{\partial\xi^l}$ forms a field of covariant vectors; we associate with the latter a field of ***unit*** covariant (contravariant) vectors $\eta_l$ $(\eta^l)$ (in terms of our metric of the tangent spaces $T_n$ of $X_n(\xi)$) by putting

$$(4.2)\qquad \eta^l=\psi(\xi_l,\dot{\xi}_l,\cdots,\overset{(M)}{\xi}_l)\frac{\partial S}{\partial\xi_l}, \qquad\Bigg|\qquad \eta_l=\phi(\xi^l,\dot{\xi}^l,\cdots,\overset{(M)}{\xi}{}^l)\frac{\partial S}{\partial\xi^l},$$

where $\psi$ $(\phi)$ is defined by

$$(4.3)\qquad \psi=\left\{\mathscr{H}(\xi_l,\dot{\xi}_l,\cdots,\overset{(M)}{\xi}_l,\frac{\partial S}{\partial\xi_l})\right\}^{-1}. \qquad\Bigg|\qquad \phi=\left\{\mathscr{H}(\xi^l,\dot{\xi}^l,\cdots,\overset{(M)}{\xi}{}^l,\frac{\partial S}{\partial\xi^l})\right\}^{-1}.$$

The $\mathscr{H}$ $(\mathscr{L})$ being supposed positively homogeneous of the *first degree in* $\eta_l$ $(\eta^l)$ (cf. (2.16), (2.36)), we have

$$(4.4)\qquad \mathscr{L}(\xi_l,\ddot{\xi}_l,\cdots,\overset{(M)}{\xi}_l,\eta^l)=1. \qquad\Bigg|\qquad \mathscr{H}(\xi^l,\ddot{\xi}^l,\cdots,\overset{(M)}{\xi}{}^l,\eta_l)=1.$$

The contravariant (covariant) vector $\eta^l$ $(\eta_l)$ corresponding to $\eta_l$ $(\eta^l)$ is given by

$$(4.5)\qquad \begin{aligned}\eta_l&=g_{lj}(\xi_l,\ddot{\xi}_l,\cdots,\overset{(M)}{\xi}_l,\eta^l)\,\eta^j\\&=\eta_l(\xi_l,\ddot{\xi}_l,\cdots,\overset{(M)}{\xi}_l).\end{aligned} \qquad\Bigg|\qquad \begin{aligned}\eta^l&=g^{lj}(\xi^l\,\ddot{\xi}^l,\cdots,\overset{(M)}{\xi}{}^l,\eta_l)\,\eta_j\\&=\eta^l(\xi^l,\ddot{\xi}^l,\cdots,\overset{(M)}{\xi}{}^l).\end{aligned}$$

Since

(4.6) $$\mathcal{H}=\eta^l\,\dot{\xi}_l, \qquad\Big|\qquad \mathcal{L}=\eta_l\,\dot{\xi}^l,$$

we have

(4.7) $$\eta^l=\frac{\partial\mathcal{H}(\xi_l,\ddot{\xi}_l,\cdots,\overset{(M)}{\xi}_l,\eta^l)}{\partial\dot{\xi}_l}. \qquad\Big|\qquad \eta_l=\frac{\partial\mathcal{L}(\xi^l,\ddot{\xi}^l,\cdots,\overset{(M)}{\xi^l},\eta_l)}{\partial\dot{\xi}^l}.$$

As (4.2) tells us, the field

$$\eta^l \qquad\Big|\qquad \eta_l$$

*is transversal to the family of hypersurfaces* (4.1).

For an arbitrary vector

$$\eta_l \qquad\Big|\qquad \eta^l$$

tangent to the member of (4.1) satisfies the condition

(4.8) $$\frac{\partial S}{\partial\xi_l}\,\eta_l=0, \qquad\Big|\qquad \frac{\partial S}{\partial\xi^l}\,\eta^l=0,$$

so that

(4.9) $$\eta_l\,\eta^l=0$$

by (4.2). From (4.7) and (4.9), we obtain

(4.10) $$\frac{\partial\mathcal{H}(\xi_l,\ddot{\xi}_l,\cdots,\overset{(M)}{\xi}_l,\eta^l)}{\partial\dot{\xi}_l}\,\eta_l=0. \qquad\Big|\qquad \frac{\partial\mathcal{L}(\xi^l,\ddot{\xi}^l,\cdots,\overset{(M)}{\xi^l},\eta_l)}{\partial\dot{\xi}^l}\,\eta^l=0.$$

We can thus construct a congruence $K$ of II–geodesic curves transversal to the family (4.1) by solving the system of first order ordinary differential equations

(4.11) $$\frac{d\xi_l}{ds}=\eta_l\,(\xi_k,\dot{\xi}_k,\cdots,\overset{(M)}{\xi}_k) \qquad\Big|\qquad \frac{d\xi^l}{ds}=\eta^l\,(\xi^k,\dot{\xi}^k,\cdots\overset{(M)}{\xi^k})$$

in (4.1), where

(4.12) $$ds=\mathcal{H}(\xi_l,\ddot{\xi}_l,\cdots,\overset{(M)}{\xi}_l,d\xi_l), \qquad\Big|\qquad ds=\mathcal{L}(\xi^l,\ddot{\xi}^l,\cdots,\overset{(M)}{\xi^l},d\xi^l),$$

and we must always have

(4.13) $$\mathcal{L}\Big(\xi^l,\ddot{\xi}^l,\cdots,\overset{(M)}{\xi^l},\frac{d\xi^l}{ds}\Big)=\mathcal{L}(\xi^l,\ddot{\xi}^l,\cdots,\overset{(M)}{\xi^l},\eta^l)$$
$$=\mathcal{H}(\xi_l,\ddot{\xi}_l,\cdots,\overset{(M)}{\xi}_l,\eta^l)=\mathcal{H}\Big(\xi_l,\ddot{\xi}_l,\cdots,\overset{(M)}{\xi}_l,\frac{d\xi_l}{ds}\Big).$$

Let $\Gamma$ represent a II-*geodesic curve* of the congruence $K$ of transversals which joins a pair of points $P_0$ and $P$ lying on two hypersurfaces of the family (4.13), which correspond to the parameter values $\Sigma_0$ and $\Sigma$ respectively. Denote by

$$(4.14)\qquad C: \qquad \xi_l=\xi_l(s) \qquad\Big|\qquad \xi^l=\xi^l(s)$$

an arbitrary curve of class $C^1$ nowhere tangent to the family (4.1), which also joins the points $P_0$ and $P$. The curve $C$ will intersect all members of (4.1) at points corresponding to the values of $\Sigma$ lying between $\Sigma_0$ and $\Sigma$; thus $\Sigma$ can be regarded as a function of $s$ such that

$$(4.15)\qquad \frac{d\Sigma}{ds}=\frac{\partial S}{\partial \xi_l}\dot{\xi}_l, \qquad\Big|\qquad \frac{d\Sigma}{ds}=\frac{\partial S}{\partial \xi^l}\dot{\xi}^l,$$

where $\dot{\xi}_l$ $(\dot{\xi}^l)$ represents the components of the tangent vectors of $C$.

Substituting from (4.2), we see that (4.15) can be written in the form

$$(4.16)\qquad \frac{d\Sigma}{ds}=\psi^{-1}\eta^l\dot{\xi}_l. \qquad\Big|\qquad \frac{d\Sigma}{ds}=\phi^{-1}\eta_l\dot{\xi}^l.$$

But by virtue of (1.18), (4.4) and (4.13), we have

$$(4.17)\qquad \begin{aligned}\eta^l&=\mathcal{H}(\xi_l,\dot{\xi}_l,\cdots,\overset{(M)}{\xi}_l)\frac{\partial\mathcal{H}(\xi_l,\dot{\xi}_l,\cdots,\overset{(M)}{\xi}_l)}{\partial\dot{\xi}_l}\\ &=\mathcal{L}(\xi^l,\dot{\xi}^l,\cdots,\overset{(M)}{\xi}{}^l,\eta^l)\frac{\partial\mathcal{H}(\xi_l,\dot{\xi}_l,\cdots,\overset{(M)}{\xi}_l)}{\partial\dot{\xi}_l}\\ &=\frac{\partial\mathcal{H}(\xi_l,\dot{\xi}_l,\cdots,\overset{(M)}{\xi}_l)}{\partial\dot{\xi}_l},\end{aligned} \qquad\Big|\qquad \begin{aligned}\eta_l&=\mathcal{L}(\xi^l,\dot{\xi}^l,\cdots,\overset{(M)}{\xi}{}^l)\frac{\partial\mathcal{L}(\xi^l,\dot{\xi}^l,\cdots,\overset{(M)}{\xi}{}^l)}{\partial\dot{\xi}^l}\\ &=\mathcal{H}(\xi_l,\dot{\xi}_l,\cdots,\overset{(M)}{\xi}_l,\eta_l)\frac{\partial\mathcal{L}(\xi^l,\dot{\xi}^l,\cdots,\overset{(M)}{\xi}{}^l)}{\partial\dot{\xi}^l}\\ &=\frac{\partial\mathcal{L}(\xi^l,\dot{\xi}^l,\cdots,\overset{(M)}{\xi}{}^l)}{\partial\dot{\xi}^l},\end{aligned}$$

so that (14.16) becomes

$$(4.18)\qquad \frac{d\Sigma}{ds}=\psi^{-1}\frac{\partial\mathcal{H}(\xi_l,\dot{\xi}_l,\cdots,\overset{(M)}{\xi}_l)}{\partial\dot{\xi}_l}\dot{\xi}_l. \qquad\Big|\qquad \frac{d\Sigma}{ds}=\phi^{-1}\frac{\partial\mathcal{L}(\xi^l,\dot{\xi}^l,\cdots,\overset{(M)}{\xi}{}^l)}{\partial\dot{\xi}^l}\dot{\xi}^l.$$

We shall now assume that the family (4.1) is such that $\psi$ $(\phi)$ is constant over each hypersurface, i.e. that $\psi$ $(\phi)$ is a function of $\Sigma$ only. Then we can show that the family (4.1) can, without loss of generality, be represented in such a way that

$$(4.19)\qquad \psi(\Sigma)=1. \qquad\Big|\qquad \phi(\Sigma)=1.$$

However, it then follows from (4.8) that the function $S(\xi_l)$ $(S(\xi^l))$ must satisfy the following first degree partial differential equation:

$$(4.20)\qquad \mathcal{L}\left(\xi_l,\dot{\xi}_l,\cdots,\overset{(M)}{\xi}_l,\frac{\partial S}{\partial\xi_l}\right)=1. \qquad\Big|\qquad \mathcal{H}\left(\xi^l,\dot{\xi}^l,\cdots,\overset{(M)}{\xi}{}^l,\frac{\partial S}{\partial\xi^l}\right)=1.$$

Clearly, this is the *Hamilton–Jacobii equation.*

The (4.2) becomes

(4.21) $$\eta^l=\frac{\partial S}{\partial \xi_l}. \qquad\Bigg|\qquad \eta_l=\frac{\partial S}{\partial \xi^l}.$$

(3.17) and (4.21) give

(4.22) $$\mathscr{H}=\eta^l\,\dot{\xi}_l=\frac{\partial S}{\partial \xi_l}\,\dot{\xi}_l=\frac{dS}{ds} \qquad\Bigg|\qquad \mathscr{L}=\eta_l\,\dot{\xi}^l=\frac{\partial S}{\partial \xi^l}\,\dot{\xi}^l=\frac{dS}{ds}$$

i. e.

(4.23) $$S=\int_{s_0}^{s}\mathscr{H}ds=\Sigma-\Sigma_0. \qquad\Bigg|\qquad S=\int_{s_0}^{s}\mathscr{L}ds=\Sigma-\Sigma_0.$$

(3.19)=(4.17) and (4.21) give

(4.24) $$\eta^l=\frac{\partial \mathscr{H}}{\partial \dot{\xi}_l}=\frac{\partial S}{\partial \xi_l}, \qquad\Bigg|\qquad \eta_l=\frac{\partial \mathscr{L}}{\partial \dot{\xi}^l}=\frac{\partial S}{\partial \xi^l},$$

from which (4.22) results by multiplication by $\dot{\xi}_l$ ($\dot{\xi}^l$).

(3.19)=(4.17) and (3.20) give

(4.25) $$\dot{\eta}^l=\frac{d}{ds}\frac{\partial \mathscr{H}}{\partial \dot{\xi}^l}=\frac{\partial \mathscr{H}}{\partial \xi_l}. \qquad\Bigg|\qquad \dot{\eta}_l=\frac{d}{ds}\frac{\partial \mathscr{L}}{\partial \dot{\xi}^l}=\frac{\partial \mathscr{L}}{\partial \xi^l}.$$

(4.25) give

(4.26) $$\frac{d}{ds}\frac{\partial \mathscr{H}}{\partial \dot{\xi}_l}-\frac{\partial \mathscr{H}}{\partial \xi_l}=0, \qquad\Bigg|\qquad \frac{d}{ds}\frac{\partial \mathscr{L}}{\partial \dot{\xi}^l}-\frac{\partial \mathscr{L}}{\partial \xi^l}=0,$$

which is nothing other than the *Euler–Lagrange equations* (3.18).

From (4.22), we obtain

(4.27) $$\begin{aligned}\dot{\mathscr{H}}&=\eta^l\,\ddot{\xi}^l+\dot{\eta}^l\,\dot{\xi}_l\\&=\frac{\partial \mathscr{H}}{\partial \dot{\xi}_l}\,\ddot{\xi}_l+\frac{\partial \mathscr{H}}{\partial \xi_l}\,\dot{\xi}_l\end{aligned} \qquad\Bigg|\qquad \begin{aligned}\dot{\mathscr{L}}&=\eta_l\,\ddot{\xi}^l+\dot{\eta}_l\,\dot{\xi}^l\\&=\frac{\partial \mathscr{L}}{\partial \dot{\xi}^l}\,\ddot{\xi}^l+\frac{\partial \mathscr{L}^l}{\partial \xi^l}\,\dot{\xi}^l\end{aligned}$$

by (3.10) and (3.19).

As (3.1), (3.7), (4.22) and (4,23) show, for

$\xi_l$ (standing in place of $x_\lambda$), | $\xi^l$ (standing in place of $x^\lambda$),

we have

$\mathcal{H}ds=\omega=dS=d\xi=cds,\ \mathcal{H}=c.$ | $\mathcal{L}ds=\omega=dS=d\xi=cds,\ \mathcal{L}=c.$

The case $M=1$ of § 4 is particularly note-worthy.

## § 5. The Relation between the Hamilton's Formalism of § 4 and the Present Author's Non-Connection Methods for Some Connection Geometries in the Large.

In T. Takasu, [5], I established *non-connection-methods* for linear connections in the large bringing respective geometries to the "Erlanger Programm", the transformation group parameters being adequate functions of the (local) coordinates and in T. Takasu, [1] I extended them further *doubly* to the case, where the transformation group parameters are adequate functions of the (local) coordinates $(x)$ as well as of $(\dot{x}, \ddot{x}, \cdots, \overset{(M)}{x})$, ($\dot{x}=dx/d\tau$, etc.; $\tau$=curve parameter). In this § 5, I will recapitulate the case of $M=1$ of I. (Doubly) extended affine geometry, II. (Doubly) extended Euclidean geometry, III. Other 20 (doubly) extended geometries indicated on p. 31 of T. Takasu, [8], IV. Geometry of Finsler–Craig–Synge–Kawaguchi spaces, all by Hamilton's Canonical Formalism, since I am now in the situation to emphasize that the *global Hamilton's canonical equations* (2.22) (which become to the) *global Lagrangian equations* (2.34) by (2.31) and (2.33), *should be utilized in the case* $M=0$, *since in this case* (1.11) *does not serve as Hamiltonian* (*Lagrangian*) *function.* [7]

### I. (Doubly) Extended Affine Geometry based on Hamilton's Canonical Formalism.

**I. 1. A New Method of Treatment of II-Geodesic Curves by Means of Hamilton's Canonical Formalism.** Consider

(I. 1) $\omega \overset{\text{def}}{=} \omega_\mu (x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})\, dx^\mu,$ | $\omega \overset{\text{def}}{=} \omega^\mu (x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})\, dx_\mu,$

$$(\lambda, \mu, \cdots=1, 2, \cdots, n),$$

which is *global* in the differentiable manifold $X_n=\bigcup_\alpha U_\alpha$ of class $C^v$,

$v$=positive integer, | $v=\infty$, | $v=\omega$,

where open subset $U_\alpha$ is the domain of the local coordinates $x_{(\alpha)}$, since (I. 1) is written in an invaricant form.

---

7) Cf. (1.11).

Let $x^\lambda = x^\lambda(\tau)$ be a parametrized curve, where $\tau$ is an arbitrary parameter (e. g. the canonical parameter (T. Takasu, [7], Art. 12; [8], Art. 14)). Set

(I. 2) $$d\xi \overset{\text{def}}{=} \omega_\mu(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})\,\dot{x}^\mu\, d\tau, \qquad d\xi \overset{\text{def}}{=} \omega^\mu(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})\,\dot{x}_\mu\, d\tau,$$

(I. 3) $$\mathfrak{L} \overset{\text{def}}{=} \omega_\mu(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda})\,\dot{x}^\mu = y_\mu \dot{x}^\mu, \qquad \mathfrak{H} \overset{\text{def}}{=} \omega^\mu(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda})\,\dot{x}_\mu = y^\mu \dot{x}_\mu,$$

where

(I. 4) $$\omega_\mu \overset{\text{def}}{=} y_\mu. \qquad \omega^\mu \overset{\text{def}}{=} y^\mu.$$

Then the Euler–Lagrangian equation for the extremal problem

(I. 5) $$\delta\int_{\tau_0}^{\tau_1} \mathfrak{L}\, d\tau = 0 \qquad \delta\int_{\tau_0}^{\tau_1} \mathfrak{H}\, d\tau = 0$$

becomes

(I. 6) $$\frac{\partial\mathfrak{L}}{\partial x^\mu} - \frac{d}{d\tau}\left[\frac{\partial\mathfrak{L}}{\partial\dot{x}^\mu} - \frac{d}{d\tau}\frac{\partial\mathfrak{L}}{\partial\ddot{x}^\mu} + \cdots + (-1)^{M-1}\frac{d^{M-1}}{d\tau^{M-1}}\frac{\partial\mathfrak{L}}{\partial\overset{(M)}{x^\mu}}\right] = 0. \qquad \frac{\partial\mathfrak{H}}{\partial x_\mu} - \frac{d}{d\tau}\left[\frac{\partial\mathfrak{H}}{\partial\dot{x}_\mu} - \frac{d}{d\tau}\frac{\partial\mathfrak{H}}{\partial\ddot{x}_\mu} + \cdots + (-1)^{M-1}\frac{d^{M-1}}{d\tau^{M-1}}\frac{\partial\mathfrak{H}}{\partial\overset{(M)}{x_\mu}}\right] = 0.$$

Define $y_\mu$ ($y^\mu$) by

(I. 7) $$y_\mu \overset{\text{def}}{=} \frac{\partial\mathfrak{L}}{\partial\dot{x}^\mu} - \frac{d}{d\tau}\frac{\partial\mathfrak{L}}{\partial\ddot{x}^\mu} + \cdots + (-1)^{M-1}\frac{d^{M-1}}{d\tau^{M-1}}\frac{\partial\mathfrak{L}}{\partial\overset{(M)}{x^\mu}} \qquad y^\mu \overset{\text{def}}{=} \frac{\partial\mathfrak{H}}{\partial\dot{x}_\mu} - \frac{d}{d\tau}\frac{\partial\mathfrak{H}}{\partial\ddot{x}_\mu} + \cdots + (-1)^{M-1}\frac{d^{M-1}}{d\tau^{M-1}}\frac{\partial\mathfrak{H}}{\partial\overset{(M)}{x_\mu}}$$

for (I. 3) anew, then (I· 3) and (I. 6) give

(I. 8) $$\frac{\partial\mathfrak{L}}{\partial x^\mu} = \dot{y}_\mu, \quad \frac{\partial\mathfrak{L}}{\partial y_\mu} = \dot{x}^\mu \qquad \frac{\partial\mathfrak{H}}{\partial x_\mu} = \dot{y}^\mu, \quad \frac{\partial\mathfrak{H}}{\partial y^\mu} = \dot{x}_\mu$$

*forming the canonical equations* (2.34) *of Lagrangian types* as in the case of (2.22). When the points $\tau_0, \tau_1$ in (2.19) and the curve $C$ passing through $\tau_0, \tau_1$ belong to one and the same domain $U_\alpha$ of local coordinates, *the* (I. 8) *are canonical equations of the local geodesic curve C, and otherwise the curve C is a* II*–geodesic* (the geodesic incl.) *curve corresponding to*

$$\omega_\mu(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x^\lambda}), \qquad \omega^\mu(x_\lambda, \dot{x}_\lambda, \cdots, \overset{(M)}{x_\lambda}),$$

*which is* (*global and*) *an extremal of* (I. 5).

**Theorem.** *For that the extremals* $\delta\mathfrak{H}(x^\lambda, \cdots, y_\lambda) = 0$ *and* $\delta\mathfrak{L}(x^\lambda, \dot{x}^\lambda, \cdots) = 0$ *coincide, it is necessary and sufficient that*

(I. 9) $$\delta(y_\mu \dot{x}^\mu) = 0.$$

**Proof.** From (2.14), we obtain

$$\delta\mathfrak{L}=\delta(y_\mu \dot{x}^\mu)=\delta y_\mu \dot{x}^\mu+y_\mu \delta\dot{x}^\mu=\dot{y}_\mu \delta\tau \dot{x}^\mu+y_\mu \delta\dot{x}^\mu=\dot{y}_\mu \delta x^\mu+y_\mu \delta\dot{x}^\mu,$$

$$\delta\mathfrak{L}(x^\mu, \dot{x}^\mu, \cdots)=\delta(y_\mu \dot{x}^\mu)-\delta\mathfrak{H}(x^\mu, y_\mu), \tag{I. 10}$$

where

$$\delta\mathfrak{H}=\delta y_\mu \dot{x}^\mu-\delta x^\mu \dot{y}_\mu. \tag{I. 11}$$

The (I. 10) gives our Theorem.

**Cor.** $\mathfrak{H}=$const. *along the extremals* $\delta\mathfrak{H}=0$.

**Proof.** From (I. 10), it follows that

$$\frac{d\mathfrak{L}}{d\tau}=\frac{d(y_\mu \dot{x}^\mu)}{d\tau}, \qquad \frac{d\mathfrak{H}(x^\mu, y_\mu)}{d\tau}=0.$$

From (3.10), (3.14), we obtain

$$d\xi^l=\omega^l=y^l_\mu \dot{x}^\mu d\tau=y^l_\mu dx^\mu=\mathfrak{L}^l d\tau=c^l \mathfrak{L} d\tau, \tag{I. 12}$$

so that

$$\begin{aligned}\xi^l&=\int y^l_\mu dx^\mu=y^l_\mu x^\mu-\int x^\mu dy^l_\mu=y^l_\mu x^\mu-\int dy^l_\mu \int dx^\mu\\ &=y^l_\mu dx^\mu-\int\int(dy^l_\mu dx^\mu),\end{aligned} \tag{I. 13}$$

the condition for that the repeated integral may be converted into the double integral, i. e. that the integrand is continuous, being evidently satisfied. Now

$$\frac{d^2\xi^l}{d\tau^2}=\frac{dy^l_\mu}{d\tau}\frac{dx^\mu}{d\tau}+y^l_\mu\frac{d^2x^\mu}{d\tau^2}. \tag{I. 14}$$

Since both terms on the right-hand side are written in invariant forms, if we take a transformation $\bar{x}^\mu=\bar{x}^\mu(x^\lambda)$ such that $d^2\bar{x}^\mu/d\tau^2=0$, from (I. 14), we must have

$$dy^l_\mu d\bar{x}^\mu=0, \tag{I. 15}$$

in which case (I. 13) becomes of the form

$$\xi^l=y^l_\mu \bar{x}^\mu+y^l_o, \qquad (y^l_o=\text{const.}).$$

Writing $h, \bar{\xi}, \xi$ and $a$ for $\mu, \xi, \bar{x}$ and $y$ respectively, we obtain the formulas of *(doubly) extended affine transformation* of the present author (T. Takasu, [6], (2.6), p. 872; [5], (3.2), p. 63):

$$\bar{\xi}^l=a^l_h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\xi^h+a^l_o, \quad (|a^l_h|\neq 0) \tag{I. 16}$$

accompanied by

$$d\bar{\xi}^l=a^l_h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})d\xi^h, \tag{I. 17}$$

$$da^l_h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})d\xi^h=0, \qquad (\text{cf. (I. 15)}), \tag{I. 18}$$

*along the* II-*geodesic line-elements.*

From (I. 16) and (I. 17), we obtain *the necessary condition*

(I. 19) $$da^l_h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\, \xi^h = 0$$

*for the* II-*geodesic line-elements.*

The $\bar{\xi}^l$ and the $\xi^l$ will be called the II-*geodesic parallel coordinates.*

Setting

(I. 20) $$ds \overset{\text{def}}{=} d\xi = \mathfrak{L}d\tau, \qquad (\text{cf. (I. 12)})$$

from (I. 12), we obtain

(I. 21) $$\xi^l = c^l(s - s_o), \qquad d\xi^l = \mathfrak{L}^l\, d\tau = c^l ds.$$

Since $d\bar{\xi}^l = \bar{c}^l ds$, $d\xi^l = c^l ds$, from (I. 17), we see that $c^l$ undergo the transformation

(I. 22) $$\bar{c}^l = d^l_h(\xi, \dot{\xi}, \cdots, \overset{(M)}{\xi})\, c^h,$$

where $\bar{c}^l$ are constant on summation with respect to $h$.

The (I. 21) tells us that the II-*geodesic curves behave as for meet and join like straight lines.* The $s$ may be called the *affine length.*

**N. B.** *The* $y^l_\mu$ in (I. 12) and the $p^l_\mu(x)$ in (1.14) *undergo the extended affine transformation* $a^l_h$. Thus there exists $a^l_k(x, \dot{x})$ such that

$$\omega^l_\mu(x, \dot{x})\, a^h_l(x, \dot{x}) = p^h_\mu(x), \qquad (M=1, \text{e. g.})$$

since multiplication factor $\omega^\mu_k(x, \dot{x})$ gives

$$a^h_k(x, \dot{x}) = \omega^\mu_k(x. \dot{x})\, p^h_\mu(x).$$

**I. 2. (Doubly) Extended Affine Geometry.** That the *totality of the* (*doubly*) *extended affine transformations* (I. 16) *forms a group* may be shown by utilizing (I. 19) quite as in p. 64 of T. Takasu, [5]. This group will be called the (*doubly*) *extended affine group* and the geometry under it the (*doubly*) *extended affine geometry.*

From § 3 ((3.1)–(3.15)), we see that *the non-connection method for the* (*doubly*) *extended affine geometry is thus deduced from the Hamilton's canonical formalism.*

## II. (Doubly) Extended Euclidean Geometry by the Hamilton's Canonical Formalism.

**II. 1. (Doubly) Extended Euclidean Geometry based on Canonical Equations of the Hamiltonian Types of II-Geodesic Curves.** When the fundamental quadratic differential form of the (doubly) extended Euclidean geometry is

$$\text{(II. 1)} \qquad ds^2 = g_{\mu\nu}(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^{(\lambda)})\, dx^\mu\, dx^\nu, \qquad (\,|g_{\mu\nu}| \neq 0),$$

it is always expressible in the form

$$\text{(II. 2)} \qquad dS^2 = \omega^l\, \omega^l, \qquad (\omega^l = \omega^l_\mu(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda)\, dx^\mu),$$

*but for undergoing (doubly) extended orthogonal transformations*

$$\text{(II. 3)} \qquad a^l_h(x^\lambda, \dot{x}^\lambda, \cdots, \overset{(M)}{x}{}^\lambda), \qquad (a^l_h\, a^l_k = \delta_{hk},\ a^h_l\, a^k_l = \delta^{hk}).$$

If we adopt (II. 2) for (II. 1), the results of I hold still and we have

$$\text{(II. 4)} \qquad \omega^2 = \omega^l\, \omega^l = d\xi^l\, d\xi^l = ds^2 = (c^l\, c^l)\, \omega^2, \qquad (\omega^l = c^l\, \omega),$$

so that

$$\text{(II. 5)} \qquad c^l\, c^l = 1.$$

The *(doubly) extended affine gronp* becomes in this case to the *(doubly) extended Euclidean group* and the *(doubly) extended affine geometry* to the *(doubly) extended Euclidean geometry* (T. Takasu, [6] ).

The (II. 2) shows further that

$$\text{(II. 6)} \qquad g_{\mu\nu} = \omega^l_\mu\, \omega^1_\nu.$$

In this way, *the Hamilton's canonical formalism leads to the (doubly) extended Euclidean geometry.*

## III. Other (Doubly) Extended Geometries by Hamilton's Canonical Formalism.

All other (doubly) extended geometries corresponding to the branches enlisted on p. 31 of T. Takasu, [8] may be treated similarly (mutatis mutandis) directly or indirectly by means of Hamilton's canonical formalism. (The detail is under preparation.)

## IV. Geometry of Finsler-Craig-Synge-Kawaguchi Spaces treated by Hamilton's Canonical Formalism.

These spaces are based on a certain integral

$$\text{(IV. 1)} \qquad s = \int_{\tau_0}^{\tau_1} F(x, \dot{x}, \cdots, \overset{(M)}{x})\, d\tau, \qquad (\dot{x} = dx \,/\, d\tau, \text{etc.})$$

satisfying the so-called Zermero's conditions for parameter-invariancy (cf. M. Kawaguchi, [1], p. 724, $*g_{ij}$):

$$\text{(IV. 2)} \quad g_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x}) = MF^2\, F_{(\mu)\nu}\, F_{(\mu)\nu} + \overset{M}{\mathfrak{E}}_\mu\, \overset{M}{\mathfrak{E}}_\nu + *\overset{1}{\mathfrak{E}}_\mu *\, \overset{1}{\mathfrak{E}}_\nu, \ (F = F(x, \dot{x}, \cdots, \overset{(M)}{x})).$$

The $ds^2$ is always expressible in the form

(IV. 3) $$ds^2=\omega^l\,\omega^l, \qquad (\omega^l=\omega^l_\mu\,(x,\dot{x},\cdots,\overset{(M)}{x})\,dx^\mu)$$

but for undergoing (doubly) extended orthogonal transformations, and thus our theory applies to the case of Finsler–Craig–Synge–Kawaguchi spaces.

## § 6. The Groups of Extended Hamilton's Canonical Transformations. [8)]

Consider the transformation

(6.1) $$\bar{q}^l=\bar{q}^l\,(q^j,p_j), \quad \bar{p}_l=\bar{p}_l\,(q^j,p_j), \qquad \Big| \qquad \bar{q}_l=\bar{q}_l\,(q_j,p^j), \quad \bar{p}^l=\bar{p}^l\,(q_j,p^j),$$

where

$q^j$ and $\bar{q}^j$ | $q_j$ and $\bar{q}_j$

are the local coordinates, ($x^\lambda\,(x_\lambda)$ in (1.0) incl.) of two points on the extremal $\Gamma$ in $X_n$ corresponding to the

Hamiltonian | Lagrangian

(6.2) $$H(q^l,\dot{q}^l,\cdots,\overset{(M)}{q^l},p_l)=L\,(q^l,\ddot{q}^l,\cdots,\overset{(M)}{q^l},\dot{q}^l) \qquad \Big| \qquad L\,(q_l,\dot{q}_l,\cdots,\overset{(M)}{q_l},p^l)=H\,(q_l,\ddot{q}_l,\cdots,\overset{(M)}{q_l},\dot{q}_l)$$

and $p_j\,(p^j)$ are defined by (6.12) for (6.2) as in the case of $\eta^l\,(\eta_l)$ in (3.19).

Set

(6.3) $$\Psi\overset{\text{def}}{=}_\Gamma\int_{\tau_0}^{\tau} L\,(q^l,\ddot{q}^l,\cdots,\overset{(M)}{q^l},\dot{q}^l)\,d\tau, \qquad \Big| \qquad \Phi\overset{\text{def}}{=}_\Gamma\int_{\tau_0}^{\tau} H\,(q_l,\ddot{q}_l,\cdots,\overset{(M)}{q_l},\dot{q}_l)\,d\tau,$$

so that

(6.4) $$d\Psi\,(q^l,p_l)=\bar{p}_l\,d\bar{q}^l-p_l\,dq^l. \qquad \Big| \qquad d\Phi\,(q_l,p^l)=\bar{p}^l\,d\bar{q}_l-p^l\,dq_l.$$

**Definition.** *The transformations* (6.1) *are said to be canonical,* when the functions

$\bar{q}^l,\bar{p}_l$ | $\bar{q}_l,\bar{p}^l$

are of class $C^2$ and when there exists a function

$\Psi\,(q^l,p_l)$, | $\Phi\,(q_l,p^l)$,

whose exact differential is given by (6.4).

The function

8) This § is different from pp. 82–99 of H. Rund, [1], in that $M\geqq 1$ and in the duality.

| | |
|---|---|
| $\Psi(q^l, p_l)$ | $\Phi(q_l, p^l)$ |

will be called the *generator* or *generating funciton* of the transformation (6.1).

If we replace $p_l\,(p^l)$ in (6.5) by means of (6.1) this would give

| | | |
|---|---|---|
| (6.7) | $\Psi(q^l, \bar{q}^l)$. | $\Phi(q_l, \bar{q}_l)$. |

[We shall show soon that (6.1) posses an inverse, so that (6.5) and (6.7) are equivalent.]

Associated with the canonical transformation (6.1), we have, quite as in the case of (3.26),

*the Hamilton's canonical equations*

| | | |
|---|---|---|
| (6.8) | $\dfrac{dp_l}{d\tau}=-\dfrac{\partial H(q^j, p_j)}{\partial q^l}$, $\dfrac{dq^l}{d\tau}=\dfrac{\partial H(q^j, p_j)}{\partial p_l}$, | $\dfrac{dp^l}{d\tau}=-\dfrac{\partial L(q_j, p^j)}{\partial q_l}$, $\dfrac{dq_l}{d\tau}=\dfrac{\partial L(q_j, p^j)}{\partial p^l}$, |

and quite as in the case of (3.20)

*the Lagrange's canonical equations*

| | | |
|---|---|---|
| (6.9) | $\dfrac{dp_l}{d\tau}=\dfrac{\partial L(q^j, p_j)}{\partial q^l}$, $\dfrac{dq^l}{d\tau}=\dfrac{\partial L(q^j, p_j)}{6p_l}$. | $\dfrac{dp^l}{d\tau}=\dfrac{\partial H(q_j, p^j)}{\partial q_l}$, $\dfrac{dq_l}{d\tau}=\dfrac{\partial H(q_j, p^j)}{\partial p^l}$. |

The *Euler–Lagrange equations* for the extremal problem

| | | |
|---|---|---|
| (6.10) | $\delta\int H d\tau=0$ | $\delta\int L d\tau=0$ |

become

| | | |
|---|---|---|
| (6.11) | $\dfrac{\partial H}{\partial q_l}-\dfrac{d}{d\tau}\Big(\dfrac{\partial H}{\partial \dot{q}_l}-\dfrac{d}{d\tau}\dfrac{\partial H}{\partial \ddot{q}_l}+\cdots+(-1)^{M-1}\dfrac{d^{M-1}}{d\tau^{M-1}}\dfrac{\partial H}{\partial \overset{(M)}{q^l}}\Big)=0.$ | $\dfrac{\partial L}{\partial q^l}-\dfrac{d}{d\tau}\Big(\dfrac{\partial L}{\partial \dot{q}^l}-\dfrac{d}{d\tau}\dfrac{\partial L}{\partial \ddot{q}^l}+\cdots+(-)^{M-1}\dfrac{d^{M-1}}{d\tau^{M-1}}\dfrac{\partial L}{\partial \overset{(M)}{q^l}}\Big)=0.$ |

Set

| | | |
|---|---|---|
| (6.12) | $p^l\overset{\text{def}}{=}\dfrac{\partial H}{\partial \dot{q}_l}-\dfrac{d}{d\tau}\dfrac{\partial H}{\partial \ddot{q}_l}+\cdots+(-)^{M-1}\dfrac{d^{M-1}}{d\tau^{M-1}}\dfrac{\partial H}{\partial \overset{(M)}{q_l}}$ | $p_l\overset{\text{edf}}{=}\dfrac{\partial L}{\partial \dot{q}^l}-\dfrac{d}{d\tau}\dfrac{\partial L}{\partial \ddot{q}^l}+\cdots+(-1)^{M-1}\dfrac{d^{M-1}}{d\tau^{M-1}}\dfrac{\partial L}{\partial \overset{(M)}{q^l}}$ |

for

(6.13) $H \overset{\text{def}}{=} p^l \dot{q}_l$ | $L \overset{\text{def}}{=} p_l \dot{q}^l$

anew, then (6.12) and (6.13) give

(6.14) $\dot{p}^l = \frac{\partial H}{\partial q_l}, \quad \dot{q}_l = \frac{\partial H}{\partial p^l}$ | $\dot{p}_l = \frac{\partial L}{\partial q^l}, \quad \dot{q}^l = \frac{\partial L}{\partial p_l}$

forming *canonical equations of Lagrange's type* (6.9) and we have

$$\delta H = \delta p^l \dot{q}_l + p^l \delta \dot{q}_l = \dot{p}^l \delta q_l + p^l \delta \dot{q}_l \qquad \Bigg| \qquad \delta L = \delta p_l \dot{q}^l + p_l \delta \dot{q}^l = \dot{p}_l \delta q^l + p_l \delta \dot{q}^l$$

i. e.

(6.15) $\delta H = \delta (p^l \dot{q}_l) + (\dot{p}^l \delta q_l - \dot{q}_l \delta p^l)$ | $\delta L = \delta (p_l \dot{q}^l) + (\dot{p}_l \delta q^l - \dot{q}^l \delta p_l)$

or

(6.16) $\delta H + \delta L = \delta (p^l \dot{q}_l),$ | $\delta L + \delta H = \delta (p_l \dot{q}^l),$

where

(6.17) $L \overset{\text{def}}{=} \dot{q}_l \delta p^l - \dot{p}^l \delta q^l,$ | $H \overset{\text{def}}{=} \dot{q}^l \delta p_l - \dot{p}_l \delta q^l,$

whence follows the *canonical eqnations of Hamilton's type* (6.8):

(6.18) $\frac{\partial L}{\partial q_l} = -\dot{p}^l, \frac{\partial L}{\partial p^l} = \dot{q}_l.$ | $\frac{\partial H}{\partial q^l} = -\dot{p}_l, \frac{\partial H}{\partial p_l} = \dot{q}^l.$

**Theorem 1⁰.** *In order that the transformation* (6.1) *be canonical, it is necessary and sufficient that the Lagrange braket relations*

(6.19) $[q^j, q^h] = 0, \quad [q^j, p_h] = \delta^j_h, \quad [p_j, p_h] = 0$ | $[q_j, q_h] = 0, \quad [q_j, p^h] = \delta^h_j, \quad [p^j, p^h] = 0$

*be satisfied identically.*

**Proof.** *Necessity.* From (6.1) and (6.5), we have

(6.20) $d\bar{q}^l = \frac{\partial \bar{q}^l}{\partial q^j} \partial q^j + \frac{\partial \bar{q}^l}{\partial p_j} dp_j,$ | $d\bar{q}_l = \frac{\partial \bar{q}_l}{\partial q_j} dq_j + \frac{\partial \bar{q}_l}{\partial p^j} dp^j,$

(62.1) $d\bar{p}_i = \frac{\partial \bar{p}_i}{\partial q^j} dq^j + \frac{\partial \bar{p}_i}{\partial p_j} dp_j$ | $d\bar{p}^i = \frac{\partial \bar{p}^i}{\partial q_j} \partial q_j + \frac{\partial \bar{p}^j}{\partial p^j} dp^j,$

(6.22) $\partial \Psi = \frac{\partial \Psi}{\partial q^j} dq^j + \frac{\partial \Psi}{\partial p_j} dp_j.$ | $d\Phi = \frac{\partial \Phi}{\partial q_j} dq_j + \frac{\partial \Phi}{\partial p^j} dp^j$

When these relations are satisfied in the condition (6.4), we obtain

$$\left(\bar{p}_i\frac{\partial \bar{q}^i}{\partial q^j}-p_j\right)dq^j+\bar{p}_i\frac{\partial \bar{p}_j}{\partial p_j}dp_j=\frac{\partial \Psi}{\partial q^j}dq^j+\frac{\partial \Psi}{\partial p_j}dp_j, \qquad \left(\bar{p}^i\frac{\partial \bar{q}_i}{\partial q^j}-p^j\right)dq_j+\bar{p}^i\frac{d\bar{p}^i}{\partial p^j}dp^j=\frac{\partial \Phi}{\partial q_j}dq_j+\frac{\partial \Phi}{\partial p^j}dp^j,$$

and thus, since the $q^j, p_j\,(q_j, p^j)$ are regarded as independent variables, we deduce the relations

$$(6.23)\quad \bar{p}_i\frac{\partial \bar{q}^i}{\partial q^h}-p_h=\frac{\partial \Psi}{\partial q^h},\quad \bar{p}_j\frac{\partial \bar{q}^j}{\partial p_h}=\frac{\partial \Psi}{\partial p_h}. \qquad \bar{p}^i\frac{\partial \bar{q}_i}{\partial q_h}-p^h=\frac{\partial \Phi}{\partial q_h},\quad \bar{p}^j\frac{\partial \bar{q}_j}{\partial p^h}=\frac{\partial \Phi}{\partial p^h}.$$

In order to find the integrability-conditions of (6.23) and (6.24), we differentiate (6.23) with respect to $q^j\,(q_j)$ and $p_h\,(p^h)$, after which (6.24) is differentiated with respect to $q^j\,(q_j)$ and $p_j\,(p^j)$. We thus find

$$(6.25)\quad \frac{\partial \bar{q}^i}{\partial q^h}\frac{\partial \bar{p}_i}{\partial q^j}+\bar{p}_i\frac{\partial^2 q^i}{\partial q^j\partial q^h}=\frac{\partial^2 \Psi}{\partial q^h\,\partial q^j}, \qquad \frac{\partial \bar{q}_i}{\partial q_h}\frac{\partial \bar{p}^i}{\partial q_j}+\bar{p}^i\frac{\partial^2 q_i}{\partial q_j\partial q_h}=\frac{\partial^2 \Phi}{\partial q_h\partial q_j},$$

$$(6.26)\quad \frac{\partial \bar{q}^i}{\partial q^j}\frac{\partial p_i}{\partial p_h}+\bar{p}_i\frac{\partial \bar{q}^i}{\partial p_h\partial q^j}-\delta^h_j=\frac{\partial^2 \Psi}{\partial q^j\,\partial p_h}, \qquad \frac{\partial \bar{q}_i}{\partial q_j}\frac{\partial \bar{p}^i}{\partial p^h}+\bar{p}^i\frac{\partial^2 q_i}{\partial p^h\partial q_j}-\delta^j_h=\frac{\partial^2 \Phi}{\partial q_j\partial p^h},$$

$$(6.27)\quad \frac{\partial \bar{q}^i}{\partial p_h}\frac{\partial \bar{p}_i}{\partial q^j}+\bar{p}_i\frac{\partial^2 \bar{q}^i}{\partial q^j\partial p_h}=\frac{\partial^2 \Psi}{\partial q^j\partial p_h}, \qquad \frac{\partial \bar{q}_i}{\partial p^h}\frac{\partial \bar{p}^i}{\partial q_j}+\bar{p}^i\frac{\partial^2 \bar{q}_i}{\partial q_j\partial p^h}=\frac{\partial^2 \Phi}{\partial q_j\partial p^h},$$

$$(6.28)\quad \frac{\partial \bar{q}^i}{\partial p_h}\frac{\partial \bar{p}_i}{\partial p_j}+\bar{p}_i\frac{\partial^2 \bar{q}^i}{\partial p_j\partial p_h}=\frac{\partial^2 \Psi}{\partial p_j\partial p_h}. \qquad \frac{\partial \bar{q}_i}{\partial p^h}\frac{\partial \bar{p}^i}{\partial p^j}+\bar{p}^i\frac{\partial^2 \bar{q}_i}{\partial p^j\partial p^h}=\frac{\partial^2 \Phi}{\partial p^j\partial p^h}.$$

On recalling the definition of the Lagrange brackets we see that (6.25) requires that

$$(6.29)\quad [q^h, p_j]\overset{\text{def}}{=}\frac{\partial \bar{q}^i}{\partial q^h}\frac{\partial \bar{p}_i}{\partial q^j}-\frac{\partial \bar{q}^i}{\partial q^j}\frac{\partial \bar{p}_i}{\partial q^h}=0, \qquad [q_h, p^j]\overset{\text{def}}{=}\frac{\partial \bar{q}_i}{\partial q_h}\frac{\partial \bar{p}^i}{\partial q_j}-\frac{\partial \bar{q}_i}{\partial q_j}\frac{\partial \bar{p}^i}{\partial q_h}=0,$$

while (6.26) and (6.27) taken together imply that

$$(6.30)\quad [q^j, p_h]\overset{\text{def}}{=}\frac{\partial \bar{q}^i}{\partial q^j}\frac{\partial \bar{p}_i}{\partial p_h}-\frac{\partial \bar{q}^i}{\partial p_h}\frac{\partial \bar{p}_i}{\partial q^j}=\delta^h_j \qquad [q_j, p^h]\overset{\text{def}}{=}\frac{\partial \bar{q}_i}{\partial q_j}\frac{\partial \bar{p}^i}{\partial p^h}-\frac{\partial \bar{q}_i}{\partial p^h}\frac{\partial \bar{p}^i}{\partial q_j}=\delta^j_h,$$

and that (6.28) can hold only when

$$(6.31)\quad [p_h, p_j]\overset{\text{def}}{=}\frac{\partial \bar{q}^i}{\partial p_h}\frac{\partial \bar{p}_i}{\partial p_j}-\frac{\partial \bar{q}^i}{\partial p_j}\frac{\partial \bar{p}_i}{\partial p_h}=0. \qquad [p^h, p^j]\overset{\text{def}}{=}\frac{\partial \bar{q}_i}{\partial p^h}\frac{\partial \bar{p}^i}{\partial p^j}-\frac{\partial \bar{q}_i}{\partial p^j}\frac{\partial \bar{p}^i}{\partial p^h}=0.$$

Thus we see that the conditions (6.19) are necessary.

*Sufficiency.* We note that (6.29) can be written as

$$0=\frac{\partial}{\partial q^h}\left(\bar{p}_i\frac{\partial\bar{q}^i}{\partial q^j}\right)-\bar{p}_i\frac{\partial^2\bar{q}^i}{\partial q^h\partial q^j}-\frac{\partial}{\partial q^j}\left(\bar{p}_i\frac{\partial\bar{q}^i}{\partial q^h}\right)+\bar{p}_i\frac{\partial^2\bar{q}^i}{\partial q^j\partial q^h}=\frac{\partial}{\partial q^h}\left(\bar{p}_i\frac{\partial\bar{q}^i}{\partial q^j}\right)-\frac{\partial}{\partial q^j}\left(\bar{p}_i\frac{\partial\bar{q}^i}{\partial q^h}\right),$$

$$0=\frac{\partial}{\partial q_h}\left(\bar{p}^i\frac{\partial\bar{q}_i}{\partial q_j}\right)-\bar{p}^i\frac{\partial^2\bar{q}_i}{\partial q_h\partial q_j}-\frac{\partial}{\partial q_j}\left(\bar{p}^i\frac{\partial\bar{q}_i}{\partial q_h}\right)+\bar{p}^i\frac{\partial^2\bar{q}_i}{\partial q_j\partial q_h}=\frac{\partial}{\partial q_h}\left(\bar{p}^i\frac{\partial\bar{q}_i}{\partial q_j}\right)-\frac{\partial}{\partial q_j}\left(\bar{p}^i\frac{\partial\bar{q}_i}{\partial q_h}\right),$$

from which we infer that there exists a function $\phi(q^j,p_j)(\bar{\phi}(q_j,p^j))$ such that

(6.32) $$\frac{\partial\phi}{\partial q^j}=\bar{p}_i\frac{\partial\bar{q}^i}{\partial q^j}. \qquad \frac{\partial\bar{\phi}}{\partial q_j}=\bar{p}^i\frac{\partial\bar{q}_i}{\partial q_j}.$$

Similarly we deduce from (6.31) that there exists a function $\psi(q^j,p_j)(\bar{\psi}(q_j,p^j))$ for which

(6.33) $$\frac{\partial\psi}{\partial p_h}=\bar{p}_i\frac{\partial\bar{q}^i}{\partial p_h}. \qquad \frac{\partial\bar{\psi}}{\partial p^h}=\bar{p}^i\frac{\partial\bar{q}_i}{\partial p^h}.$$

Finally, we see that the condition (6.30) is equivalent to

(6.34) $$\delta_j^h=\frac{\partial}{\partial p_h}\left(\bar{p}_i\frac{\partial\bar{q}^i}{\partial q^j}\right)-\frac{\partial}{\partial q^j}\left(\bar{p}_i\frac{\partial\bar{q}^i}{\partial p_h}\right), \qquad \delta_h^j=\frac{\partial}{\partial p^h}\left(\bar{p}^i\frac{\partial\bar{q}_i}{\partial q_j}\right)-\frac{\partial}{\partial\bar{q}_j}\left(\bar{p}^i\frac{\partial\bar{q}_i}{\partial p^h}\right),$$

and this gives in terms of (6.32) and (6.33):

$$\delta_j^h=\frac{\partial}{\partial p_h}\left(\frac{\partial\phi}{\partial q^j}\right)-\frac{\partial}{\partial q^j}\left(\frac{\partial\psi}{\partial p_h}\right), \qquad \delta_h^j=\frac{\partial}{\partial p^h}\left(\frac{\partial\bar{\phi}}{\partial q_j}\right)-\frac{\partial}{\partial q_j}\left(\frac{\partial\bar{\psi}}{\partial p^h}\right),$$

or

$$\frac{\partial}{\partial p_h}\left(\frac{\partial\phi}{\partial q^j}-p_j\right)=\frac{\partial}{\partial q^j}\left(\frac{\partial\psi}{\partial p_h}\right). \qquad \frac{\partial}{\partial p^h}\left(\frac{\partial\bar{\phi}}{\partial q_j}-p^j\right)=\frac{\partial}{\partial q_j}\left(\frac{\partial\bar{\psi}}{\partial p^h}\right).$$

This again ensures the existence of a function $\Psi(q^j,p_j)(\Phi(q_j,p^j))$ such that

(6.35) $$\frac{\partial\Psi}{\partial q^j}=\frac{\partial\phi}{\partial q^j}-p_j, \quad \frac{\partial\Psi}{\partial p_h}=\frac{\partial\psi}{\partial p_h}. \qquad \frac{\partial\Phi}{\partial q_j}=\frac{\partial\bar{\phi}}{\partial q_j}-p^j, \quad \frac{\partial\Phi}{\partial p^h}=\frac{\partial\bar{\psi}}{\partial p^h}.$$

Now, from (6.32) and (6.33), we have

$$\bar{p}_i\,d\bar{q}^i-p_i\,dq^i=\bar{p}_i\frac{\partial\bar{q}^i}{\partial q^j}dq^j+\bar{p}_i\frac{\partial\bar{q}^i}{\partial p_j}dp_j-p_j\,dq^j=\left(\frac{\partial\phi}{\partial q^j}-p_j\right)dq^j,+\frac{\partial\psi}{\partial p_j}dp_j,$$

$$\bar{p}^i\,d\bar{q}_i-p^i\,dq_i=\bar{p}^i\frac{\partial\bar{q}_i}{\partial q_j}dq_j+\bar{p}^i\frac{\partial\bar{q}_i}{\partial p^j}dp^j-p^j\,dq_j=\left(\frac{\partial\bar{\phi}}{\partial q_j}-p^j\right)dq_j+\frac{\partial\bar{\psi}}{\partial p^j}dp^j,$$

which is the defining relation (6.4) of a canonical transformation. Thus we see that the conditions (6.19) are sufficient.

**Lemma 1°.** *The Jacobian $J$ of* (6.19) *is non-vanishing.*

**Proof** (for the left-hand side).

$$J=\det\begin{vmatrix} \dfrac{\partial\bar{q}^i}{\partial q^r} & \dfrac{\partial\bar{p}_h}{\partial q^r} \\[2ex] \dfrac{\partial\bar{q}^i}{\partial p_s} & \dfrac{\partial\bar{p}_h}{\partial p_s}\end{vmatrix}. \tag{6.36}$$

Here we have chosen our notation such that $r, s$ indicate the $r$–th and $(n+s)$–th rows respectively, while $i$ and $h$ indicate the $i$–th and $(n+h)$–th columns. In this determinant we interchange the first $n$ columns with the second $n$ columns, afrer which the rows are similarly interchanged. This involves an even number of operations, giving

$$J=\det\begin{vmatrix} \dfrac{\partial\bar{p}_i}{\partial p_r} & \dfrac{\partial\bar{q}^h}{\partial p_r} \\[2ex] \dfrac{\partial\bar{p}_i}{\partial q^s} & \dfrac{\partial\bar{q}^h}{\partial q^s}\end{vmatrix}.$$

In this the rows and columns are interchanged so that

$$J=\det\begin{vmatrix} \dfrac{\partial\bar{p}_r}{\partial p_i} & -\dfrac{\partial\bar{p}_r}{\partial q^h} \\[2ex] -\dfrac{\partial\bar{q}^s}{\partial p_i} & \dfrac{\partial\bar{q}^s}{\partial q^h}\end{vmatrix}, \tag{6.37}$$

where the minus signs, which have been appended to all the terms of the top right-hand and bottem left-hand parts of the determinant, do not affect its value since its order is even, namely $2n$.

Multiplication of (6.36) by (6.37) yields $J^2$. Since the $(r, i)$ element of $J^2$ in its top left-hand corner is the product of the row of (6.36) and the columns of (6.37), we see that this element is

$$\frac{\partial\bar{q}^i}{\partial q^r}\frac{\partial\bar{p}_i}{\partial p_i}+\cdots+\frac{\partial\bar{q}^n}{\partial q^r}\frac{\partial\bar{p}_n}{\partial p_i}-\frac{\partial\bar{p}_i}{\partial q^r}\frac{\partial\bar{q}^i}{\partial p_i}-\cdots-\frac{\partial\bar{p}_n}{\partial q^r}\frac{\partial\bar{q}^n}{\partial p_i}$$
$$=\frac{\partial\bar{q}^j}{\partial q^r}\frac{\partial\bar{p}_j}{\partial p_i}-\frac{\partial\bar{q}^j}{\partial p_i}\frac{\partial\bar{p}_j}{\partial q^r}=[q^r, p_i].$$

It is similarly found that the other parts of $J^2$ are also such brackets, so that we have

$$J^2=\det\begin{vmatrix} [q^r, p_i] & [q^h, q^r] \\ [p_s, p_i] & [q^h, p_s]\end{vmatrix}, \tag{6.38}$$

in which the indices $r, h, i, s$ have the same significance as in (6.36).

The relation (6.38) is an *identity*, which holds for any transformation of the type (6.1). If, in particular, this transformation happens to be canonical, we may apply the Lagrange bracket relations (6.10) to the entities (6.38) by virtue of Theorem 1⁰. We then obtain the determinant of the unit $2n\times 2n$ matrix, from which we infer that $J^2=1$. Thus Lemma 1⁰ is proved.

**Cor. 1⁰.** *Every canonical transformation possesses an inverse.*

**Cor. 2⁰.** By Lemma 1⁰, we can solve equations (6.1) for $q^i, p_i\,(q_i, p^i)$ as functions of $\bar{q}^j, \bar{p}_j\,(\bar{q}_j, \bar{p}^j)$.

(6.39) $q^i=q^i\,(\bar{q}^j, \bar{p}_j),\quad p_i=p_i\,(\bar{q}^j, \bar{p}_j),$ | $q_i=q_i\,(\bar{q}_j, \bar{p}^j),\quad p^i=p^i\,(\bar{q}_j, \bar{p}^j),$

and hence write the generating functions (6.6) in the forms

(6.40) $\bar{\Psi}\overset{\text{def}}{=}\bar{\Psi}\,(q^i\,(\bar{q}^i\,(\bar{q}^j, \bar{p}_j), p_i\,(\bar{q}^j, \bar{p}_j))=\bar{\Psi}\,(\bar{q}^i, \bar{p}_j).$ | $\bar{\Phi}\overset{\text{def}}{=}\bar{\Phi}\,(q_i\,(\bar{q}_j, \bar{p}^j), p^j\,(\bar{q}_j, \bar{p}^j))=\bar{\Phi}\,(\bar{q}_j, \bar{p}^j).$

**Theorem 2⁰.** *The totality of all canonical transformations* (6.1) *in the* $2n$ *variables*

$(q^j, p_j)$ | $(q_j, p^j)$

*constitutes a group.*

**Proof.** Cor 1⁰ and Cor. 2⁰ hold. And moreover, it is almost obvious that the definition of canonical transformations that the product of two such transformations is again canonical, while the identity transformation is clearly also of this kind. Thus Theorem 2⁰ holds.

**Theorem 3⁰.** *In order that the transformɒtion* (6.1) *be canonical, it is necessary and sufficient that the relations*

(6.41) $\dfrac{\partial \bar{q}^i}{\partial q^h}=\dfrac{\partial p_h}{\partial \bar{p}_i},\ \dfrac{\partial \bar{q}^i}{\partial p_h}=-\dfrac{\partial q^h}{\partial \bar{p}_i},$ | $\dfrac{\partial \bar{q}_i}{\partial q_h}=\dfrac{\partial p^h}{\partial \bar{p}^i},\ \dfrac{\partial \bar{q}_i}{\partial p^h}=-\dfrac{\partial q_h}{\partial \bar{p}^i},$

(6.42) $\dfrac{\partial \bar{p}_i}{\partial q^h}=\dfrac{\partial p_h}{\partial \bar{q}^i},\ \dfrac{\partial \bar{p}_i}{\partial p_h}=\dfrac{\partial q^h}{\partial \bar{q}^i}$ | $\dfrac{\partial p^i}{\partial q_h}=-\dfrac{\partial p^h}{\partial \bar{q}_i},\ \dfrac{\partial \bar{p}^i}{\partial p^h}=\dfrac{\partial q_h}{\partial \bar{q}_i}$

*hold.*[9)]

**Proof** (for the left-hand side). *Necessity.* Using the generating functions $\bar{\Psi}\,(\bar{\Phi})$ instead of (6.5) we can now repeat the argument leading to Theorem 1⁰ with the roles

9) L. P. Eisenhart-M. S. Knebelman. [1].

of the two sets of variables $(q^j, p_j), (\bar{q}^j, \bar{p}_j)$ interchanged, and in this manner obtain Lagrange bracket relations corresponding to (6.29), (6.30), (6.31) subject to this interchange. Thus it is found that the canonical transformations (6.1) is also characterized by the conditions:

$$(6.43) \qquad [\bar{q}^h, \bar{q}^j] \overset{\text{def}}{=} \frac{\partial q^i}{\partial \bar{q}^h}\frac{\partial p_i}{\partial \bar{q}^j} - \frac{\partial q^i}{\partial \bar{q}^j}\frac{\partial p_i}{\partial \bar{q}^h} = 0,$$

$$(6.44) \qquad [\bar{q}^j, \bar{p}_h] \overset{\text{def}}{=} \frac{\partial q^i}{\partial \bar{q}^j}\frac{\partial p_i}{\partial \bar{p}_h} - \frac{\partial q^i}{\partial \bar{p}_h}\frac{\partial p_i}{\partial \bar{q}^j} = \delta^h_j,$$

$$(6.45) \qquad [\bar{p}_h, \bar{p}_j] \overset{\text{def}}{=} \frac{\partial q^i}{\partial \bar{p}_h}\frac{\partial p_i}{\partial \bar{p}_j} - \frac{\partial q^i}{\partial \bar{p}_j}\frac{\partial p_i}{\partial \bar{p}_h} = 0.$$

Furthermore, when the equations (6.39) are substituted in (6.1), we obtain $2n$ identities in the $\bar{q}^i, \bar{p}_i$. Differentiation of these with respect to $\bar{q}^k$ and $\bar{p}_k$ yields the relations

$$(6.46) \qquad \delta^i_k = \frac{\partial \bar{q}^i}{\partial q^j}\frac{\partial q^j}{\partial \bar{q}^k} + \frac{\partial \bar{q}^i}{\partial p_j}\frac{\partial p_j}{\partial \bar{q}^k}, \quad 0 = \frac{\partial \bar{p}_i}{\partial q^j}\frac{\partial q^j}{\partial \bar{q}^k} + \frac{\partial \bar{p}_i}{\partial p_j}\frac{\partial p_j}{\partial \bar{q}^k},$$

$$(6.47) \qquad \delta^k_i = \frac{\partial \bar{p}_i}{\partial q^j}\frac{\partial q^j}{\partial \bar{p}_k} + \frac{\partial \bar{p}_i}{\partial p_j}\frac{\partial p_j}{\partial \bar{p}_k}, \quad 0 = \frac{\partial \bar{q}^i}{\partial q^j}\frac{\partial q^j}{\partial \bar{p}_k} + \frac{\partial \bar{q}^i}{\partial p_j}\frac{\partial p_j}{\partial \bar{p}_k},$$

together with a similar set, in which $(q^i, p_i), (\bar{q}^i, \bar{p}_i)$ are interchanged.

The existence of the inverse (6.39) of $(\bar{q}^i, \bar{p}_i)$ allows us to express the necessary and sufficient conditions that a transformation be canonical in another manner. From (6.39) we firstly have

$$(6.48) \qquad dq^h = \frac{\partial q^h}{\partial \bar{q}^i} d\bar{q}^i + \frac{\partial q^h}{\partial \bar{p}_i} d\bar{p}_i,$$

$$(6.49) \qquad dp_h = \frac{\partial p_h}{\partial \bar{q}^i} d\bar{q}^i + \frac{\partial p_h}{\partial \bar{p}_i} d\bar{p}_i.$$

Secondly, let us multiply (6.11) by $\partial \bar{p}_i/\partial p_h$, (6.12) by $\partial \bar{q}^i/\partial p_h$, and subtract the resulting equations from each other. Assuming that (6.1) is canonical transformation, so that (6.10) may be applied, we find

$$(6.50) \qquad \begin{aligned} \frac{\partial \bar{p}_i}{\partial p_h} d\bar{q}^i - \frac{\partial \bar{q}^i}{\partial p_h} d\bar{p}_i &= \left(\frac{\partial \bar{q}^i}{\partial q^j}\frac{\partial \bar{p}_i}{\partial p_h} - \frac{\partial \bar{q}^i}{\partial p_h}\frac{\partial \bar{p}_i}{\partial q^j}\right) dq^j \\ &\quad + \left(\frac{\partial \bar{q}^i}{\partial p_j}\frac{\partial \bar{p}_i}{\partial p_h} - \frac{\partial \bar{q}^i}{\partial p_h}\frac{\partial \bar{p}_i}{\partial q^j}\right) dp_j \\ &= [q^j, p_h]\, dq^j + [p_j, p_h]\, dp_j = \delta^h_j\, dq^j = dq^h. \end{aligned}$$

Similarly, multiplying (6.20) by $\partial \bar{p}_i / \partial q^h$, (6.21) by $\partial \bar{q}^i / dq^h$, and again subtract the resulting equation from each other, we obtain

$$\frac{\partial \bar{p}^i}{\partial q^h} d\bar{q}^i - \frac{\partial \bar{q}^i}{\partial q^h} d\bar{p}_i = \left( \frac{\partial \bar{q}^i}{\partial q^j} \frac{\partial \bar{p}_i}{\partial q^h} - \frac{\partial \bar{q}^i}{\partial q^h} \frac{\partial \bar{p}_i}{\partial q^j} \right) dq^j \tag{6.51}$$

$$+ \left( \frac{\partial \bar{q}^i}{\partial p^j} \frac{\partial \bar{p}_j}{\partial q^h} - \frac{\partial \bar{q}^i}{\partial q^h} \frac{\partial \bar{p}_i}{\partial p_j} \right) dp_j$$

$$= [q^j, q^h]\, dq^j + [p_j, q^h]\, dp_j = -\delta^j_h\, dp_j = -dp_h.$$

A comparson of (6.48), with (6.50), (6.49) with (6.51) now yields

$$\frac{\partial q^h}{\partial \bar{q}_i} d\bar{q}^i + \frac{\partial q^h}{\partial \bar{p}_i} d\bar{p}_i = \frac{\partial \bar{p}_i}{\partial p_h} d\bar{q}^i - \frac{\partial \bar{q}^i}{\partial p_h} d\bar{p}_i,$$

$$\frac{\partial p_h}{\partial \bar{q}^i} d\bar{q}^i + \frac{\partial p_h}{\partial \bar{p}_i} d\bar{p}_i = \frac{\partial \bar{p}_i}{\partial q^h} d\bar{q}^i + \frac{\partial \bar{q}^i}{\partial q^h} d\bar{p}_i.$$

Now since $(\bar{q}^i, \bar{p}_i)$ can be regarded as a set of independent variables, we may infer from these identities that

$$\frac{\partial \bar{q}^i}{\partial q^h} = \frac{\partial p_h}{\partial \bar{p}_i}, \quad \frac{\partial \bar{q}^i}{\partial p_h} = -\frac{\partial q^h}{\partial \bar{p}_i}, \tag{6.52}$$

$$\frac{\partial \bar{p}_i}{\partial q^h} = -\frac{\partial p_h}{\partial \bar{q}^i}, \quad \frac{\partial \bar{p}_i}{\partial p_h} = \frac{\partial q^h}{\partial \bar{q}^i}. \tag{6.53}$$

*These relations are a neccssary consequence of the necessary conditions* (6.10) *for the canonical transformation* (6.1).

*Sufficiency.* Suppose conversely that (6.52) and (6.53) are valid. In the Lagrange bracket $[q^j, p_h]$ we substitute for $\partial \bar{p}_i/\partial p_h$ from (6.53) and for $\partial \bar{q}^i/\partial p_h$ from (6.52). We thus obtain

$$[q^j, p_h] \overset{\text{def}}{=} \frac{\partial \bar{q}^i}{\partial q^j} \frac{\partial \bar{p}_i}{\partial p_h} - \frac{\partial \bar{q}^i}{\partial p_h} \frac{\partial \bar{p}_i}{\partial q^j} = \frac{\partial \bar{q}^i}{\partial q^j} \frac{\partial q^h}{\partial \bar{q}^h} + \frac{\partial q^h}{\partial \bar{p}_i} \frac{\partial \bar{p}_i}{\partial q^j} = \delta^h_j,$$

the last step resulting from the inverse of (6.46). This relation is the second of the sufficient conditions (6.10) for the canonical transformation (6.1). It may be shown similarly that the first and the third conditions (6.10) are consequences of (6.52) and (6.53). Thus we see that (6.52) and (6.53) are sufficient conditions for the sufficient conditions (6.10) for the canonical transformations (6.1). It therefore follows from Theoren $1^0$ that (6.52) and (6.53) are necessary and sufficient for that (6.1) be canonical.

**Theorem $4^0$.** *In order that the transformation* (6.1) *be canonical, it is necessary and sufficient that the following Poisson bracket relations hold*:

$$(\bar{p}_j, \bar{p}_h) = 0, \ (\bar{q}^h, \bar{p}_j) = \delta^h_j, \ (\bar{q}^j, \bar{q}^h) = 0. \qquad \Big| \qquad (\bar{p}^j, \bar{p}^h) = 0, \ (\bar{q}_h, \bar{p}^j) = \delta^j_h, \ (\bar{q}_j, \bar{q}_h) = 0. \tag{6.54}$$

**Proof** (for the left-hand side).

*Necessity.* In the definition (6.49) of $[\bar{q}^h, \bar{q}^j]$ let us replace all terms according to (6.52) and (6.53), which gives

$$[\bar{q}^h, \bar{q}^j] = \frac{\partial \bar{p}^h}{\partial p_i}\left(-\frac{\partial \bar{p}_j}{\partial q^i}\right) - \frac{\partial \bar{p}_j}{\partial p_i}\left(-\frac{\partial \bar{p}_h}{\partial q^i}\right),$$

so that if we define the Poisson brackets $(\bar{p}_h, \bar{p}_j)$ according to the definition as

$$(6.55) \qquad (\bar{p}_h, \bar{p}_j) = \frac{\partial \bar{p}_h}{\partial q^i}\frac{\partial \bar{p}_j}{\partial p_i} - \frac{\partial \bar{p}_h}{\partial p_i}\frac{\partial \bar{p}_j}{\partial q^i},$$

we obtain

$$[\bar{q}^h, \bar{q}^j] = (\bar{p}_h, \bar{p}_j).$$

Similarly two other relations are found, which are again adjoin to the last result:

$$(6.56) \qquad [\bar{q}^j, \bar{q}^h] = (\bar{p}_j, \bar{p}_h),\ [\bar{q}^j, \bar{p}_h] = (\bar{q}^h, \bar{p}_j),\ [\bar{p}_j, \bar{p}_h] = (\bar{q}^j, \bar{q}^h).$$

It therefore follows from (6.43) – (6.45) that for a canonical transformation, we must have

$$(6.57) \qquad (\bar{p}_j, \bar{p}_h) = 0,\ (\bar{q}^h, \bar{p}_j) = \delta^h_j,\ (\bar{q}^j, \bar{q}^h) = 0.$$

Thus the conditions (6.54)=(6.57) are necessary.

*Sufficiency.* Conversely, let us suppose that we are given a transformation (6.54)= (6.57) for which these relations are satisfied, Multiplying the second of (6.57) by $\partial q^k/\partial \bar{p}_j$, we obtain

$$(6.58) \qquad \frac{\partial q^k}{\partial \bar{p}_h} = \frac{\partial \bar{q}^h}{\partial q^j}\frac{\partial \bar{p}_j}{\partial p_i}\frac{\partial q^k}{\partial \bar{p}_i} - \frac{\partial \bar{q}^k}{\partial p_i}\frac{\partial \bar{p}_j}{\partial q^t}\frac{\partial q^k}{\partial \bar{p}_j}.$$

But in analogy with (6.43) we have

$$(6.59) \qquad 0 = \frac{\partial q^k}{\partial \bar{p}_j}\frac{\partial \bar{p}_j}{\partial p_i} + \frac{\partial q^k}{\partial \bar{q}^j}\frac{\partial \bar{q}^j}{\partial p_i},\quad \delta^k_i = \frac{\partial q^k}{\partial \bar{p}_j}\frac{\partial \bar{p}_j}{\partial q^i} + \frac{\partial q^k}{\partial \bar{q}^j}\frac{\partial \bar{q}^j}{\partial q^i},$$

and when these identities are substituted in the first and second terms on the right-hand side of (6.58), we find

$$\frac{\partial q^k}{\partial \bar{p}_h} = -\frac{\partial \bar{q}^h}{\partial q^i}\frac{\partial q^k}{\partial \bar{q}^j}\frac{\partial \bar{q}^j}{\partial p_i} - \frac{\partial \bar{q}^k}{\partial p_i}\left(\delta^k_i - \frac{\partial q^k}{\partial \bar{q}^j}\frac{\partial \bar{q}^j}{\partial q^i}\right) = -\frac{\partial \bar{q}^h}{\partial p_k} + \frac{\partial q^k}{\partial \bar{q}^j}(\bar{q}^j, \bar{q}^h).$$

But by hypothesis the Poisson bracket on the right-hand side vanish (cf. the third equation of (6.57)=(6.54), so that this equation simply reduces to the second relation of (6.41). Similarly, the first of (6.41), together with both sets of (6.42), may be derived from (6.57)=(6.54). Thus (6.57)=(6.54) is sufficient.

**Cor.** *In order that the transformation* (6.1) *be canonical, it is necessary and sufficient that the following Poisson bracket relations hold:*

(6.60) $$(p_j, p_h)'=0,\ (q^h, p_j)'=\delta^h_j,\quad (q^j, q^h)'=0, \qquad\Big|\qquad (p^j, p^h)'=0,\ (q_h, p^j)'=\delta^j_h,\quad (q_j, q_h)'=0,$$

*where the dashes on the Poisson brackets indicate that these are to be formed with respect to the* $(\bar q^i, \bar p_i)$, *i. e. for two functions* $f(\bar q^i, \bar p_i), g(\bar q^i, \bar p_i)$ *we write*

(6.61) $$(f, g)'=\frac{\partial f}{\partial \bar q^i}\frac{\partial g}{\partial \bar p_i}-\frac{\partial f}{\partial \bar p_i}\frac{\partial g}{\partial \bar q^i}.$$

**Proof** (for the left-hand side). (6.60) means merely interchanging the roles of the two sets of variables $(q^i, p_i), (\bar q^i, \bar p_i)$, so that we may infer that a canonical transformation is also characterized by the equations (6.60).

**Theorem 5⁰.** *In order that the transformation* (6.1) *be canonical, it is necessary and sufficient that it leaves invariant the Poisson bracket of an arbitrary pair of functions of the* 2*n variables* $(q^i, p_i)$.

**Proof** (for the left-hand side). First of all, let $F(q^i, p_i), G(q^i, p_i)$ be two arbitrary functions of class $C^2$ in $q^i, p_i$.

*Necessity.* The transforms $\bar F(\bar q^j, \bar p_j), \bar G(\bar q^j, \bar p_j)$ under the inverse (6.39) of (6.1) are

$$\bar F=\bar F(\bar q^j, \bar p_j)=F(q^j(\bar q^h, \bar p_h), p_j(\bar q^h, \bar p_h)),$$
$$\bar G=\bar G(\bar q^j, \bar p_j)=G(q^j(\bar q^h, \bar p_h), p_j(\bar q^h, \bar p_h)).$$

The Poisson bracket $(F, G)$ can therefore be expressed as

(6.62) $$\begin{aligned}(F, G)&=\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i}-\frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i}\\ &=\Big(\frac{\partial \bar F}{\partial \bar q^h}\frac{\partial \bar q^h}{\partial q^i}+\frac{\partial \bar F}{\partial \bar p_h}\frac{\partial \bar p_h}{\partial q^i}\Big)\Big(\frac{\partial \bar G}{\partial \bar q^j}\frac{\partial \bar q^j}{\partial p_i}+\frac{\partial \bar G}{\partial \bar p_j}\frac{\partial \bar p_j}{\partial p_i}\Big)\\ &\quad-\Big(\frac{\partial \bar F}{\partial \bar q^h}\frac{\partial \bar q^h}{\partial p_i}+\frac{\partial \bar F}{\partial \bar p_h}\frac{\partial \bar p_h}{\partial p_i}\Big)\Big(\frac{\partial \bar G}{\partial \bar q^j}\frac{\partial \bar q^j}{\partial q^i}+\frac{\partial \bar G}{\partial \bar p_j}\frac{\partial \bar p_j}{\partial q^i}\Big)\\ &=\frac{\partial \bar F}{\partial \bar q^h}\frac{\partial \bar G}{\partial \bar q^j}(\bar q^h, \bar q^j)-\frac{\partial \bar F}{\partial \bar p_h}\frac{\partial \bar G}{\partial \bar q^j}(\bar q^j, \bar p_h)\\ &\quad+\frac{\partial \bar F}{\partial \bar q^h}\frac{\partial \bar G}{\partial \bar p_j}(\bar q^h, \bar p_j)-\frac{\partial \bar F}{\partial \bar p_h}\frac{\partial \bar G}{\partial \bar p_j}(\bar p_j, \bar p_h).\end{aligned}$$

If (6.1) be canonical, we may apply (6.54) to (6.62). Hence by (6.54), the (6.62) becomes to

$$(F, G)=-\frac{\partial \bar F}{\partial \bar p_h}\frac{\partial \bar G}{\partial \bar q^j}\delta^j_h+\frac{\partial \bar F}{\partial \bar q^h}\frac{\partial \bar G}{\partial \bar p_j}\delta^h_j=\frac{\partial \bar F}{\partial \bar q^j}\frac{\partial \bar G}{\partial \bar p_j}-\frac{\partial \bar F}{\partial \bar p_j}\frac{\partial \bar G}{\partial \bar q^j},$$

i. e.

(6.63) $$(\bar F, G)=(\bar F, \bar G)'.$$

This invariance under a canonical transformation is thus necessary.

*Sufficiency.* Conversely, suppose that the transformation (6.1) is such that (6.63) is always satisfied, In this equation let us put $\bar{F}=\bar{q}^i, \bar{G}=\bar{p}_h$, noting the identitty

$$(\bar{F}, \bar{G})'=\frac{\partial \bar{q}^i}{\partial \bar{q}^j}\frac{\partial \bar{p}_h}{\partial \bar{p}_j}-\frac{\partial \bar{q}^i}{\partial \bar{p}_j}\frac{\partial \bar{p}_h}{\partial \bar{q}^j}=\delta^i_j\,\delta^j_h=\delta^i_h, \tag{6.64}$$

and thus we deduce that

$$(\bar{q}^i, \bar{p}_h)=(\bar{q}^i, \bar{p}_h)'=\delta^i_h.$$

Similarly we find that the other two sets of equations in (6.54) are satisfied. Hence by Theorem $4^0$, the condition is sufficient.

**N. B.** Because of identities such as (6.64) it is now clear that *Theorem* $4^0$ *is a special case of Theorem* $5^0$. From the latter we can now deduce a result which represents one of the most significant properties of canonical transformation.

**Theorem $6^0$.** *Any canonical transformation* (6.1) *leaves invariant the canonical equations*

$$q^i=\frac{\partial H}{\partial p_i},\ \dot{p}_i=-\frac{\partial H}{\partial q^i}. \quad \Bigg| \quad q_i=\frac{\partial L}{\partial p^i},\ \dot{p}^i=-\frac{\partial L}{\partial q_i}. \tag{6.65}$$

**Proof** (for the left-hand side). We can write the canonical equations in Poisson bracket form as follows.

$$\dot{q}^i=\delta^i_j\frac{\partial H}{\partial p_j}=\frac{\partial q^i}{\partial q^j}\frac{\partial H}{\partial p_j}-\frac{\partial q^i}{\partial p^j}\frac{\partial H}{\partial q^j}=(q^i, H), \tag{6.66}$$

$$\dot{p}_i=-\delta^j_i\frac{\partial H}{\partial q^j}=\frac{\partial p_i}{\partial q^j}\frac{\partial H}{\partial p_j}-\frac{\partial p_i}{\partial p_j}\frac{\partial H}{\partial q^j}=(p_i, H). \tag{6.67}$$

But according to Theorem $5^0$, or, more specifically to (6.63), the Poisson brackets on the right-hand sides of (6.66) and (6.67) transform as follows under a canonical transformation :

$$(\bar{q}^i, \bar{H})'=(q^i, H), \qquad (\bar{p}_i, \bar{H})'=(p_i, H),$$

while the left-hand side simply becomes to $\dot{\bar{q}}^i$ and $\dot{\bar{p}}_i$.

Thus the equations (6.66), (6.67) are invariant, what proves our assertion.

**Remark $1^0$.** *The canonical equations* (6.65) *constitute an infinitesimal canonical transformation.*

**Proof.** (6.4) tells

$$d\Psi=\bar{p}_l\,d\bar{q}^l-p_l\,dq^l=(p_l+\dot{p}_l\,d\tau)\,d\,(q^l+\dot{q}^l\,d\tau)-p_l\,dq^l$$
$$=(\dot{q}^l\,dp_l-\dot{p}_l\,dq^l)d\tau=\frac{\partial H}{\partial p_l}dp_l+\frac{\partial H}{\partial q^l}\,dq^l=dH$$

by (6.8).

**Remark 2⁰.** One would obtain families of canonical equations of the form:

(6.68) $$\bar{q}^i = \bar{q}^i(s, q^j, p_j), \qquad \bar{p}_i = \bar{p}_i(s, q^j, p_j).$$

*All our previous results hold still for such families, in particular* Theorem $5^0$, in which the functions $F$ and $G$ may also depend explicitly on $s$. Thus Theorem $6^0$ is valid even if the Hamiltonian $H$ is explicitly dependent on this parameter,

**Remark $3^0$.** *It is not difficult to prove the inverse of Theorem* $6^0$, namely that *any transformation, which leaves invariant the canonical equations* (6.1) *for any arbitrary Hamiltonian function must be canonical.*

**Theorem $7^0$.** *If a canonical transformation*

(6.69) $$\begin{matrix} \bar{q}^\alpha = \bar{q}^\alpha(q^n, q^\beta, p_n, p_\beta), & \bar{q}^n = \bar{q}^n(q^n, q^\beta, p_n, p_\beta), \\ \bar{p}_\alpha = \bar{p}_\alpha(q^n, q^\beta, p_n, p_\beta), & \bar{p}_n = \bar{p}_n(q^n, q^\beta, p_n, p^\beta) \end{matrix} \quad (\alpha = 1, 2, \cdots, n-1),$$

*with generating function $\Psi$ is a t-independent transformation, in the sense that it satisfies the conditions*

(6.70) $$\frac{\partial \bar{p}_\alpha}{\partial q^n} = 0, \quad \frac{\partial \bar{p}_n}{\partial q^n} = 0, \quad \frac{\partial \bar{q}^\alpha}{\partial q^n} = 0, \qquad \bar{q}^n = q^n,$$

*it is equivalent to a canonical transformation of the type*

(6.71) $$\bar{q}^\alpha = \bar{q}^\alpha(q^\beta, p^\beta), \quad \bar{p}_\alpha = \bar{p}_\alpha(q^\beta, p_\beta),$$

*between the* $2(n-1)$ *variables* $(q^\alpha, p_\alpha), (\bar{q}^\alpha, \bar{p}_\alpha)$, *the generating function of the latter transformation being given by*

(6.72) $$\psi = \psi(q^\alpha, p_\alpha) = \Psi + kq^n,$$

*where $k$ is an arbitrary constant.*

**Proof.** Adopting our original $(t, q^\alpha) \equiv (q^n, q^\alpha)$ - notation, let us write (6.1) in the form

(6.73) $$\begin{matrix} \bar{q}^\alpha = \bar{q}^\alpha(q^m, q^\beta, p_n, p_\beta), \bar{q}^n = \bar{q}^n(q^n, q^\beta, p_n, p_\beta), \\ \bar{p}_\alpha = \bar{p}_\alpha(q^n, q^\beta, p_n, p_\beta), \bar{p}_n = \bar{p}_n(q^n, q^\beta, p_n, p_\beta). \end{matrix}$$

This $t$-independent transformation satisfies the conditions (6.70).

Since (6.73) is supposed to represent a canonical transformation, we may apply (6.41) and (6.42). From the first two equations of (6.70), we obtain

(6.74) $$\frac{\partial \bar{p}_k}{\partial q^n} = -\frac{\partial p_n}{\partial \bar{q}^k} = 0, \qquad (k = 1, 2, \cdots, n).$$

The third equation of (6.70) yields similarly

(6.75) $$\frac{\partial \bar{q}_n}{\partial q^n}=\frac{\partial p_n}{\partial \bar{p}_\alpha}=0,$$

which, taken together with (6.74), shows that $p_n$ is at most a function of $\bar{p}_n$. However, from the last equation of (6.70) and (6.41) we deduce that

(6.76) $$\frac{\partial \bar{q}^n}{\partial q^n}=\frac{\partial p_n}{\partial \bar{p}_n}=1,$$

so that

(6.77) $$p_n=\bar{p}_n+k,$$

in which $k$ is some constant. Furthermore putting $k=n$ in (6.23) we find

$$\bar{p}_\alpha\frac{\partial \bar{q}^\alpha}{\partial q^n}+\bar{p}_n\frac{\partial \bar{q}^n}{\partial q^n}=p_n+\frac{\partial \Psi}{\partial q_n},$$

where we recall that $\Psi=\Psi(q^n, q^\alpha, p_n, p_\alpha)$ is the generating function of the canonical transformation. We therefore deduce from (6.70) and (6.77) that

(6.78) $$\frac{\partial \Psi}{\partial q^\alpha}=-k.$$

We obtain similarly from (6.24)

(6.79) $$\bar{p}_h\frac{\partial \bar{q}^h}{\partial p_n}=\frac{\partial \Psi}{\partial p_n}.$$

But from (6.41) we have

(6.80) $$\frac{\partial \bar{q}^h}{\partial p_n}=-\frac{\partial q^n}{\partial \bar{p}_h},$$

while, by hypothesis, $q^n$ occurs only in the equation of (6.73), so that the inverse of the last of (6.70) is simply $q^n=\bar{q}^n$, which in turn implies that $\partial q^n/\partial \bar{p}_n=0$. Hence (6.80) gives

(6.81) $$\frac{\partial \bar{q}^n}{\partial p_n}=0, \ (h=1, 2, \cdots, n),$$

and (6.79) becomes

(6.82) $$\frac{\partial \Psi}{\partial p_n}=0.$$

Combining (6.78) with (6.82), we see that $\Psi$ must possess the form

(6.83) $$\Psi(q^j, p_j)=\phi(q^\alpha, p_\alpha)-kq^n,$$

where $\psi$ is some function of class $C^2$ in the $2(n-1)$ variables. Finally, we deduce as above from (6.42) that

$$\frac{\partial \bar{p}_\alpha}{\partial p_n} = \frac{\partial q^n}{\partial \bar{q}^\alpha} = 0. \tag{6.84}$$

It follows from (6.81) and (6.84) that the functions $\bar{q}^\alpha, \bar{p}_\alpha$ do not contain $p_n$ explicitly. Thus, by virtue of (5.70), the transformation (6.69) can now be written as

$$\bar{q}^\alpha = \bar{q}^\alpha (q^\beta, p_\beta), \quad \bar{p}_\alpha = \bar{p}_\alpha (q^\beta, p_\beta), \tag{6.85}$$

to which (6.77) may be adjoint. Furthermore, in the relation (6.4) written in the form

$$\bar{p}_\alpha \, d\bar{q}^\alpha + \bar{p}_n \, d\bar{q}^n - p_\alpha \, dq^\alpha - p_n \, dq^n = d\Psi,$$

let us substitute from (6.70), (6.77) and (6.83), which gives

$$\bar{p}_\alpha \, d\bar{q}^\alpha - k \, d\bar{q}^n - p_\alpha \, dq^\alpha = d\psi - k \, dq^n,$$

or

$$\bar{p}_\alpha \, d\bar{q}^\alpha - p_\alpha \, dq^\alpha = d\psi. \tag{6.86}$$

This result shows that the relations (6.85) constitute a canonical transformation between the variables $(q^\alpha, p_\alpha), (\bar{q}^\alpha, \bar{p}_\alpha)$. Thus Theorem $7^0$ is proved.

**N. B.** This Theorem enables us to apply our general formalism to *t-independent canonical transformation* (6.85), *which is thus charaeterized by the Poisson brackeet relations*

$$(\bar{q}^\alpha, q^\beta) = 0, \qquad (\bar{q}^\alpha, \bar{p}_\beta) = \delta^\alpha_\beta, \qquad (\bar{p}_\alpha, \bar{p}_\beta) = 0. \tag{6.87}$$

*These relations hold for all values of t.*

## REFERENCES

E. Cartan.

[1] Leçons sur les invariants intégraux. Herrmann, Paris (1922), reprinted 1958.

G. A. Briss.

[1] Jacobi's condition for problems in the calculus of vsriations in parametric forms. Trans. Amer. Math. Soc. 17 (1916) 195–206.

[2] The problem of Lagrange in the calculus of variations. Amer. J. Math. 52 (1930), 673–744.

[3] The calculus of variations for multiple integrals. Amer. Math. Monthly. 49 (1942), 79–89. [MR 3, 248].

[4] Lectures on the Calculus of Variations. Univ. of Chicago Press, Chicago (1946). [MR 8, 22].

C. Carathéodory.

[1] Über die diskontinuirlichen Lösungen in der Variationsrechnung. Diss. Göttingen (1904).

[2] Über die starken Maxima und Minima bei einfachen Integralen. Math. Ann. 62 (1906), 449-503.

[3] Über den Wiederkehrsatz von Poincaré. Sitzungsber. Preuss. Akad. Wiss. (1919) 580-584.

[4] Über die kanonischen Veränderlichen in der Variationsrechnung der mehrfachen Integrale. Math. Ann. 85 (1922), 78-88.

[5] Über ein Reziprozitätsgesetz der verallgemeinerten Legendreschen Transformation. Math. Ann. 86 (1922), 272-275.

[6] Die Methode der geodätischen Äquidistanten und das Problem von Lagrange. Acta Math. 47 (1926), 199-236.

[7] Über die Variationsrechnung bei mehrfachen Integralen. Acta Szeged Sect. Scient Math. 4 (1929), 193-216.

[8] Bemerkung über die Eulerschen Differentialgleichungen der Variationsrechnung. Nachr. Ges. Wiss. Göttingen, Math.-phys. Kl. H. 1 (1931), 40-42. [Zbl. 2, 268].

[9] Über die Einteilung der Variationsprobleme von Lagrange nach Klassen. Comment. Math. Helv., 5 (1933), 1-19. [Zbl. 6, 60].

[10] Die Theorie der zweiten Variationsprobleme von Lagrange. Sitzungsber. Bayer. Akad. Wiss., Math.-Naturwiss. Abt. (1932), 99-114.

[11] Sur les équations de la méchanique. Actes Congr. Interbalcan. Math. Athènes (1934), 211-214.

[12] Variationsrechnung und partielle Differentialgleichungen erster Ordnung. Teubner, Leipzig and Berlin (1935). [Zbl. 11. 356].

[13] Geometrische Optik. Ergebn. Math. 4573-680. Sqringer, Berlin (1937). [Zbl. 17,382].

[14] Gesammelte mathematische Schriften, C. H. Beck, München, 5 vol. (1954).

L. P. Eisenhart-S. Knebelman.

[1] Invariant theory of homogeneous contact transformations. Ann. Math. 37 (1936), 747-765.

H. Rund.

[1] Application des méthodes de la géométrie métrique généralisée à la dynamique théorique. Géométrie différentielle, Colloq. Int. du Centre National de la Recherche Scientifique. Strassbourg (1953), 41-51. C. N. R. S. Paris (1953). [MR 15, 659].

M. Kawaguchi.

[1] An introduction to the theory of higher order spaces. I. The theory of Kawaguchi spaces. RAAG Mem., I, E. (1962), 718-734.

A. Kawaguchi.

[1] View on Higher Order Geometry of Connections. TENSOR, 1 (1938), 13-18; 2 (1939),

29–45; 3 (1940), 68–72: 4 (1941), 66–68.

[ 2 ] Über nicht-holonome allgemeine metrische Geometrie. Math. Nachr. 11, 61–81 (1954). [MR 15, 898].

[ 3 ] Über allgemeine nicht-holonome und dissipiative dynamische Systeme. Bericht von der Riemann-Tagung 1954, Schr. Forschungsinst. f. Math. 1 (1957), 269–279. (MR 19, 477).

[ 4 ] The Differential Geometry of Finsler Sqaces, Springer, Berlin-Göttingen-Heidelberg (1959). [MR 21, 827].

[ 5 ] On Carathéodory's method of "equivalent integrals" in the calculus of variations. Nederl. Akad. Wetensck. Proc. Ser. A 62 (Indag. Mafh. 21) (1959), 135–141. [MR 23A, 91].

[ 6 ] The theory of problems in the calculus of variations whose Lagrangian functions involves second order derivatives; a new approach. Ann. Mat. pura. Appl. (4) 55 (1960), 77–104. [MR 25, 292].

[ 7 ] Note on Lagrangian formalism in relativistic mechaniques. Nuovo Cimento (10) 23 (1962), 227–232. [MR 25, 556].

[ 8 ] On the Weierstrass excess functton of parameter-invariant multiple integrals in the calculus of variations. Tydskr. Naturwetensk. 3 (1963), 118–179 (1964).

[ 9 ] Canonical formalism for parameter-invariant integrals in the calculus of variations, whose Lagrange functtions involve second order derivatives. Ann. Mat. pura Appl. (4) 63 (1964), 99–107. [MR 29, 751].

[10] Canonical formalism and the relativistic wave equation. Perspectives in Geometry and Relativity. Festschrift, Univ. of of Indiana Press (1966).

[11] Variational Principles for combined vector and tensor fields. Abh. math. Sem. Univ. Hamburg, 29 (1966), 243–262.

[12] The Hamilton-Jacobi Theory in the Calculus of Variations. (1965). The University Mathematics Series, Univ. of Southampton. D. van Nostrand Company, Ltd., London-Toronto-New York, Princeton, New Jersey.

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[ 1 ] Non-connection Methods for Some Connection Geometries based on Canonical Equations of Hamiltonian Types of II-Geodesic Curves. Proc. Japan Acad. 42 (1966), 539–544.

[ 2 ] Canonical Equations for Force Lines. Tensor. N. S. 18 (1967), 162–167.

[ 3 ] The Relativity Theory in the Einstein Space under the Extended Lorentz Transformation Group. Prcc. Japan Acad. 39 (1963), 620–625.

[ 4 ] A New Theory of Relativity under the Non-Locally Extended Lorentz Transformation Group. Proc. Japan Acad. 40 (1964), 691–696.

[ 5 ] Non-connection methods for linear connections in the large. Yokohama Math. J. 11 (1964), 57–91.

[ 6 ] Doubly extended geometries by non-connection methods. Proc. Japan Acad. 41 (1965), 869–874.

[ 7 ] A Theory of Extended Lie Transformation Groups. Annali di Mat. pura ed applicata. Ser. 454 (1963), 247–333.

[ 8 ] A Theory of Doubly Extended Lie Transformation Groups. Yokohama Math. J. 13

(1966), 31–120.

[9] A New Theory of Relativity under the 5–Dimensionally Extended Lorentz Transformation Group with Oriented II–Geodesic Spheres as Particle Models. NEWSLETTER (Research Assoc. of Applied Geometry), No. 84 (1968), 8–10.

[10] A Global Theory of Relativity under the 5–Dimensionally Doubly Extended Lorentz Transformation Group with Oriented II–Geodesic Spheres as Particle Models and its Hamilton's Canonical Formalism. Under preparation for Tôhoku Sci. Rep. Ser. I.

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