A NOTE ON SIMPLICIAL COLLAPSING

By

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(Received October 4, 1967)

In this paper we will let L denote a simplicial complex, K a subcomplex of L, S^n a simplicial complex consisting of all i-simplexes of an $n+1$ simplex, $i \leqq n$, and $S^n K$ the simplicial complex which is the join of S^n and K. We denote the r-th derived subdivision of L by $L^{(r)}$ and denote a simplicial collapse of L onto K by $L \searrow K$ [1].

Theorem I. *$L^{(r)} \searrow K^{(r)}$ if and only if $(S^0 L)^{(r)} \searrow (S^0 K)^{(r)}$.*

Before proceeding with the proof we give the following definision. Let a and b be the vertices of S^0. We will define an $n+1$ cell C_i in $(S^0 L)^{(r)}$ containing the n-simplex $\varDelta_i$ of $L^{(r)}$ and vertices a, b by induction on n and r as follows:

(i) If $r=0$ and $\varDelta_i$ is an n-simplex of $L^{(0)}$, then C_i will be the join of $\varDelta_i$ with a and b.

(ii) If the $C's$ have been defined for $r-1$ and $\varDelta_i$ is an o-simplex of $L^{(r)}$, then $\varDelta_i$ is in the interior of some $\varDelta_j$ of $L^{(r-1)}$. Hence define C_i to be the 1-cell defined by a, b, and the vertices of the first derived subdivision of C_j in the interior of C_j.

(iii) If the $C's$ have been defined for $r-1$ and all $n-1$-simplexes of $L^{(r)}$ and $\varDelta_i$ is an n-simplex of $L^{(r)}$, then define C_i to be the $n+1$ cell bounded by $\cup C_j$, where C_j is the cell containing $\varDelta_j$ and $\varDelta_j$ is on the boundary of $\varDelta_i$.

We will first assume that $(S^0 L)^{(r)} \searrow (S^0 K)^{(r)}$ by the sequence of elementary simplicial collapses $(S^0 L)^{(r)} = K'_0 \searrow K'_1 \searrow K'_2 \cdots \searrow K'_m = (S^0 K)^{(r)}$. For each elementary simplicial collapse $K'_i \searrow K'_{i+i}$, through a simplex $\varDelta'_i$ from a simplex $\varDelta'_j$, there exist $C_i, C_j, \varDelta_i$, and $\varDelta_j$ such that (interior of $\varDelta_i$) $\cup$ (interior of $\varDelta'_i$) $\subset$ (interior of C_i), (interior of $\varDelta_j$) $\cup$ (interior of $\varDelta'_j$) $\subset$ (interior of C_j), and $\varDelta_i, \varDelta_j$, are simplexes of $L^{(r)}$. Hence we can show that $L^{(r)}$ simplicially collapses to $K^{(r)}$ by $L^{(r)} = K_0 \searrow K_1 \searrow K_2 \cdots \searrow K_m = K^{(r)}$, where $K_i \searrow K_{i+1}$ denotes the elementary simplicial collapse of $\varDelta_i$ from $\varDelta_j$ or $K_i = K_{i+1}$ if $\varDelta_i$ has been previously collapsed. If we could not collapse $\varDelta_i$ from $\varDelta_j$ then there would exist a simplex $\varDelta_k$ of $L^{(r)}$ such that:

(1) $\varDelta_j$ is a face of $\varDelta_k$ and $\varDelta_k \neq \varDelta_i$.

(2) $C_i \cap C_k = C_j$ and no simplex of C_k has been collapsed.

Thus $\varDelta'_i$ could not have been collapsed from $\varDelta'_j$ as $\varDelta'_j$ is on the boundary of C_k. However this contradicts the fact that $K'_i \searrow K'_{i+1}$ is an elementary simplicial collapse of $\varDelta'_i$

from Δ'_j.

We shall now assume that $L^{(r)} \searrow K^{(r)}$ by the seqnence of elementary simplicial collapses $L^{(r)} = L_0 \searrow L_1 \searrow L_2 \cdots \searrow L_{m'} = K^{(r)}$. For each elementary simplicial collapse $L_i \searrow L_{i+1}$ of the n-simplex Δ_i from Δ_j we will define the following simplicial collapses of $(S^0 L)^{(r)}$:

(i) Let Δ_k be an $n+1$-simplex of C_i having a as a vertex.

(ii) Define $L'_{i_0} \searrow L'_{i_1}$ (where $L'_{i_0} = \cup C_c$ and Δ_c is a simplex of L_i) by collapsing Δ_k from its n dimensional face on C_j.

(iii) If A_p is a face of Δ_k having a as a vertex, then $(C_i - \cup \Delta_p)$ is a cone from the vertex b and having the n-face of Δ_k opposite a as base. Let A_b be the subcone of $(C_i - \cup A_p)$ having vertex b and base the uuion of all $n-1$ simplexes of Δ_k not in C_j and not having a as a vertex.

(iv) Define $L'_{i_1} \searrow L'_{i_2} \searrow L'_{i_3} \cdots \searrow L'_{i_s}$ to be the sequence of elementary simplicial collapsings of $(C_i - \cup \Delta_p)$ to A_b.

The sequence of elementary simplicial collapsings

$$\begin{aligned}(S^0 L)^{(r)} &= L'_{0_0} \searrow L'_{0_1} \cdots \searrow L'_{i_0} \searrow L'_{i_1} \cdots \searrow L'_{i_s} \\ &= L'_{i+l_0} \searrow L'_{i+l_1} \cdots \searrow (S^0 K)^{(r)}\end{aligned}$$

defines a simplicial collapse of $(S^0 L)^{(r)}$ onto $(S^0 K)^{(r)}$.

By a similar argument we can prove the following.

Theorem 2. *$(S^n L)^{(r)} \searrow (S^n K)^{(r)}$ if and only if $L^{(r)} \searrow K^{(r)}$.*

REFERENCES

1. E. C. Zeeman, *Seminar on Combinatorial Topology*, Mimeographed Notes, Inst. Hautes Etude Sci., 1963.

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