SOME METHODS OF FORMAL PROOFS II [1)]

[Generalization of the satisfiability definition]

By

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The syntactic picture of the generalization of the satisfiability definition considered in [3] [2)] and recalled on p 36 gives a generalization of usual proof rules in notion of sequents according to which it is constructed a generalized diagram, s. [2].

In view of cited papers the proof of adequacy of semantic and syntactic considerations is also given.

Examples to different sequent proof rules are given in [8], [9].

We consider the first-order functional calculus [3]-[6] based on alternative+, negation′ and general quantifier Π with free variables $x, x_1, \cdots$, apparent variables $a, a_1, \cdots$, relations signs $f_1^1, \cdots, f_q^1, \cdots, f_1^t, \cdots, f_q^t$ [f_i^m—of m arguments] and expressions $E, F, E_1, F_1, \cdots$.

Other more important notations written shortly : $\{i_l\}$ denotes the sequence $i_1, \cdots, i_l$; $\{i_{W(E)}\}$— sequence of all different indices of free variables occurring in E ; $w(E)$— the maximal number of indices of free [$p(E)$—apparent] variables occurring in E ; $\{F_q^t\}$— the sequence $F_1^1, \cdots, F_q^1, \cdots, F_1^t, \cdots, F_q^t$; $n(E) = \max \{p(E)+w(E), \max \{i_{W(E)}\}\}$; $Q, Q_1, \cdots$— non-empty sets of tables of the same rank ; $Q(k)$—elements of Q have the rank k ; $A, A_1, \cdots$—sets of indecomposable formulas [i. e. atomic formulas with their negations] whose indices of free variables are $\leqslant k$ for which $E \epsilon A . \equiv . E' \bar{\epsilon} A$ [they are called sets of the rank k] ; $\Gamma, \Gamma_1, \cdots$— arbitrary sets of formulas ; $X, Y, X_1, Y_1, \cdots$—models M or sets A ; $M/s_1, \cdots, s_k/ = <D_k, \{\phi_q^t\}> . \equiv . \{(M = <D, \{F_q^t\}>) \wedge (\phi_j^i(r_1, \cdots, r_i) . \equiv . F_j^i(s_{r_1}, \cdots, s_{r_i}), i = 1, \cdots, t, \text{ and } j = 1, \cdots, q)\}$; $E \epsilon A/s_1, \cdots, s_k/ . \equiv . E(x_{s_1}/x_1) \cdots (x_{s_k}/x_k) \epsilon A$ [3)] ; $A/s_1, \cdots, s_k/$ is restricted to indecomposable formulas ; $X/\{s_k\} = X/s_1, \cdots, s_k/$ (in three last definitions occur homomorphisms) ; $X \epsilon Y[k] . \equiv . (\exists \{s_k\})(X = Y/\{s_k\})$; $C\{E\}$—the set of all parts of E ; $\Gamma(\{i_l\})$— the sets of all formulas

1) The paper is connected with my lectures on J. Slupecki's seminar in 1956/7 and on meetings of Polish Mathematical Society in 1957 year at Wroclaw ; one is independent on the paper I.

2) It is a simple modification of the generalized satisfiability definition considered in [3] which gives the same generalized sequent proof rules.

The syntactic picture of this generalized satisfiability definition in many valued Boolean propositional calculus with quantifiers enables a simulteneously proof Godel-Skolem-Lovenheim-Herbrand's theorems for all this calculus including first-order functional calculus, s. [12].

3) $E(x/y)$—substitution x for y with known restrictions.

belonging to Γ whose free variables have indices occurring in $\{i_l\}$; $M\{E\}=0$, i.e. E′ is true in the model M; $M\{E\{s_k\}\}=0$, i.e. $\{s_k\}$ are elements of the domain of M, x_j are names of s_j and $\{s_k\}$ do not satisfy E in the model M; let $T=<D_k, \{F_q^t\}>$, T, A — have the same rank k and for each $m_1, \cdots, m_j \leqslant k$ and $j \leqslant t, i \leqslant q : F_i^j(m_1, \cdots, m_j). \equiv . f_i^j(x_{m_1}, \cdots, x_{m_j}) \epsilon A$ and $\sim F_i^j(m_1, \cdots, m_j). \equiv . f_i^{j'}(x_{m_1}, \cdots, x_{m_j}) \epsilon A$ — such T is called the description of A; $R(M). \equiv . (s_1)(s_2)\{M/s_1/=M/s_2/) \rightarrow (s_1=s_2)\}$ — each model M may be extended to a model M_1 such that $R(M_1)$, by means of a denumerable sequence of monadic relations, s. e. g. [11].

Of course:

L. 1. $X/\{s_k\}/\{j_m\}=X/\{s_{j_m}\}$, s. [1].

L. 2. If T_1 is the description of A_1 and T_2 is the description of A_2 and both tables have the same rank, then $T_1/\{j_m\}=T_2/\{j_m\} . \equiv . A_1/\{j_m\}=A_2/\{j_m\}$.

For an arbitrary Q such that $Q(k)$, for an arbitrary $T=<D_k, \{F_q^t\}> \epsilon Q$, for an arbitrary formula F and arbitrary $\{i_l\}$ such that $\{i_{w(F)}\} \subset \{i_l\}, l+p(F) \leqslant k$ we introduce the following inductive definition of the functional V:

(1d) $V\{k, Q, T, \{i_l\}, f_j^m(x_{r_1}, \cdots, x_{r_m})\}=1. \equiv . F_j^m(r_1, \cdots, r_m)$,

(2d) $V\{k, Q, T, \{i_l\}, F'\}=1. \equiv . \sim V\{k, Q\, T, \{i_l\}, F\}=1. \equiv . V\{k, Q, T, \{i_l\}, F\}=0$,

(3d) $V\{k, Q, T, \{i_l\}, F+G\}=1. \equiv . V\{k, Q, T, \{i_l\}, F\}=1 \vee V\{k, Q, T, \{i_l\}, G\}=1$.

(4d) $V\{k, Q, T, \{i_l\}, \Pi aF\}=1. \equiv . (i)(T_1)\{i \leqslant k) \wedge (T_1 \epsilon Q) \wedge (T_1/\{i_l\}=T_2/\{i_l\}) \Rightarrow$
$V\{k, Q, T_1, \{i_l\}, i, F(x_i/a)\}=1\}$.

D. 1. $N(k, Q, G). \equiv . (\{i_l\})(T)\{(l+p(F)<k) \wedge (\{i_{w(G)}\} \subset \{i_l\})$
$\Rightarrow (V\{k, Q, T, \{i_l\}, G\}=1. \equiv . V\{k, Q, T, \{i_l\}, i, G\}=1)\}$.

D. 2. $E \epsilon P(k, Q, T, \{i_l\}). \equiv . (\exists G)\{(G \epsilon C\{E\}) \wedge (N(k, Q, G)$
$\Rightarrow V\{k, Q, T, \{i_l\}, E\}=1)\}$.

D. 3. $E \epsilon P\{k\}. \equiv . (Q)(T)\{Q(k) \wedge (T \epsilon Q) \Rightarrow (E \epsilon P(k, Q, T, \{i_{w(E)}\})))\}$.

D. 4. $E \epsilon P. \equiv . (\exists k)\{(k \geqslant n(E)) \wedge (E \epsilon P\{k\})\}$.

The relation $N(k, Q, G)$ is invariant respectively to the sequences $\{i_l\}$ and it holds for all quantifierless formulas G.

Definitions (1d)—(4d) are generalizations of the satisfiability definition in the domain of natural numbers $1, \cdots, k$; the general case is analogic and remains for readers, s. [3], [5], [6].

If we assume that Q is one- elementing, then (4d) is the usual satisfiability

1) The reader will omit this extension by means of extending the family $Q=M[k]$ in T. 1 according to properties needed in T. 5 s. [12].

definition in the domain $1, \cdots, k$.

If M is a model and $Q=M[k]$, then elements of Q are submodels of M, the number i in (4d) is the name of an arbitrary element of the domain of M.

Of course:

(4d') $V\{k, Q, T, \{i_l\}, \Pi aF\}=0. \equiv .(\exists i)(\exists T_1)\{(i \leqslant k)\wedge(T_1 \epsilon Q)\wedge(T_1/\{i_l\}=T/\{i_l\}) \wedge V\{k, Q, T_1, \{i_l\}, i, F(x_i/a)\}=0\}$.

(5d') $V\{k, Q, T, \{i_l\}, \Sigma aF\}=0. \equiv .(i)(T_1)\{(i \leqslant k)\wedge(T_1 \epsilon Q)\wedge(T_1/\{i_l\}=T/\{i_l\}) \Rightarrow V\{k, Q, T_1, \{i_l\}, i, F(x_i/a)\}=0\}$.

L. 3. If $T/\{i_l\}=T^0/\{i_l\}$, then:

$V\{k, Q, T, \{i_l\}, E\}=1. \equiv . V\{k, Q, T^0, \{i_l\}, E\}=1.$

The proof of L. 3. is easy and inductivel on the length of the formula E, s. L. 3. in [3] and L. 14. in [6].

T. 1. If E is an alternative of formulas of the form $\Sigma a_1 \cdots \Sigma a_{j-1} \Pi a_j G$, for some quantifierless and variable-free $G, F \epsilon C\{E\}$, $M\{E\}=0$, $k \geqslant n(E)$, $Q=M[k]$, $T \epsilon Q$, $\{i_{w(F)}\} \subset \{i_l\}$, then:

(1) If $l+p(F) \leqslant k$, $M\{F\{i_l\}\}=0$ and $M/\{s_{i_l}\}=T/\{i_l\}$, then $V\{k, Q, T, \{i_l\}, F\}=0$ and for each $H \epsilon C\{E\}$ we have $N(k, Q, H)$ and $E \bar{\epsilon} P$.

(2) If $R(M)$, $M/\{s_{i_l}\}=T/\{i_l\}$, then for an arbitrary formula F:

$$M\{F\{s_{i_l}\}\}=0. \equiv . V\{k, Q, T, \{i_l\}, F\}=0.$$

The inductive proof of T. 1. is almost identical with the proof of T. 2. in [3], [5] and [10] and analogic to the proof of T. 2. in [4]; we give it in [10]. We point out here that in the proof of (1) we use (4d') for $i \bar{\epsilon} \{i_l\}$

T. 2. If $E_1, \cdots, E_r$ is a formalized proof of the formula E and $k \geqslant \max\{n(E_1), \cdots, n(E_r)\}$, then $E_j \epsilon P\{k\}, j=1, \cdots, r$.

From T. 1. and T. 2. follows [s. also the construction of Skolem's normal forms]:

T. 3. A formula E is a thesis iff $E \epsilon P$.

We recall that for normal formulas E we received more strong theorem given in [3] namely that for ones we can replace D. 2. by:

D. 2'. $E \epsilon P(n, Q, T, \{i_l\}). \equiv . N(Q, n, E) \Rightarrow V\{n, Q, T, \{i_l\}, E\}=1$ and the second equivalence in D. 1. we can replace by the implication $\Rightarrow$.

In order to give sequent proof rules we introduce certain additional definitions, s. [2]:

A sequence of formulas is called fundamental iff E and E' occur in the sequence.

For each k, Γ, F and $\{i_l\}$:

1. $\boldsymbol{x}_i$ means the first variable x_i such that $i \leqslant k$ and $F(x_i/a) \bar{\epsilon} \Gamma$.
2. $\boldsymbol{x}_i'$ means the first variable x_i such that $i \leqslant k$ and i does not belong to $\{i_l\}$.

We explain the meaning of the following equivalence schemas written shortly, s. figure p. 5 :

$\dfrac{\Gamma_1}{\dfrac{\Gamma_2}{\cdots\cdots}}$ —from Γ_1 follows Γ_2 and from Γ_2 follows $\cdots$ etc. ; such schema is called a diagram [we assume $\Gamma_1, \Gamma_2, \cdots$ –non–empty] .

$\dfrac{\Gamma}{\Gamma_1 \mid \Gamma_2}$ —from Γ follows Γ_1 or Γ_2 ; the rule determines two diagrams ; the prolongation of Γ is Γ_1 in the first diagram and Γ_2 in the second one.

All above schemas determine one last line and the following

$\dfrac{\Gamma}{\Gamma_1 \overset{|}{\circ}\, \Gamma_2 \overset{|}{\circ} \cdots}$ —determines many last lines according to the number of $\overset{|}{\circ}$ and it means: from Γ follows Γ_1 and Γ_2 and$\cdots$; the prolongation of Γ is Γ_1 and the prolongation of Γ_i is Γ_{i+1}.

The given schemas are called proof rules. Composition of such proof rules according to a diagram is called a generalized diagram or a generalized tree ; all proof rules we apply in a generalized diagram which describes the work of ones, s. figure p. 39.

In order to give proof rules for a given natural number k we assume here that we ordered all sequences $\{i_l\}$ such that $\{i_l\} \leqslant k$ and then :

According to the interpretation E as 0 and according to the generalized satisfiability definition we apply to an arbitrary formula E–called a topformula–the following proof rules :

$(A)\ \dfrac{\Gamma, F+G}{\Gamma, F, G}$; $(AN)\ \dfrac{\Gamma, (F+G)'}{\Gamma, F' \mid \Gamma, G'}$; $(N)\ \dfrac{\Gamma, F''}{\Gamma, F}$; $(\Pi_1)\ \dfrac{\Gamma, (\Pi a\, F)'}{\Gamma, (\Pi a\, F)', F'(\boldsymbol{x}_i/a)}$

—if $i=k$, then we do not apply further the rule to the formula $(\Pi_a F)'$ with explanation given below.

$(\Pi_2)\ \dfrac{\Gamma, \Pi_a F}{\Gamma \overset{|}{\circ}\, \Gamma(\{i_l\}), F(\boldsymbol{x}'_i/a) \overset{|}{\circ} \cdots}$ —we begin to apply the right hand side for $l=w(F)$ and afterwards we apply only the right hand side to the next sequence $\{i_l\}$ of numbers$\leqslant k$ in the given order such that $\{l_l\} \supset \{i_{w(F)}\}$ and $l+p(F) \leqslant k$ which determines a new last line till the last such sequence ; columns determined here by $\Gamma(\{i_l\}), F\boldsymbol{x}'_i/a)$ must be equal with Γ on indecomposable formulas with free variables of indices $\{i_l\}$ [it suffices to assume the property for last lines of these columns].

A generalized diagram is correct iff for its two columns Γ_1, Γ_2 :

1. If $\Gamma_1 / \{i_l\} = \Gamma_2 / \{i_l\}$,[1] $\{i_l\} \supset \{i_{w(F)}\}$, $l+p(F) \leqslant k$, then :

1) I. e. columns Γ_1 and Γ_2 are equal on indecomposable formulas with indices of free variables belonging to $\{i_l\}$.

a/ If $\Pi a F$ occurs in the column Γ'_1, then $\Pi a F$ occurs in the colum Γ'_2 (it means that the property (Π_2) must hold for Γ'_1 and Γ'_2);

b/ If $(\Pi a F)'$ occurs in the column Γ'_1, then $(\Pi a F)'$ occurs in the column Γ'_2;

2. If Γ is a column and $F \in C\{E\}$, then either F belongs to Γ or F' belongs to Γ.

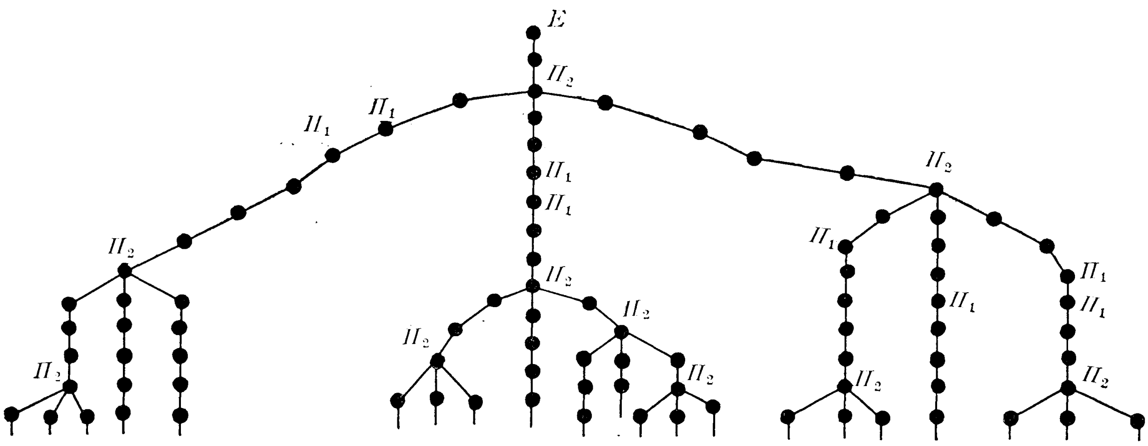
The above points 1 and 2 mean that if for a generalized diagram points 1 and 2 are not fulfilled, then we add to suitable columns respective formulas, i. e. in point 1a) we add the formula $\Pi a F$, in point 2 formulas F or F' and afterwards we act according to the introduced sequent rules.

In the following we consider only correct diagrams.

In the case of a classical diagram we assume that all columns are equal, i. e. we have only one column [1]; therefere all assumptions about columns are here less and we receive an usual sequent proof, s. [2]; thus our proof rules are generalizations of classical ones.

According to the considered proof rules each formula E determines a generalized diagram composed of columns with the main top E.

Work scheme of a generalized diagram



Each column determines a new last line; lines are denoted by Circles; (Π_1) and (Π_2) denote applications of the rules (Π_1) and (Π_2) respectively; dots denote prolongation of the diagram aecording to the proof rules described abeve and properties 1–2 of a correct diagram.

T. 4. If for each $k \geqslant n(E)$ all lines of each column of a certain generalized diagram of E are not fundamental, then E is not a thesis.

1) The precisation of the rule (Π_2) in this case remains for readers.

Proof: In order to prove T. 4. for a given formula E we consider a natural number $k \geqslant n(E)$ and the generalized diagram of E with properties described in the theorem. Each last line we consider as a set A of formulas of the rank k (completion of the last line to a set of formulas of the rank k is here arbitrary) and to each set A we attribute the description T of negated indecomposable formulas belenging to A (therefore A and T have the rank k) and the family of all such $T's$ creates the set Q of tables of the rank k [1].

We point out, each last line A determines the described table T and a column $\mathfrak{T}$ with the basis A and the top E.

We prove by induction on the length of a formula H:

(1) If $H \epsilon \mathfrak{T}$, then $V\{k, Q, T, \{i_l\}, H\}=0$, for each $\{i_l\} \supset \{i_{w(H)}\}$ such that $l+p(H) \leqslant k$.

For atomic formulas and their negation (1) holds by the assumption.

Let (1) hold for formulas of the length $< m$; we shall prove it for formulas H of the length m.

We consider here three cases:

1. $H=F+G$, for some F, G,
2. $H=F'$, for some F,
3. $H=\Pi a\, F$, for some F.

In the case $H=F+G \epsilon \mathfrak{T}$ by virtue of (A) we receive $F, G \epsilon \mathfrak{T}$; therefore by the inductive assumption $V\{k, Q, T, \{i_{l'}\}, F\}=0$, for each $\{i_{l'}\} \supset \{i_{w(F)}\}$, $l+p(F) \leqslant k$ and $V\{k, Q, T, \{i_{l''}\}, G\}=0$, for each $\{i_{l''}\} \supset \{i_{w(G)}\}$, $l''+p(G) \leqslant k$; therefore by (3d') $V\{k, Q, T, \{i_l\}, F+G\}=0$, for each $\{i_l\} \supset \{i_w(F+G)\}$, $l+p(F+G) \leqslant k$, which proves (1) in the firrst case.

In the case $H=F'$ we consider three cases:

(1°) $F=F_1'$, for some F_1,

(2°) $F=F_1+G_1$, for some F_1, G_1,

(3°) $F=\Pi a\, F_1$, for some F_1.

In the case $F=F_1'$ we have by assumption $H=F_1'' \epsilon \mathfrak{T}$; therefore by (N) we have $F_1 \epsilon \mathfrak{T}$. Hence by means of the inductive assumption $V\{k, Q, T, \{i_l\}, F_1\}=0$, for each $\{i_l\} \supset \{i_{w(F_1)}\}$, $l+p(F_1) \leqslant k$ and because $w(F_1)=w(H)$ we have also $V\{k, Q, T, \{i_l\}, H\}=0$, for each $\{i_l\} \supset \{i_{w(H)}\}$, $l+p(H) \leqslant k$, which proves (1) in the case (1°).

1) In the classical case we must also consider $k=\aleph_0$ and then the diagram has only one column and T is the description of negations of indecomposable formulas belonging to the column.

In the case $F=F_1+G_1$ we have by assumption $H=(F_1+G_1)'\epsilon\mathfrak{T}$; therefore by (AN) $F_1'\epsilon\mathfrak{T}$ or $G_1'\epsilon\mathfrak{T}$.

We consider here the case $F_1'\epsilon\mathfrak{T}$; the case $G_1'\epsilon\mathfrak{T}$ is analogic.

From the above by means of the inductive assumption $V\{k, Q, T, \{i_{l'}\}, F_1'\}=0$, for each $\{i_{l'}\}\supset\{i_{w(F'_1)}\}$, $l'+p(F_1')\leqslant k$; therefore by (2d) $V\{k, Q, T, \{i_{l'}\}, F_1\}=1$ for each $\{i_{l'}\}\supset\{i_{w(F_1)}\}$, $l'+p(F_1)\leqslant k$, and by (3d) and (2d) we obtain respectively $V\{k, Q, T, \{i_l\}, (F_1+G_1)'\}=0$, for each $\{i_l\}\supset\{i_{w(F_1+G_1)}\}, l+p((F_1+G_1)')\leqslant k$, i. e. $V\{k, Q, T, \{i_l\}, H\}=0$, for each $\{i_l\}\supset\{i_{w(H)}\}, l+p(H)\leqslant k$, what proves (1) in the case (2°).

In the case $F=\Pi a F_1$ we have by assumption $(\Pi a F_1)'\epsilon\mathfrak{T}$; therefore by the property 1b) of the correct diagram and (Π_1) for each $\{i_l\}\supset\{i_{w(F_1)}\}, l+p(F_1)\leqslant k$, for every $i\leqslant k$, and for each $\mathfrak{T}_1$ if $\mathfrak{T}_1/\{i\}=\mathfrak{T}/\{i_l\}$, then $(\Pi a F_1)'\epsilon\mathfrak{T}_1$ and $F_1'(x_i/a)$ $\epsilon\mathfrak{T}_1$; hence by the construction of Q, L. 2. and the inductive hypothesis for each $\{i_l\}\supset\{i_{w(F_1)}\}, l+p(F_1)\leqslant k$, for every $i\leqslant k$ and for each $T_1\epsilon Q$, if $T_1/\{i_l\}=T/\{i\}$, then $V\{k, Q, T_1, \{i_l\}, i, F_1'(x_i/a)\}=0$ and $V\{k, Q, T_1, \{i_l\}, i, F_1(x_i/a)\}=1$. Therefere by virtue of (4d) $V\{k, Q, T, \{i_l\}, \Pi a F_1\}=1$, for each $\{i_l\}\supset\{i_{w(F_1)}\}$, $l+p(F_1)\leqslant k$, and therefore by (2d) $V\{k, Q, T, \{i_l\}, H\}=0$, for each $\{i\}\supset\{i_{w(H)}\}$, $l+p(H)\leqslant k$, what proves (1) in the case (3°).

In the last case $H=\Pi a F\epsilon\mathfrak{T}$. Hence in view of the construction of the generalized diagram and (Π_2) for each $\{i_l\}\supset\{i_{w(F)}\}, l+p(F)\leqslant k$, there exists $\mathfrak{T}_1$ such that $\mathfrak{T}/\{i_l\}$ and $F(\boldsymbol{x}_i'/a)\epsilon\mathfrak{T}_1$. Hence in view of the definition of Q, L. 2., and the indudtive hypothesis for each $\{i_l\}\supset\{i_{w(F)}\}, l+p(F)\leqslant k$, there exists $i\leqslant k$ and $T_1\epsilon Q$ such that $T_1/\{i_l\}=T/\{i_l\}$ and $V\{k, Q, T_1, \{i_l\}, i, F(x_i/a)\}=0$.

Thus by virtue of (4d') $V\{k, Q, T, \{i_l\}, \Pi a F\}=0$, for each $\{i_l\}\supset\{i_{w(\Pi aF)}\}, l+p$ $(\Pi a F)<k$, and also $V\{k, Q, T, \{i\}, H\}=0$, for each $\{i_l\}\supset\{i_{w(H)}\}, l+p(H)\leqslant k$, what proves (1) in the last case (3).

Thus we closed the inductive proof of (1).

Therefore for formulas H belonging to the generalized diagram we proved $N(k, Q, H)$.

If now $H\epsilon C\{E\}$, then in view of the construction of a correct generalized diagram, property 2, either H belongs to each column of the generalized diagram or H' belongs to the same column.

Therefore in view of the above we have $N(k, Q, H)$ for each $H\epsilon C\{E\}$.

Beeause E belongs to the diagram, therefore even for each $T\epsilon Q$ we have $V\{k, Q, T, \{i_l\}, E\}=0$, for each $\{i_l\}\supset\{i_{w(E)}, l+p(E)\leqslant k$, and therefore $E\bar{\epsilon}P\{k\}$.

From the above and the assumption we obtain $E\bar{\epsilon}P$ and therefore in view of

T. 2. E is not a thesis.

According to our explanation, s. footnotes on p. 37, the proof also holds in the classical case (then we have only one column).

T. 5. If a line of a certain column of each generalized diagram of E for certain $k \geqslant n(E)$ is fundamental, then E is a thesis.

Proof. In contrary, if E is not a thesis, then according to T. 1. $E \bar{\epsilon} P$; therefere for each $k \geqslant n(E)$ there exists such set Q of tables of the rank k and there exists a table $T \epsilon Q$ such that for each $\{i_l\} \supset \{i_{n(E)}\}$ we have $V\{k, Q, T, \{i\}, E\} = 0$ [1] and for each $G \epsilon C\{E\}$: $N(k, Q, G)$. Then the generalized satisfiability definition determines here a generalized diagram analogic to sequent proof rules which correspond to the above the diagram has no fundamental line in contrary to the assumption of T. 5.

From T. 4. and T. 5. follows:

T. 6. A formula E is a thesis iff its each generalized diagram has a fundamental line for certain $k \geqslant n(E)$.

Other sequent proof rules based on my papers are considered in [7]-[9]; they different kind corresponds to different characterizations (satisfiability definitions) of theses of the first order functional calculus presented in my papers.

Examples are given in [8] and [9].

1) with the remark given in the proof of T. 1. (4d').

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