A NECESSARY AND SUFFICIENT CONDITION FOR UNIQUENESS OF OPTIMAL STRATEGIES IN A 2-PERSON, O-SUM MATRIX GAME

By

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I. Introduction. A 2–person zero–sum matrix game Γ is defined by an m×n matrix A, the set X of m–dimensional probability vectors x (strategies for player one, P_1), set Y of n–dimensional probability vectors y (strategies for player two, P_2), and a payoff function ϕ defined on $X\times Y$ by $\phi(x,y)=xAy^t$. A pair (x',y') of strategies for P_1 and P_2 respectively is said to be a pair of optimal strategies (for P_1 and P_2) if, for all $x\epsilon X, y\epsilon Y$, $\phi(x',y)\geq\phi(x',y')\geq\phi(x,y')$; in this case, $v=\phi(x',y')$ is called the value of the game Γ. An index i is essential for P_1 if there is an optimal strategy for $P_1, x'=(x_1', x_2', \cdots, x_m')\epsilon X$, such that $x_i'>0$; an index j is essential for P_2 if there is an optimal strategyfor $P_2, y'=(y_1' y_2', \cdots, y_n')\epsilon Y$, such that $y_j'>0$.

We present a necessary and sufficient condition that an optimal $x'\epsilon X$ for P_1 be unique for P_1, and that an optimal $y'\epsilon Y$ for P_2 be unique for P_2.

II. Let (x',y') be a pair of optimal strategies for P_1 and P_2, respectively. Let $I=\{i;i=1,2,\cdots,m\}, J=\{j;j=1,2,\cdots,n\}$, $I'=\{i\epsilon I; x_i'>0\}, \sim I'=\{i\epsilon I; i\bar{\epsilon} I'\}, J'=\{j\epsilon J; j$ is essential for $P_2\}$.

Theorem: *x' is unique for P_1 if and only if for every $I''\subseteq I' (I''\neq\phi)$ there do not exist m-dimensional probability vectors p and r such that for all $i\epsilon\sim I''=I-I'', p_i=0$; for all $i\epsilon I'', r_i=0$,* and *for all $j\epsilon J'$*

$$\sum_{i\epsilon I''} a_{ij} p_i = \sum_{i\epsilon\sim I''} a_{ij} r_i.$$

Proof: Suppose there exist vectors p, r satisfying the above conditions; then let $P=\{i\epsilon I;\ p_i>0\}, R=\{i\epsilon I; r_i>0\}$, and $Q=\{i\epsilon I; p_i=0$ and $r_i=0\}$.

(1) Since, for each $j\epsilon J'$, $\sum_{i\epsilon P} a_{ij} p_i = \sum_{i\epsilon R} a_{ij} r_i$ and

$$\max_{i\epsilon P}\{p_i\}>0,\quad \max_{i\epsilon R}\{r_i\}>0,\quad \min_{i\epsilon P}\{x_i'\}>0,\quad \max_{i\epsilon R}\{x_i'\}<1,$$

there is a $d_1>0$ and $d_2>0$ such that

$$d_1\leq(\min_{i\epsilon P}\{x_i'\})/(\max_{i\epsilon P}\{p_i\}),\qquad d_2\leq(1-\max_{i\epsilon R}\{x_i'\})/(\max_{i\epsilon R}\{r_i\}).$$

Let $d=\min(d_1, d_2)$ and

$$x_i''=x_i' \qquad \text{for } i\epsilon Q$$
$$x_i''=x_i'+r_i d \qquad \text{for } i\epsilon R$$
$$x_i''=x_i'-p_i d \qquad \text{for } i\epsilon P.$$

Then x'' is a probability vector of dimension m, and we now need only to show that x'' is also optimal for P_1.

A lemma from *Gale and Sherman* [1] asserts

x is optimal for P_1 if and only if for each $j\epsilon J'$,

$$\sum_{i\epsilon I} a_{ij} x_i = v.$$

But, for each $j\epsilon J'$, $\sum_{i\epsilon I} a_{ij} x_i'=v$ and, for each $j\epsilon J'$ $\sum_{i\epsilon I} a_{ij} x_i''=\sum_{i\epsilon I} a_{ij} x_i'+\sum_{i\epsilon R} a_{ij} r_i d-\sum_{i\epsilon P} a_{ij} p_i d$. It then follows from (1) that $\sum_{i\epsilon I} a_{ij}x_i''=\sum_{i\epsilon I} a_{ij}x_i'=v$ for each $j\epsilon J'$ and thus $x''\neq x'$ is also optimal for P_1.

If, on the other hand, x' is not unique for P_1, let x', x'' be optimal strategies for $P_1, x'\neq x''$; let $Q=\{i\epsilon I; x_i'=x_i''\}$, $P=\{i\epsilon I; x_i'>x_i''\}$, and $R=\{i\epsilon I; x_i'<x_i''\}$.

Then, for each $j\epsilon J''$,

$$\sum_{i\epsilon I} a_{ij} x_i'=\sum_{i\epsilon I} a_{ij} x_i''=v \text{ and } \sum_{i\epsilon Q} a_{ij} x_i'=\sum_{i\epsilon Q} a_{ij} x_i'', \text{ so that}$$

$$\sum_{i\epsilon P} a_{ij}(x_i'-x_i'')=\sum_{i\epsilon R} a_{ij}(x_i''-x_i'). \tag{2}$$

Also, since $\sum_{i\epsilon\sim Q} x_i' = \sum_{i\epsilon\sim Q} x_i''$, we have $\sum_{i\epsilon P}(x_i'-x_i'')=\sum_{i\epsilon R}(x_i''-x_i')=b$.

Let

$$p_i=(x_i'-x_i'')/b \qquad \text{for } i\epsilon P$$
$$p_i=0 \qquad \text{for } i\epsilon Q\cup R;$$

let

$$r_i=(x_i''-x_i')/b \qquad \text{for } i\epsilon R$$
$$r_i=0 \qquad \text{for } i\epsilon P\cup Q.$$

Clearly, $p=(p_1, \cdots, p_m)$ and $r=(r_1. \cdots, r_m)$ are m-dimensional probability vectors; for all $i\epsilon\sim P, p_i=0$, and for all $i\epsilon P\subseteq I', r_i=0$. It follows from (2) that, for each $j\epsilon J'$,

$$\sum_{i\epsilon P} a_{ij}p_i=\sum_{i\epsilon P} a_{ij}(x_i'-x_i'')/b=\sum_{i\epsilon R} a_{ij}(x_i''-x_i')/b=\sum_{i\epsilon R} a_{ij} r_i,$$

so that the vectors p and r satisfy the conditions of our theorem, and the proof is complete.

The corresponding result and proof for uniqueness of optimal strategy for P_2 is similar.

References

[1] Gale, D. and S. Sherman. ***Solutions of Finite Two–Person Games.*** Annals of Mathcmatics Study 24, pp. 37–49. Princeton University Press. 1950.

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