THE CHORDAL AND HOROCYCLIC PRINCIPAL CLUSTER SETS OF A CERTAIN HOLOMORPHIC FUNCTION*

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(Received July 29, 1968)

Let Γ be the unit circle and D be the open unit disk in the complex plane. If $f(z)$ is a single-valued complex-valued function defined in D, and if $\zeta\epsilon\Gamma$, we are interested in the following four cluster sets of f at ζ (for the general theory of cluster sets, see [5]):

$C_{\mathfrak{B}}(f,\zeta)$, the inner angular cluster set, is defined as

$$C_{\mathfrak{B}}(f,\zeta)=\bigcap_{\Delta} C_{\Delta}(f,\zeta),$$

where Δ ranges over the set of *Stolz* angles at ζ.

$C_{\mathfrak{H}}(f,\zeta)$, the inner horocyclic angular cluster set, is defined as

$$C_{\mathfrak{H}}(f,\zeta)=\bigcap_{H} C_{H}(f,\zeta),$$

where H ranges over the set of horocyclic angles at ζ (see [2]).

$\Pi_{\chi}(f,\zeta)$, the chordal principal cluster set, is defined as

$$\Pi_{\chi}(f,\zeta)=\bigcap_{\chi} C_{\chi}(f,\zeta),$$

where χ ranges over the set of all chords at ζ.

$\Pi_{\omega}(f,\zeta)$, the horocyclic principal cluster set, is defined as

$$\Pi_{\omega}(f,\zeta)=\bigcap_{h} C_{h}(f,\zeta),$$

where h ranges over the set of all right and left horocycles at ζ (see [2]).

when we say that almost every point of Γ has a certain property, we mean that the exceptional set has linear Lebesgue measure zero; and when we say that nearly every point of Γ has a certain property, we mean that the exceptional set is of linear first Baire category.

Dragosh [4, Theorem 4] has shown that if $f(z)$ is an arbitrary function in D, then

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* Supported by the U. S. Army Research Office-Durham.

$$C_{\mathfrak{B}}(f,\zeta) \subseteqq C_{\mathfrak{B}}(f,\zeta)$$

for almost every and nearly every point $\zeta \epsilon \Gamma$. This naturally suggests the question of whether analogously

$$\Pi_{\chi}(f,\zeta) \subseteqq \Pi_{\omega}(f,\zeta) \tag{1}$$

for every ζ belonging to some sort of subset of Γ. Again *Dragosh* [4, Remark 7] has shown that even if $f(z)$ is holomorphic in D, (1) need not hold for almost every point $\zeta \epsilon \Gamma$; and he asks whether (1) holds for nearly every $\zeta \epsilon \Gamma$ in case $f(z)$ is meromorphic in D. The following result (wherein we use the symbol $\subset$ to denote proper set inclusion) answers this question in the negative.

Theorem. *There exists a holomorphic function $f(z)$ in D such that*

$$\Pi_{\omega}(f,\zeta) \subset \Pi_{\chi}(f,\zeta), \tag{2}$$

for nearly every and almost every $\zeta \epsilon \Gamma$.

Proof. For every ternary fraction

$$t = 0.\, t_1\, t_2\, t_3 \cdots$$

in which each t_j is either 0 or 2, we denote by

$$b(t) = 0.\, b_1\, b_2\, b_3 \cdots$$

the binary fraction such that, for $j = 1, 2, 3, \cdots$,

$$b_j = \begin{cases} 0 & \text{if } t_j = 0, \\ 1 & \text{if } t_j = 2. \end{cases}$$

The set of all such ternary fractions t is the Cantor middle-thirds set T, and the set of corresponding binary fractions $b(t)$ is the closed unit interval.

Let $T^* = T - \{1\}$, and for $t \epsilon T^*$ let $h_r^+(\zeta)$ be the right horocycle at the point

$$\zeta = \zeta(t) = e^{2\pi b(t) i} \epsilon \Gamma$$

with radius

$$r = \frac{1}{2}\left(1 - \sin\frac{\pi t}{2}\right).$$

If $t \geqq \frac{7}{9}$, then $t \geqq b(t)$; $h_r^+(\zeta)$ has a diameter of length $2r$, and since

$$1 - 2r = \sin\frac{\pi t}{2} > \cos 2\pi t \geqq \cos 2\pi b(t),$$

$h_r^+(\zeta)$ does not intersect $h_{\frac{1}{2}}^+(1)$. It follows that the set

$$P = \{h_r^+(\zeta(t)) : t \epsilon T^*\}$$

is a set of disjoint right horocycles with the property that at every point of Γ there is a single right horocycle belonging to P, except at the points of an enumerable everywhere dense subset E of Γ, at each point of which there are two right horocycles belonging to P.

For every $\zeta\epsilon E$, if $h^+_{r_1}(\zeta)$ and $h^+_{r_2}(\zeta)$ are the two right horocycles at ζ belonging to P, define

$$\tilde{h}(\zeta)=h^+_{\frac{r_1+r_2}{2}}(\zeta),$$

and let

$$Q=\{\tilde{h}(\zeta):\zeta\epsilon E\}.$$

According to [3], there exists a holomorphic function $f(z)$ in D such that, for every $h\epsilon P$,

$$\lim_{\substack{|z|\to 1\\ z\epsilon h}} f(z)=0,$$

and, for every $h\epsilon Q$,

$$\lim_{\substack{|z|\to 1\\ z\epsilon h}} f(z)=\infty.$$

Consequently, for every $\zeta\epsilon\Gamma$, $\{0,\infty\}\subset C(f,\zeta)$ and $\Pi^+_\omega(f,\zeta)\subseteqq\{0\}$. This implies that no point of Γ is a right horocyclic Meier point of f [2, p. 6], and hence [2, Theorem 6] nearly every point of Γ is a right horocyclic Plessner point of f.

Let $\zeta\epsilon\Gamma$ be a right horocyclic Plessner point of f that is not an ambiguous point of f, and suppose that χ is a chord at ζ such that $\infty\notin C_\chi(f,\zeta)$. If h is the right horocycle at ζ that belongs to P, then $\infty\notin C_h(f,\zeta)$. By the Gross–Iversen theorem, ∞ is an asymptotic value of f at ζ, which contradicts our assumption that ζ is not an ambiguous point of f. Therefore $\infty\,\epsilon\,\Pi_\chi(f,\zeta)$, and since f has at most enumerably many ambiguous points [1], (2) is satisfied at nearly every point of Γ. *Dragosh's* argument [4, Theorem 8 and Remark 7] shows that (2) is also satisfied at almost every point of Γ.

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