A NOTE CONCERNING CERTAIN SUBCOMPLEXES OF TRIANGULATED MANIFOLDS

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1. Introduction.

In section 2 a local flatness criterion is established for some special manifold pairs by a refinement of some of our earlier methods. This is of interest principally because it implies two seemingly unrelated *Schoenflies* type of results. These are the author's 5-dimensional polyhedral *Schoenflies* theorem [6] and *Newman's* result that an $(n-1)$-star sphere which is a subpolyhedron of an n-star sphere S, is flat in S [3]. This new theorem admits an elementary combinatorial verification, although we must appeal to our own work [4, 5, 6] and *M. Brown's* paper [1] in which it is shown that "locally bicollared implies bicollared".

In section 3 we extend to triangulated manifolds with boundary a result of ours which dealt with simplex links in unbounded manifolds [Theorem 7 and Corollary, 4]. This fills a gap in the literature, since the only result in print which deals with behaviour at the boundary was observed by *Kwun and Raymond* [Corollary 4. 4, 2]. They noticed that for a vertex v in the boundary, the open star of v is a topological half space. Like our earlier theorem, the extension holds for all simplexes.

2. A Schoenflies theorem for L-complexes

We shall follow notations and definitions previously adopted in [4, 5, 6].

By an *L-complex* we shall mean a complex K which has the property that for each of its simplexes σ, the suspension of $Lk(K, \sigma)$ is topologically a sphere. In our nomenclature, each link of K is a suspension sphere. It is easily verified that K and each of its links is a manifold. This apparently artificial concept may be seen to be intermediate in terms of generality, between those of star manifold and triangulated manifold.

Lemma. *Let K be a suspension sphere which is a subcomplex and $(n-1)$-submanifold of the star n-sphere S. If for each vertex v of K, $Lk(K, v)$ is flat in $Lk(S, v)$, then K is flat in S.*

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Proof. By definition (see p. 512 of [6]) $(Lk(S, v), Lk(K, v)) \approx (S(Lk(K, v)), Lk(K, v))$. Let $p \circ Lk(K, v)$ be the closure of a complementary domain of $Lk(K, v)$ in $Lk(S, v)$. Then, as in the proof of Theorem 1 of [6], we have $v \circ p \circ Lk(K, v) \approx CS(Lk(K, v)) \approx C(S^{n-1}) = I^n$. Therefore K is locally flat and hence locally bicollared in S. By [1], K is bicollared in S. From Lemma 1 of [6], it follows that K is flat in S.

Theorem 1. *Let K be an n-dimensional L-complex which is a subpolyhedron of the star $(n+1)$-manifold M. Then K is locally flat in M. If in addition, M is a topological sphere and K, a suspension sphere, then K is flat in M.*

Proof. By induction on n. It is seen to be trivially true for $n=0$.

Let us assume the theorem for $n-1$. If v is a vertex of K, then $Lk(K, v)$ is an L-complex and suspension sphere and $Lk(M, v)$ is a star sphere. From the induction, we may conclude that $Lk(K, v)$ is flat in $Lk(M, v)$. The above Lemma implies that $N(K, v)$ is locally flat in $N(M, v)$. Accordingly K is locally flat in M. If M is a sphere and K is a suspension sphere, again the Lemma implies that K is flat in M.

Remark. That Newman's result may be derived from Theorem 1 is obvious. Now suppose that K is a topological 4-sphere which is a subpolyhedron of the star 5-sphere M. If σ is a k-simplex of K with $k>0$, then $Lk(K, \sigma)$ is a topological sphere. (See section I. of [6]) The links in K of vertices are clearly suspension spheres. (By Theorem 4 of [4]) Thus K and M satisfy the conditions of Theorem 1. This demonstrates that the 5-dimensional polyhedral Schoenflies theorem is also a consequence of the above.

3. Links in triangulated manifolds with boundary.

In past work we have generally restricted our investigations to unbounded manifolds. It is natural to wonder what sort of modifications must be made for the theory of bounded manifolds. It should be satisfying that, as our intuition might lead us to believe, the required changes are relatively minor.

Theorem 2. *Let M be a triangulated n-manifold with boundary. Suppose σ is a k-simplex of M. If $\sigma \cap Int\ M \neq \phi$ then $S^k \circ Lk(M, \sigma) \approx S^n$. If $\sigma \subseteq \partial M$ then $S^k \circ LK(M, \sigma) \approx I^n$.*

Proof. Consider the natural injection of M in N, where N is the double of M, By the Corollary to Theorem 7 of [4], $S^k \circ Lk(N, \sigma) \approx S^n$.

If σ meets the interior of M, it is not difficult to verify that $Lk(N, \sigma) = Lk(M, \sigma)$. This finishes the first case.

Next we suppose that $\sigma \subseteq \partial M$. Since the latter is a manifold, $S^k \circ Lk(\partial M, \sigma)$ is an $(n-1)$-sphere. On the other hand ∂M is locally flat in M. This means that $N(\partial M, \sigma)$ is locally flat in $N(M, \sigma)$. As a result, $S^k \circ Lk(\partial M, \sigma)$ is a flat $(n-1)$-sphere in the n-

sphere $S^k \circ Lk(N, \sigma)$. Consequently $I^n \approx S^k \circ Lk(M, \sigma)$, for the latter set corresponds to the closure of a complementary domain of $S^k \circ Lk(\partial M, \sigma)$ in $S^k \circ Lk(N, \sigma)$.

Bibliography.

1. M. Brown, "*Locally flat imbeddings of topological manifolds*", Ann. of Math. (2) 75 (1962), 331–341.
2. K. W. Kwun and F. Raymond, "*Manifolds which are joins*", Trans. Amer. Math. Soc. 111 (1964), 108–120.
3. M. Newman, "*On the division of Euclidean n-space by topological* $(n-1)$*-spheres*", Proc. Roy. Soc. London 257 (1960), 1–12.
4. R. H. Rosen, "*Stellar neighborhoods in polyhedral manifolds*", Proc. Amer. Math. Soc. 14 (1963), 401–406.
5. ——, "*Polyhedral neighborhoods in triangulated manifolds*", Bull. Amer. Math. Soc. 69 (1963), 359–361,
6. ——, "*The five dimensional polyhedral Schoenflies theorem*", Ibid 70 (1964), 511–516.

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