CERTAIN DOUBLE WHITTAKER TRANSFORMS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

By

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1. INTRODUCTION. *Rainville* [16, p. 104], and *Abdul-Halim and Al-Salam* [1] have shown that the single and double *Euler* transformations of the hypergeometric function ${}_pF_q$ are effective tools for augmenting its parameters. The double *Laplace* transforms of the ${}_pF_q$-function, given by *Singh* [17] and *Jain* [14], have similar interesting properties.

Recently, *Srivastava and Singhal* [20] have discussed a double *Meijer* transform of the generalized hypergeometric function in the form

$$(1.1)\qquad \int_0^\infty\int_0^\infty x^{\alpha-1}\,y^{\beta-1}\,(x+y)^{\mu+\frac{1}{2}}e^{-\frac{1}{2}\lambda(x+y)}\,K_\nu\left[\tfrac{1}{2}\lambda\,(x+y)\right]$$

$$\cdot\,{}_pF_q\left[\begin{matrix}a_1,\cdots,a_p\,;\\ b_1,\cdots,b_q\,;\end{matrix}\ t\,x^s\,y^k\,(x+y)^r\right]dx\,dy$$

$$=\frac{\sqrt{\pi}\,\Gamma(\alpha+\beta+\mu\pm\nu+\frac{1}{2})}{\lambda^{\alpha+\beta+\mu+\frac{1}{2}}\,\Gamma(\alpha+\beta+\mu+1)}\,B(\alpha,\beta)\,{}_{p+3s+3k+2r}F_{q+2s+2k+r}\left[\begin{matrix}a_1,\cdots,a_p,\\ b_1,\cdots,b_q,\end{matrix}\right.$$

$$\left.\begin{matrix}\Delta(s,\alpha),\,\Delta(k,\beta),\,\Delta(s+k+r,\alpha+\beta+\mu\pm\nu+\frac{1}{2})\,;\\ \Delta(s+k,\alpha+\beta),\ \Delta(s+k,\alpha+\beta+\mu+1)\,;\end{matrix}\ t\delta\left(\frac{s+k+r}{\lambda}\right)^{s+k+r}\right]$$

where r,s,k are non-negative integers; $R_e(\alpha)>0$, $R_e(\beta)>0$, $R_e(\lambda)>0$ and $R_e(\alpha+\beta+\mu\pm\nu+\frac{1}{2})>0$; $\Delta(s,\alpha)$ stands for the set of s parameters

$$\frac{\alpha}{s},\ \frac{\alpha+1}{s},\ \frac{\alpha+2}{s},\cdots,\ \frac{\alpha+s-1}{s},$$

and for the sake of brevity, the gamma-product

$$\Gamma(\alpha+\beta)\,\Gamma(\alpha-\beta)$$

is denoted by

$$\Gamma(\alpha\pm\beta),$$

and

$$\delta=\frac{s^s\,k^k}{(s+k)^{s+k}}\,.$$

In this paper we present a double *Whittaker* transform of the generalized hypergeometric functions which leads to yet another interesting process of augmenting the parameters in the ${}_pF_q$-function. As a first instance of the usefulness of the generalized operator which we introduce in the next section, we give an alternative derivation of *Kummer's* theorem and *Weisner's* bilateral generating function, and discuss its numerous other applications to certain classical polynomials and *Appell's* functions.

In the various formulae the parameters, occurring in the operators employed, are restricted for reasons of convergence and term-by-term integration.

2. THE GENERAL FORMULAE. We start with the following known result [10, p. 177]

$$\int_0^\infty \int_0^\infty \phi(x+y)\, x^{\alpha-1} y^{\beta-1}\, dx\, dy = B(\alpha, \beta) \int_0^\infty \phi(z)\, z^{\alpha+\beta-1} dz$$

which holds when $R_e(\alpha)>0$ and $R_e(\beta)>0$.

Following the technique of term-by-term integration it is easy to prove that, if $R_e(\alpha)>0$, $R_e(\beta)>0$ and if r, s, k are non-negative integers, then inside the region of convergence of the resulting series

$$\int_0^\infty \int_0^\infty \phi(x+y)\, x^{\alpha-1} y^{\beta-1}\, {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix}\; t\, x^s y^k (x+y)^r\right] dx\, dy \tag{2.1}$$

$$= B(\alpha, \beta) \int_0^\infty \phi(z)\, z^{\alpha+\beta-1}\, {}_{p+s+k}F_{q+s+k}\left[\begin{matrix} a_1, \cdots, a_p, \Delta(s, \alpha), \Delta(k, \beta); \\ b_1, \cdots, b_q, \Delta(s+k, \alpha+\beta); \end{matrix}\; t\,\delta\, z^{s+k+r}\right] dz,$$

where, as before,

$$\delta = \frac{s^s k^k}{(s+k)^{s+k}}.$$

In particular, if we let

$$\phi(z) = e^{-\lambda z} z^\sigma E[c, d :: \lambda z]$$

in terms of *MacRobert's* E-function, and invoke the formula [15, p. 396]

$$\int_0^\infty e^{-at} t^{b-1} E[c, d :: at]\, dt = \frac{\Gamma(c)\Gamma(d)\Gamma(b+c)\Gamma(b+d)}{a^b\, \Gamma(b+c+d)}$$

where $R_e(a)>0$, $R_e(b+c>0$, $R_e(b+d)>0$, we obtain

$$\int_0^\infty \int_0^\infty x^{\alpha-1} y^{\beta-1} e^{-\lambda(x+y)} (x+y)^\sigma E[c, d :: \lambda(x+y)]\; {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix}\; t x^s y^k (x+y)^r\right] dx dy \tag{2.2}$$

$$= \frac{\Gamma(c)\Gamma(d)\Gamma(\alpha+\beta+\sigma+c)\Gamma(\alpha+\beta+\sigma+d)}{\lambda^{\alpha+\beta+\sigma}\, \Gamma(\alpha+\beta+\sigma+c+d)} B(\alpha, \beta).$$

$${}_{p+3s+3k+2r}F_{q+2s+2k+r}\left[\begin{matrix} a_1, \cdots, a_p, \Delta(s, \alpha), \Delta(k, \beta), \Delta(s+k+r, \alpha+\beta+\sigma+d); \\ b_1, \cdots, b_q, \Delta(s+k, \alpha+\beta), \Delta(s+k+r, \alpha+\beta+\sigma+c+d); \end{matrix}\; t\,\delta\,\delta'\right],$$

provided the double integral converges, $R_e(\alpha)>0$, $R_e(\beta)>0$, $R_e(\lambda)>0$, $R_e(\alpha+\beta+\sigma+c)>0$, $R_e(\alpha+\beta+\sigma+d)>0$, and

$$\delta'=\left(\frac{s+k+r}{\lambda}\right)^{s+k+r}.$$

Since [11, p. 205]

$$E\left[\tfrac{1}{2}-\mu-\nu, \tfrac{1}{2}-\mu+\nu :: z\right]=\Gamma\left(\tfrac{1}{2}-\mu\pm\nu\right) z^{-\mu} e^{\frac{1}{2}z}\, W_{\mu,\nu}(z),$$

in terms of *Whittaker* function, he formula (2. 2) shows that inside the region of convergence of the double integral,

$$(2.3)\quad \int_0^\infty\int_0^\infty x^{\alpha-1} y^{\beta-1}(x+y)^\sigma e^{-\frac{1}{2}\lambda(x+y)} W_{\mu,\nu}[\lambda(x+y)]\cdot {}_pF_q\left[\begin{matrix}a_1,\cdots,a_p\,;\\ b_1,\cdots,b_q\,;\end{matrix}\; tx^s y^k (x+y)^r\right]dx\,dy$$

$$=\frac{\Gamma(\alpha+\beta+\sigma\pm\nu+\frac{1}{2})}{\lambda^{\alpha+\beta+\sigma}\,\Gamma(\alpha+\beta+\sigma-\mu+1)}B(\alpha,\beta)\cdot {}_{p+3s+3k+2r}F_{q+2s+2k+r}\left[\begin{matrix}a_1,\cdots,a_p, \Delta(s,\alpha), \Delta(k,\beta),\\ b_1,\cdots,b_q, \Delta(s+k,\alpha+\beta),\end{matrix}\right.$$

$$\left.\begin{matrix}\Delta(s+k+r,\alpha+\beta+\sigma\pm\nu+\frac{1}{2})\,;\\ \Delta(s+k+r,\alpha+\beta+\sigma-\mu+1)\,;\end{matrix}\; t\delta\delta'\right],$$

where r, s, k are non-negative integers, $R_e(\alpha)>0$, $R_e(\beta)>0$, $R_e(\lambda)>0$, $R_e(\alpha+\beta+\sigma\pm\nu+\frac{1}{2})>0$, and the resulting hypergeometric series converges.

When $\mu=0$, our formula (2. 3) corresponds to (1. 1), since

$$W_{0,\mu}(z)=\left[\frac{z}{\pi}\right]^{\frac{1}{2}} K_\mu\left[\frac{1}{2}z\right];$$

and if in addition, we set $\nu=\pm\frac{1}{2}$ and make use of the relation

$$\mathbf{K}_{+\frac{1}{2}}(z)=\left[\frac{\pi}{2z}\right]^{\frac{1}{2}} e^{-z},$$

the special case $\sigma=-\frac{1}{2}$ of (2. 3) will lead us to the earlier results of *Jain* [14] and *Singh* [17].

In terms of the operator

$$(2.4)\quad \Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\{\ \}\equiv\frac{\Gamma(\alpha+\beta+\sigma-\mu+1)}{\Gamma(\alpha+\beta+\sigma\pm\nu+\frac{1}{2})}[B(\alpha,\beta)]^{-1}\cdot\int_0^\infty\int_0^\infty x^{\alpha-1}y^{\beta-1}(x+y)^\sigma e^{-\frac{1}{2}(x+y)}$$

$$\cdot\, W_{\mu,\nu}[x+y]\{\ \}\,dx\,dy,$$

(2. 3) with $\lambda=1$ and $r=0$ assumes the form

$$(2.5)\quad \Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\left\{{}_pF_q\left[\begin{matrix}a_1,\cdots,a_p\,;\\ b_1,\cdots,b_q\,;\end{matrix}\; t x^s y^k\right]\right\}$$

$$={}_{p+3s+3k}F_{q+2s+2k}\left[\begin{matrix}a_1,\cdots,a_p, \Delta(s,\alpha), \Delta(k,\beta), \Delta(s+k,\alpha+\beta+\sigma\pm\nu+\frac{1}{2})\,;\\ b_1,\cdots,b_q, \Delta(s+k,\alpha+\beta),\ \Delta(s+k,\alpha+\beta+\sigma-\mu+1)\,;\end{matrix}\; ts^s k^k\right],$$

and it follows that

$$\Omega^{(0,\nu)}_{(\alpha,\beta,\sigma)}\{\ \}\equiv\Omega^{(\nu)}_{(\alpha,\beta,\sigma)}\{\ \} \tag{2.6}$$

where $\Omega^{(\nu)}_{(\alpha,\beta,\mu)}$ denotes the operator introduced earlier by *Srivastava and Singhal* [20, § 1].

The following operational relations are immediate consequences of (2.5): -

$$\Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\{1\}=1. \tag{2.7}$$

$$\Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\{e^{xt}\}={}_3F_2\left[\begin{matrix}\alpha,\alpha+\beta+\sigma\pm\nu+\frac{1}{2};\\ \alpha+\beta,\alpha+\beta+\sigma-\mu+1;\end{matrix}\ t\right]. \tag{2.8}$$

$$\Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\{{}_1F_0[a;-;xt]\}={}_4F_2\left[\begin{matrix}a,\alpha,\alpha+\beta+\sigma\pm\nu+\frac{1}{2};\\ \alpha+\beta,\alpha+\beta+\sigma-\mu+1;\end{matrix}\ t\right], \tag{2.9}$$

a being a non-positive integer.

$$\Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\{{}_1F_1[a;b;xt]\}={}_4F_3\left[\begin{matrix}a,\alpha,\alpha+\beta+\sigma\pm\nu+\frac{1}{2};\\ b,\alpha+\beta,\alpha+\beta+\sigma-\mu+1;\end{matrix}\ t\right]. \tag{2.10}$$

$$\Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\{{}_1F_1[a;b;t(x+y)]\}={}_3F_2\left[\begin{matrix}a,\alpha+\beta+\sigma\pm\nu+\frac{1}{2};\\ b,\alpha+\beta+\sigma-\mu+1;\end{matrix}\ t\right]. \tag{2.11}$$

$$\Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\{{}_0F_1[-;a;xt]\}={}_3F_3\left[\begin{matrix}\alpha,\alpha+\beta+\sigma\pm\nu+\frac{1}{2};\\ a,\alpha+\beta,\ \alpha+\beta+\sigma-\mu+1;\end{matrix}\ t\right]. \tag{2.12}$$

$$\Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\{{}_0F_1[-;a;t(x+y)]\}={}_2F_2\left[\begin{matrix}\alpha+\beta+\sigma\pm\nu+\frac{1}{2};\\ a,\alpha+\beta+\sigma-\mu+1;\end{matrix}\ t\right]. \tag{2.13}$$

$$\Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\{{}_2F_1[a,b;c;xt]\}={}_5F_3\left[\begin{matrix}a,b,\alpha,\alpha+\beta+\sigma\pm\nu+\frac{1}{2};\\ c,\alpha+\beta,\alpha+\beta+\sigma-\mu+1;\end{matrix}\ t\right], \tag{2.14}$$

where a or b is a non-positive integer.

$$\Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\{{}_2F_2[a,b;c,d;xt]\}={}_5F_4\left[\begin{matrix}a,b,\alpha,\alpha+\beta+\sigma\pm\nu+\frac{1}{2};\\ c,d,\alpha+\beta,\alpha+\beta+\sigma-\mu+1;\end{matrix}\ t\right]. \tag{2.15}$$

$$\Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\{x^m y^n(x+y)^p\}=\frac{(\alpha)_m(\beta)_n(\alpha+\beta+\sigma\pm\nu+\frac{1}{2})_{m+n+p}}{(\alpha+\beta)_{m+n}(\alpha+\beta+\sigma-\mu+1)_{m+n+p}}. \tag{2.16}$$

In particular, we shall need the following results.

$$\Omega^{(1,\frac{1}{2})}_{(\alpha',\beta',-1)}\{(ax)^{\mu+\nu-\lambda}J_\lambda(2ax)\}=\frac{a^{\mu+\nu}(\alpha')_{\mu+\nu}}{\Gamma(\lambda+1)}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}(\alpha'+\mu+\nu),\frac{1}{2}(\alpha'+\mu+\nu+1);\\ \lambda+1;\end{matrix}\ -4a^2\right]. \tag{2.17}$$

$$\Omega^{(1,\frac{1}{2})}_{(\alpha',\beta',-1)}\{J_\mu(ux)J_\nu(vx)\}=\frac{\Gamma(\alpha'+\mu+\nu)}{\Gamma(\alpha')\Gamma(\mu+1)\Gamma(\nu+1)}(\tfrac{1}{2}u)^\mu(\tfrac{1}{2}v)^\nu \tag{2.18}$$

$$\cdot F_4[\tfrac{1}{2}(\alpha'+\mu+\nu),\tfrac{1}{2}(\alpha'+\mu+\nu+1);\mu+1,\nu+1;-u^2,-v^2],$$

where F_4 is *Appell's* double hypergeometric function of the fourth type defined by means of [18, p. 211]

$$F_4[a,b;c,c';x,y]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(c')_n}\frac{x^m}{m!}\frac{y^n}{n!},$$

provided, for convergence, $|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}<1$.

(2.19) $\Omega^{(1,\frac{1}{2})}_{(\alpha',\beta',-1)}\{(ux)^m L^{(\alpha)}_n(ux)\}=u^m(\alpha')_m P^{(\alpha,\alpha'-\alpha+m-n-1)}_n(1-2u)$.

(2.20) $\Omega^{(1,\frac{1}{2})}_{(\alpha',\beta',-1)}\{(ux)^l(uy)^m L^{(\alpha)}_n[u(x+y)]\}=u^{l+m}(\alpha')_l(\beta')_m P^{(\alpha,\alpha'+\beta'+l+m-n-\alpha-1)}_n(1-2u)$.

(2.21) $\Omega^{(1,\frac{1}{2})}_{(\alpha',\beta',-1)}\{L^{(\mu)}_m(ux)L^{(\nu)}_n(vy)\}=P^{(\mu,\alpha'-\mu-m-1)}_m(1-2u)\,P^{(\nu,\beta'-\nu-n-1)}_n(1-2v)$.

(2.22) $\Omega^{(1,\frac{1}{2})}_{(\alpha',\beta',-1)}\{L^{(\mu)}_m(ux)L^{(\nu)}_n(vx)\}=\dfrac{(\mu+1)_m(\nu+1)_n}{m!\,n!}$.

$\cdot F_2[\alpha',-m,-n;\mu+1,\nu+1;u,v]$,

where F_2 is the *Appell* function [18, p. 211]

$$F_2[a,b,b';c,c';x,y]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_m(c')_n\,m!\,n!}x^m y^n,$$

which converges when $|x|+|y|<1$.

3. ALTERNATIVE DERIVATION OF KUMMER'S FIRST THEOREM.

From (2.16) we have

$$\Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\{e^{-t(x+y)}\,{}_1F_1[a;b;t(x+y)]\}$$
$$=\sum_{m=0}^{\infty}\frac{(\alpha+\beta+\sigma\pm\nu+\frac{1}{2})_m}{(\alpha+\beta+\sigma-\mu+1)_m}\frac{(-t)^m}{m!}\sum_{n=0}^{\infty}\frac{(-m)_n(a)_n}{n!\,(b)_n},$$

and summing the inner series by means *Vandermonde's* theorem, we find that

$$\Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\{e^{-t(x+y)}\,{}_1F_1[a;b;t(x+y)]\}={}_3F_2\left[\begin{matrix}\alpha+\beta+\sigma\pm\nu+\frac{1}{2},b-a;\\ \alpha+\beta+\sigma-\mu+1,b;\end{matrix}\;-t\right]$$
$$=\Omega^{(\mu,\nu)}_{(\alpha,\beta,\sigma)}\{{}_1F_1[b-a;b;-t(x+y)]\},$$

in view of (2.11)

Therefore, by an appeal to *Lerch's* theorem we finally have

(3.1) $e^{-z}\,{}_1F_1[a;b;z]={}_1F_1[b-a;b;-z]$

which is *Kummer's* first theorem [16, p. 125].

4. APPLICATIONS TO CERTAIN CLASSICAL POLYNOMIALS.

Consider the formula [8, p. 30]

$$L^{(\alpha-n)}_n(x)=\sum_{r=0}^{n}(-)^r\binom{n}{r}L^{(\alpha)}_r(x).$$

On changing x to xt, if we operate on both the sides by

$$\Omega^{(\beta+1,\frac{1}{2})}_{(\alpha,\beta,0)}$$

we get from (2.11) that

$$P_n^{(\alpha-n,\beta)}(1-2t)=\sum_{r=0}^{n}(-)^{n-r}\binom{n}{r}P_r^{(\alpha,\beta-r)}(1-2t) \tag{4.1}$$

which leads us to

$$P_n^{(\alpha,\beta+n)}(x)=\sum_{r=0}^{n}(-)^{r}\binom{n}{r}P_{n-r}^{(\alpha+n,\beta+r)}(x) \tag{4.2}$$

when the order of summation in (4.1) is reversed.

On the other hand, from the known result [8, p. 30]

$$L_n^{(\alpha)}(x)=\sum_{r=0}^{n}\binom{n}{r}L_r^{(\alpha-r)}(x)$$

we similarly arrive at

$$P_n^{(\alpha+n,\beta)}(x)=\sum_{r=0}^{n}\binom{n}{r}P_{n-r}^{(\alpha+r,\beta+n)}(x) \tag{4.3}$$

which corresponds to the case $\mu=n$ of *Carlitz's* formula [6, p. 377 (4.7)]. From (4.2) and (4.3) it follows that

$$\sum_{r=0}^{n}\binom{n}{r}P_r^{(\alpha-r,\beta+n)}(x)=\sum_{r=0}^{n}(-)^{n-r}\binom{n}{r}P_r^{(\alpha+n,\beta-r)}(x). \tag{4.4}$$

Next we consider the formula [16, p. 209]

$$L_n^{(\alpha)}(xy)=\sum_{k=0}^{n}\frac{(\alpha+1)_n}{(n-k)!(\alpha+1)_k}y^k(1-y)^{n-k}L_k^{(\alpha)}(x),$$

and our formula (2.10) yields

$$P_n^{(\alpha,\beta-n)}(1-2xy)=\sum_{k=0}^{n}\frac{(\alpha+1)_n}{(n-k)!(\alpha+1)_k}y^k(1-y)^{n-k}P_k^{(\alpha,\beta-k)}(1-2x). \tag{4.5}$$

Similarly, the known result [12, p. 262]

$$e^t(xt)^{-\frac{1}{2}\alpha}J_\alpha(2\sqrt{xt})=\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(n+\alpha+1)}L_n^{(\alpha)}(x)$$

gives us

$$e^t\,{}_1F_1\left[\begin{matrix}\alpha+\beta+1;\\ \alpha+1;\end{matrix}\ -xt\right]=\sum_{n=0}^{\infty}\frac{t^n}{(\alpha+1)_n}P_n^{(\alpha,\beta-n)}(1-2x), \tag{4.6}$$

which leads to *Feldheim's* formula [13, p. 120]

$$\sum_{n=0}^{\infty}\frac{2^n t^n}{(\alpha+1)_n}P_n^{(\alpha,\beta-n)}(x)=e^{(x+1)t}\,{}_1F_1\left[\begin{matrix}-\beta;\\ \alpha+1;\end{matrix}\ (1-x)t\right], \tag{4.7}$$

in view of (3.1).

The formula [11, p. 189]

$$\sum_{n=0}^{\infty} L_n^{(\alpha-n)}(x)\, t^n = (1+t)^\alpha\, e^{-xt}$$

when operated upon by

$$\Omega_{(\alpha+\beta+1,\ \beta,-1)}^{(1,\ \frac{1}{2})}$$

gives us

$$(4.8) \qquad \sum_{n=0}^{\infty} t^n\, P_n^{(\alpha-n,\ \beta)}(x) = (1+t)^\alpha\, [1-\tfrac{1}{2}\, t\,(x-1)]^{-\alpha-\beta-1}$$

also due to Feldheim [13, p. 120].

Since

$$P_n^{(\alpha,\ \beta)}(-z) = (-)^n\, P_n^{(\beta,\ \alpha)}(z)$$

(4.8) assumes the form [13, p. 120]

$$(4.9) \qquad \sum_{n=0}^{\infty} t^n\, P_n^{(\alpha,\ \beta-n)}(x) = (1-t)^\beta\, [1-\tfrac{1}{2}\, t\,(x+1)]^{-\alpha-\beta-1}$$

In view of (2.6) and (2.21) the known relation [16, p. 209]

$$L_n^{(\alpha+\beta+1)}(x+y) = \sum_{k=0}^{n} L_k^{(\alpha)}(x)\, L_{n-k}^{(\beta)}(y)$$

readily gives us

$$(4.10) \qquad \sum_{r=0}^{n} P_r^{(\alpha,\ \beta-r)}(x)\, P_{n-r}^{(\gamma+n,\ \gamma+r)}(y)$$

$$= \binom{\alpha+\delta+n+1}{n} F_1\, [-n, \alpha+\beta+1, \gamma+\delta+n+1\, ; \alpha+\delta+2\, ; \tfrac{1}{2}(1-x), \tfrac{1}{2}(1-y)],$$

and from (2.14) and [16, p. 283]

$$\frac{(2x)^n}{n\,!} = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(\nu+n-2k)}{k\,!\ (\nu)_{n-k+1}}\, C_{n-2k}^{(\nu)}(x)$$

we similarly have

$$(4.11) \qquad \frac{2^n}{n\,!}\, Y_n^{(\nu)}(x) = \sum_{k=0}^{[\frac{1}{2}n]} \frac{(\nu+2n+\frac{1}{2}-2k)\,(2\nu+2n+1)_{n-2k}}{k\,!\,(\nu+n+\frac{1}{2})_{n-k+1}}\, Y_{n-2k}^{(2\nu+2n)}(-\tfrac{1}{2}x),$$

where $Y_n^{(a)}(x), n \geqq 0$ are the *Bessel* polynomials defined as

$$Y_n^{(\alpha)}(x) = {}_2F_0\, [-n, \alpha+n+1\, ; -\, ; -\tfrac{1}{2}\, x\,].$$

The formula [11, p. 214]

$$x^\rho = \Gamma(\alpha+\rho+1) \sum_{n=0}^{\infty} \frac{(-\rho)_n}{\Gamma(\alpha+n+1)}\, L_n^{(\alpha)}(x)$$

with the aid of (2.19) yields

$$(4.12)\qquad \left(\frac{1-t}{2}\right)^{\rho}=\frac{\Gamma(1+\alpha+\rho)}{(1+\alpha+\beta)_{\rho}}\sum_{n=0}^{\infty}\frac{(-\rho)_n}{\Gamma(\alpha+n+1)}P_n^{(\alpha,\beta-n)}(t),$$

and from [16, p. 210]

$$(1-t)^{-\lambda}\,{}_1F_1\left[\lambda;1+\alpha;-\frac{xt}{1-t}\right]=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{(1+\alpha)_n}L_n^{(\alpha)}(x)\,t^n$$

it similarly follows that [11, p. 85]

$$(4.13)\qquad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n,b;\\ c;\end{matrix}\,x\right]t^n=(1-t)^{-\lambda}\,{}_2F_1\left[\begin{matrix}\lambda,b;\\ c;\end{matrix}-\frac{xt}{1-t}\right].$$

On the other hand, the generating function relation [16, p. 209]

$$\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)\,t^n=(1-t)^{-1-\alpha}exp\left(-\frac{xt}{1-t}\right)$$

when operated upon by

$$\Omega_{(\alpha',\beta',-1)}^{(1,\frac{1}{2})}$$

gives us

$$(4.14)\qquad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_3F_2\left[\begin{matrix}-n,b,\mu;\\ c,\lambda;\end{matrix}\,x\right]t^n=(1-t)^{-\lambda}\,{}_2F_1\left[\begin{matrix}b,\mu;\\ c;\end{matrix}-\frac{xt}{1-t}\right].$$

This formula provides a generalization of (4.13) to which it reduces when $\mu=\lambda$.

A repeated application of this process will finally lead us to

$$(4.15)\qquad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_{p+1}F_q\left[\begin{matrix}-n,a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\,x\right]t^n$$

$$=(1-t)^{-\lambda}\,{}_{p+1}F_q\left[\begin{matrix}\lambda,a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}-\frac{xt}{1-t}\right],$$

a formula due to *Chaundy* [9, p. 62 (i)].

On applying our formulae (2.16) and (2.21) to

$$L_m^{(\alpha)}(u)\,L_m^{(\alpha)}(v)=\frac{\Gamma(1+\alpha+m)}{m!}\cdot$$

$$\cdot\sum_{r=0}^{m}\frac{(uv)^r}{r!\,\Gamma(1+\alpha+r)}L_{m-r}^{(\alpha+2r)}(u+v)\cdot$$

due to *Bailey* [3], we get

$$(4.16)\qquad P_m^{(\alpha,\beta)}(x)\,P_m^{(\alpha,\gamma)}(y)=\binom{\alpha+m}{m}\sum_{r=0}^{\infty}\frac{(\alpha+\beta+m+1)_r(\alpha+\gamma+m+1)_r}{(\alpha+1)_r}\cdot$$

$$\cdot\frac{(xy)^r}{r!}F_1\left[-m+r,\alpha+\beta+m+r+1,\alpha+\gamma+m+r+1;\alpha+2r+1;\tfrac{1}{2}(1-x),\tfrac{1}{2}(1-y)\right],$$

which provides an inverse of our earlier result (4.10); and in view of (2.19) the known relation [7, p. 39]

$$L_n^{(\beta)}(u)=\frac{(m+n)!}{n!}\sum_{r=0}^{m+n}(-)^r\frac{(1+\alpha+\beta+2r)(1+\alpha)_r(1+\beta+r)_{n-r}}{r!(1+\alpha+\beta+r)_{m+n+1}}\cdot$$
$$\cdot\,{}_3F_2\left[\begin{matrix}-r,-m,1+\alpha+\beta+r;\\-m-n,1+\alpha;\end{matrix}\right]u^r L_{m+n-r}^{(1+\alpha+\beta+2r)}(u)$$

gives us

$$P_n^{(\beta,1+\alpha+\delta+m)}(x)=\frac{(m+n)!}{m!}\cdot \tag{4.17}$$
$$\cdot\sum_{r=0}^{m+n}(-)^r\frac{(1+\alpha+\beta+2r)(1+\alpha)_r(1+\beta+r)_{n-r}(2+\alpha+\beta+m+n+\delta)_r}{r!(1+\alpha+\beta+r)_{m+n+1}}$$
$$\cdot\,{}_3F_2\left[\begin{matrix}-r,-m,1+\alpha+\beta+r;\\-m-n,1+\alpha;\end{matrix}\right]\left[\tfrac{1}{2}(1-x)\right]^r P_{m+n-r}^{(1+\alpha+\beta+2r,\delta)}(x).$$

Since [7, p. 37]

$$\sum_{m=0}^{k}(-)^m\frac{(x+y)^m(\frac{1}{2}z)^m}{(k-m)!\,\Gamma(1+\alpha+m)\,\Gamma(1+\beta+m)}P_m^{(\alpha,\beta)}\left[\frac{1+xy}{x+y}\right]$$
$$=\sum_{r=0}^{k}(-)^r\frac{\Gamma(1+\alpha+\beta+r)\,r!\,(1+\alpha+\beta+2r)}{\Gamma(1+\alpha+r)\,\Gamma(1+\beta+r)\,\Gamma(2+\alpha+\beta+k+r)}z^r$$
$$\cdot\,P_r^{(\alpha,\beta)}(x)\,P_r^{(\alpha,\beta)}(y)\,L_{k-r}^{(1+\alpha+\beta+2r)}(z),$$

it follows that

$$\sum_{m=0}^{k}(-)^m\frac{(2+\alpha+\beta+\delta+k)_m[\frac{1}{2}(x+y)]^m[\frac{1}{2}(1-z)]^m}{(k-m)!\,\Gamma(1+\alpha+m)\,\Gamma(1+\beta+m)}P_m^{(\alpha,\beta)}\left[\frac{1+xy}{x+y}\right] \tag{4.18}$$
$$=\sum_{r=0}^{k}(-)^r\frac{(1+\alpha+\beta+2r)\,r!\,\Gamma(1+\alpha+\beta+r)\,\Gamma(2+\alpha+\beta+\delta+k)}{\Gamma(1+\alpha+r)\,\Gamma(1+\beta+r)\,\Gamma(2+\alpha+\beta+k+r)}\left[\tfrac{1}{2}(1-z)\right]^r$$
$$\cdot\,P_r^{(\alpha,\beta)}(x)\,P_r^{(\alpha,\beta)}(y)\,P_{k-r}^{(1+\alpha+\beta+2r,\delta)}(z),$$

and finally the known relation [7, p. 38]

$$L_k^{(\beta)}[\tfrac{1}{2}x(1+y)]=\sum_{r=0}^{k}(-)^r\frac{(1+\alpha+\beta+2r)(1+\beta+r)_{k-r}}{(1+\alpha+\beta+r)_{k+1}}x^r$$
$$\cdot\,L_{k-r}^{(1+\alpha+\beta+2r)}(x)\,P_r^{(\alpha,\beta)}(y)$$

similarly gives us the elegant formula

$$P_n^{(\alpha,1+\beta+\beta')}[1-\tfrac{1}{2}(1-x)(1+y)] \tag{4.19}$$
$$=\sum_{r=0}^{n}(-)^r\frac{(1+\alpha+\beta+2r)(1+\beta+r)_{n-r}(2+\alpha+\beta+\beta'+n)_r}{(1+\alpha+\beta+r)_{n+1}}$$
$$\cdot\,[\tfrac{1}{2}(1-x)]^r P_{n-r}^{(1+\alpha+\beta+2r,\beta')}(x)\,P_r^{(\alpha,\beta)}(y)$$

which does not appear to have been noticed earlier

5. SOME GENERATING RELATIONS.

In the *Bateman* generating function [16, p. 256]

$$ {}_0F_1[-;1+\alpha;\tfrac{1}{2}(x-1)t]\,{}_0F_1[-;1+\beta;\tfrac{1}{2}(x+1)t]=\sum_{n=0}^{\infty}\frac{t^n}{(1+\alpha)_n(1+\beta)_n}P_n^{(\alpha,\beta)}(x)$$

if we replace t by tuv and operate upon both sides by

$$\Omega_{(\alpha',\beta',-1)}^{(1,\frac{1}{2})},$$

we shall get

$$\text{(5.1)}\qquad \sum_{n=0}^{\infty}\frac{(\gamma)_n(\delta)_n}{(1+\alpha)_n(1+\beta)_n}t^n P_n^{(\alpha,\beta)}(x)=F_4[\gamma,\delta;1+\alpha,1+\beta;\tfrac{1}{2}(x-1)t,\tfrac{1}{2}(x+1)t],$$

a formula due to *Brafman* (Cf., e. g., [16, p. 271]), where F_4 is an *Appell* function defined in § 2.

If, however, we replace t by tu and regard u as the variable, the Bateman generating function will give us

$$\text{(5.2)}\qquad \Psi_2[\delta;1+\alpha,1+\beta;\tfrac{1}{2}(x-1)t,\tfrac{1}{2}(x+1)t]=\sum_{n=0}^{\infty}\frac{(\delta)_n}{(1+\alpha)_n(1+\beta)_n}t^n P_n^{(\alpha,\beta)}(x),$$

where Ψ_2 is a confluent hypergeometric function in two arguments defined by *Humbert* in the form [11, p. 225]

$$\Psi_2[\alpha;\beta,\gamma;x,y]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}}{(\beta)_m(\gamma)_n}\frac{x^m}{m!}\frac{y^n}{n!}.$$

6. BILATERAL GENERATING FUNCTIONS.

Let us consider the *Hille-Hardy* formula [16, p. 212]

$$(1-t)^{-1-\alpha}\exp\left[-\frac{t(x+y)}{1-t}\right]{}_0F_1\left[-;1+\alpha;\frac{xyt}{(1-t)^2}\right]$$

$$=\sum_{n=0}^{\infty}\frac{n!}{(1+\alpha)_n}L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)t^n,$$

which can be written as

$$(1-t)^{-1-\alpha}\exp\left[-\frac{tx}{1-t}\right]\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-)^r\frac{(ty)^{r+s}x^s}{r!\,s!\,(1-t)^{r+s}(1+\alpha)_s}$$

$$=\sum_{n=0}^{\infty}\frac{n!}{(1+\alpha)_n}L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)t^n.$$

Changing y to uy and operating by

$$\Omega_{(c,\beta,-1)}^{(1,\frac{1}{2})}$$

we get

$$(6.1)\qquad (1-t)^{-1-\alpha}\, exp\left[-\frac{tx}{1-t}\right]\sum_{s=0}^{\infty}\frac{(\beta)_s\,(tux)^s}{s\,!\,(1+\alpha)_s\,(1-t)^s}\left[1+\frac{tu}{1-t}\right]^{-\beta-s}$$

$$=\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\,{}_2F_1\left[\begin{matrix}-n,\beta\,;\\ 1+\alpha\,;\end{matrix}\,u\right]t^n,$$

which finally simplifies to the well-known formula

$$(6.2)\qquad (1-t)^{-1+\alpha+\beta}\, exp\left[-\frac{tx}{1-t}\right](1-t+ut)^{-\beta}\,{}_1F_1\left[\beta\,;1+\alpha\,;\frac{xut}{(1-t)(1-t+ut)}\right]$$

$$=\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\,{}_2F_1\left[\begin{matrix}-n,\beta\,;\\ 1+\alpha\,;\end{matrix}\,u\right]t^n,$$

due to *Weisner* [22].

Further, if in (6.2) we change x to xv and operate once again by

$$\Omega_{(c,\beta,-1)}^{(1,\frac{1}{2})},$$

we obtain yet another formula due to *Weisner* [22], viz.

$$(6.3)\qquad (1-t)^{\beta+\gamma-\delta}(1-t+ut)^{-\beta}(1-t+vt)^{-\gamma}\,{}_2F_1\left[\begin{matrix}\beta,\gamma\,;\\ \delta\,;\end{matrix}\,\frac{uvt}{(1-t+ut)(1-t+vt)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(\delta)_n}{n\,!}\,{}_2F_1\left[\begin{matrix}-n,\beta\,;\\ \delta\,:\end{matrix}\,u\right]{}_2F_1\left[\begin{matrix}-n,\gamma\,;\\ \delta\,;\end{matrix}\,v\right]t^n.$$

Since

$${}_2F_1\,[a,b\,;c;z]=(1-z)^{c-a-b}\,{}_2F_1\,[c-a,c-b\,;c\,;z],$$

on changing the notations slightly, we have

$$(6.4)\qquad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n\,!}\,{}_2F_1\left[\begin{matrix}-n,\alpha\,;\\ \lambda\,;\end{matrix}\,u\right]{}_2F_1\left[\begin{matrix}\lambda+n,\beta\,;\\ \lambda\,;\end{matrix}\,v\right]t^n$$

$$=(1-t)^{\beta-\lambda}(1-v-t)^{\alpha-\beta}\,[1-(1-u)\,t-v]^{-\alpha}$$

$$\cdot\,{}_2F_1\left[\begin{matrix}\alpha,\lambda-\beta\,;\\ \lambda\,;\end{matrix}\,\frac{uvt}{(1-t)\,\{1-(1-u)\,t-v\}}\right],$$

which in view of

$$(6.5)\qquad {}_2F_1\left[\begin{matrix}a,b\,;\\ c\,;\end{matrix}\,z\right]=(1-z)^{-a}\,{}_2F_1\left[\begin{matrix}a,c-b\,;\\ c\quad;\end{matrix}\,z\right]$$

assumes the known form [1, p. 60]

$$(6.6)\qquad (1-t)^{\alpha+\beta-\lambda}(1-t-u)^{-\beta}(1-t+vt)^{-\alpha}\,\cdot{}_2F_1\left[\begin{matrix}\alpha,\beta\,;\\ \lambda\,;\end{matrix}\,\frac{uvt}{(1-t-u)(1-t+vt)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n\,!}\,{}_2F_1\left[\begin{matrix}\lambda+n,\beta\,;\\ \lambda\quad;\end{matrix}\,u\right]{}_2F_1\left[\begin{matrix}-n,\alpha\,;\\ \lambda\quad;\end{matrix}\,v\right]t^n.$$

Further we consider the formula [1, p. 57]

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, {}_2F_1\left[\begin{matrix}-n,\gamma\,; \\ \delta\,;\end{matrix}\, 1+\frac{1}{t}\right] t^n$$

$$=\frac{\Gamma(\delta)\,\Gamma(\delta+\alpha-\gamma)}{\Gamma(\delta+\alpha)\,\Gamma(\delta-\gamma)}(1+t)^{\alpha} e^{-xt}\, {}_1F_1\left[\begin{matrix}\gamma\,; \\ \delta+\alpha\,;\end{matrix}\, -x(1+t)\right],$$

change x to xt, operate by

$$\Omega^{(1,\frac{1}{2})}_{(\alpha',\beta',-1)}$$

and make use of (6.5); we shall thus arrive at the result

$$\sum_{n=0}^{\infty}(u-1)^n P_n^{(\alpha-n,\beta+n)}\left[\frac{1+u}{1-u}\right] {}_2F_1\left[\begin{matrix}-n,\delta-\gamma\,; \\ \delta\,;\end{matrix}\, t\right] \tag{6.7}$$

$$=\frac{\Gamma(\delta)\,\Gamma(\delta+\alpha-\gamma)}{\Gamma(\delta+\alpha)\,\Gamma(\delta-\gamma)}\, t^{\alpha}(1-u+ut)^{-\beta}\, {}_2F_1\left[\begin{matrix}\beta,\gamma\,; \\ \delta+\alpha\,;\end{matrix}\, -\frac{ut}{1-u+ut}\right].$$

7. OPERATIONAL FORMULAE ASSOCIATED WITH APPELL'S FUNCTIONS.

From the definition of *Humbert's* function, we have

$$\Omega^{(1,\frac{1}{2})}_{(\alpha,\beta,-1)}\{\Psi_2(a\,;b,c\,;ux,vy)\}=F_2[a,\alpha,\beta\,;b,c;u,v]. \tag{7.1}$$

Similarly

$$\Omega^{(1,\frac{1}{2})}_{(\alpha,\beta,-1)}\{\Phi_2(a,b\,;c\,;ux,vy)\}=F_3[a,b,\alpha,\beta\,;c\,;u,v], \tag{7.2}$$

$$\Omega^{(1,\frac{1}{2})}_{(\alpha,\beta,-1)}\{\Psi_2(a\,;b,c\,;ux,vx)\}=F_4[a,\alpha\,;b,c\,;u,v], \tag{7.3}$$

$$\Omega^{(1,\frac{1}{2})}_{(\alpha,\beta,-1)}\{\Phi_1(a,b\,;c;u,vy)\}=F_1[a,b,\beta\,;c\,;u,v], \tag{7.4}$$

$$\Omega^{(1,\frac{1}{2})}_{(\alpha,\beta,-1)}\{\Psi_1(a,b\,;c,d\,;u,vy)\}=F_2[a,b,\beta\,;c,d\,;u,v], \tag{7.5}$$

$$\Omega^{(1,\frac{1}{2})}_{(\alpha,\beta,-1)}\{\Xi_1(a,b,c\,;d\,;u,vy)\}=F_3[a,b,c,\beta\,;d\,;u,v] \tag{7.6}$$

and

$$\Omega^{(1,\frac{1}{2})}_{(\alpha,\beta,-1)}\{\Xi_2(a,b\,;c;ux,vy)\}=F_1[\alpha,a,b\,;c;u,v], \tag{7.7}$$

where F_1, F_2, F_3 and F_4 are *Appell's* double hypergeometric functions and $\Phi_1, \Phi_2, \Psi_1, \Psi_2$, Ξ_1 and Ξ_2 are their confluent functions defined in [11, pp. 224-226]. Furthermore, the formula [21, p. 147]

$$J_{\mu}(x)J_{\nu}(x)=\frac{(\frac{1}{2}x)^{\mu+\nu}}{\Gamma(1+\mu)\,\Gamma(1+\nu)}\, {}_2F_3\left[\begin{matrix}\frac{1}{2}(\mu+\nu+1),\frac{1}{2}(\mu+\nu+2)\,; \\ \mu+1,\nu+1,\ \mu+\nu+1\,;\end{matrix}\, -x\right]$$

in view of (2.16) and (2.18) yields

$$(7.8)\qquad {}_4F_3\left[\begin{matrix}\tfrac{1}{2}(\mu+\nu+1),\tfrac{1}{2}(\mu+\nu+2),\lambda,\lambda+\tfrac{1}{2};\\ \mu+1,\nu+1,\mu+\nu+1;\end{matrix}\ 4z\right]=F_4\left[\lambda,\lambda+\tfrac{1}{2};\mu+1,\nu+1;z,z\right];$$

and the formula [16, p. 106]

$${}_1F_1\left[\alpha;\beta;x\right]{}_1F_1\left[\alpha;\beta;-x\right]={}_2F_3\left[\begin{matrix}\alpha,\beta-\alpha;\\ \beta,\tfrac{1}{2}\beta,\tfrac{1}{2}(\beta+1);\end{matrix}\ \frac{1}{4}x^2\right]$$

in a similar manner furnishes

$$(7.9)\qquad {}_4F_3\left[\begin{matrix}\alpha,\beta-\alpha,\tfrac{1}{2}\gamma,\tfrac{1}{2}(\gamma+1);\\ \beta,\tfrac{1}{2}\beta,\tfrac{1}{2}(\beta+1)\quad;\end{matrix}\ \frac{1}{4}u^2\right]=F_2\left[\gamma,\alpha,\alpha;\beta,\beta;u,-u\right].$$

The above cases of reducibility of *Appell's* F_2 and F_4 are well-known.

Next we consider [2]

$${}_1F_1\left[a;c;(u+v)\right]=\sum_{r=0}^{\infty}\frac{(a)_r(c-a)_r}{r!(c+r-1)_r(c)_r}(uv)^r.$$
$$\cdot\,{}_1F_1\left[a+r;c+2r;u\right]{}_1F_1\left[a+r;c+2r;v\right]$$

which in view of (2.10) and (2.16) gives us [4, p. 255]

$$(7.10)\qquad {}_2F_1\left[\begin{matrix}a,\alpha;\\ c\ ;\end{matrix}\ (u+v)\right]=\sum_{r=0}^{\infty}\frac{(a)_r(c-a)_r(\alpha)_{2r}}{r!(c+r-1)_r(c)_{2r}}\ .$$
$$\cdot\,F_2\left[\alpha+2r,a+r,a+r;c+2r,c+2r;u,v\right]$$

On the other hand the formula [2]

$${}_1F_1\left[a;c;u\right]{}_1F_1\left[a;c;v\right]=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(c-a)_r}{r!(c)_r(c)_{2r}}(uv)^r.$$
$$\cdot\,{}_1F_1\left[a+r;c+2r;(u+v)\right],$$

furnishes us with [4, p. 255]

$$(7.11)\qquad F_2\left[\alpha,a,a;c,c;u,v\right]=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(c-a)_r(\alpha)_{2r}}{r!(c)_r(c)_{2r}}(uv)^r.$$
$$\cdot\,{}_2F_1\left[\begin{matrix}a+r,\alpha+2r;\\ c+2r\quad;\end{matrix}\ (u+v)\right],$$

which provides the inverse of (7.10).

Lastly, on changing u to ux, and v to vx, if we treat u and v as the variables, we get

$$(7.12)\qquad {}_2F_1\left[\begin{matrix}a,\alpha;\\ c\ ;\end{matrix}\ x\right]{}_2F_1\left[\begin{matrix}a,\beta;\\ c\ ;\end{matrix}\ x\right]$$
$$=\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(c-a)_r(\alpha)_r(\beta)_r}{r!(c)_r(c)_{2r}}x^{2r}\cdot{}_2F_1\left[\begin{matrix}a+r,\alpha+\beta+2r;\\ c+2r\quad;\end{matrix}\ x\right],$$

a formula due to *Burchnall and Chaundy* [5, p. 114].

8. SOME IDENTITIES.

The identity [19, p. 249]

$$x^{\nu+1} J_\mu(2ax) = a^{\mu+1} \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n+1)\,\Gamma(\mu+\nu+n+1)}{n!} \cdot$$

$$\cdot J_{\mu+\nu+2n+1}(2x)\ {}_2F_1\left[\begin{matrix} -n, \mu+\nu+n+1\,; \\ \mu+1\,; \end{matrix}\ a^2\right]$$

on being multiplied by $a^{\nu+1}$ throughout, with the aid of (2. 17), readily gives

$$(8.1)\qquad {}_2F_1\left[\begin{matrix} \mu+\nu+2, \mu+\nu+\frac{3}{2}\,; \\ \mu+1\,; \end{matrix}\ xz\right]$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\mu+\nu+n+1)\,\Gamma(\mu+1)\,(\alpha+\mu+\nu+1)_{2n}}{n!\,\Gamma(\mu+\nu+2n+1)} \left(-\frac{1}{4}x\right)^n$$

$$\cdot\, {}_2F_1\left[\begin{matrix} \frac{1}{2}(2n+\alpha+\mu+\nu+1), \frac{1}{2}(2n+\alpha+\mu+\nu+2)\,; \\ \mu+\nu+2n+1\,; \end{matrix}\ x\right]$$

$$\cdot\, {}_2F_1\left[\begin{matrix} -n, \mu+\nu+n+1\,; \\ \mu+1\,; \end{matrix}\ z\right],$$

and the formula [19, p. 429]

$$(ax)^{\mu+\nu-\lambda} J_\mu(2ax) = \frac{(2a)^{\mu+\nu}\,\Gamma(\mu+1)\,\Gamma(\nu+1)}{\Gamma(\lambda+1)\,\Gamma(\mu+\nu+1)} \cdot$$

$$\cdot \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n)\,\Gamma(\mu+\nu+n)}{n!} J_{\mu+n}(x) J_{\nu+n}(x)$$

$$\cdot\, {}_4F_3\left[\begin{matrix} -n, \mu+1, \nu+1, \mu+\nu+n\,; \\ \lambda+1, \frac{1}{2}(\mu+\nu+1), \frac{1}{2}(\mu+\nu+2)\,; \end{matrix}\ a^2\right],$$

in view of (2. 17) and (2. 18) yields

$$(8.2)\qquad {}_2F_1\left[\begin{matrix} \mu+\nu+1, \mu+\nu+\frac{1}{2}\,; \\ \lambda+1\,; \end{matrix}\ xz\right]$$

$$= \frac{1}{\Gamma(\mu+\nu+1)} \sum_{n=0}^{\infty} \frac{(\mu+\nu+2n)\,\Gamma(\mu+\nu+n)\,(\alpha+\mu+\nu)_{2n}}{n!\,(\mu+1)_n\,(\nu+1)_n} \left(-\frac{1}{4}x\right)^n$$

$$\cdot\, {}_4F_3\left[\begin{matrix} -n, \mu+1, \nu+1, \mu+\nu+n\,; \\ \lambda+1, \frac{1}{2}(\mu+\nu+1), \frac{1}{2}(\mu+\nu+2)\,; \end{matrix}\ z\right]$$

$$\cdot\, {}_4F_3\left[\begin{matrix} \frac{1}{2}(\mu+\nu+2n+1), \frac{1}{2}(\mu+\nu+2n+2), \frac{1}{2}(\alpha+\mu+\nu+2n), \frac{1}{2}(\alpha+\mu+\nu+2n+1)\,; \\ \mu+\nu+1, \mu+\nu+1, \mu+\nu+2n+1\,; \end{matrix}\ x\right].$$

Next we observe that if in the formula [1, p. 61]

$$(1-x)^{-a}(1-y)^{-a} = \sum_{r=0}^{\infty} (-)^r \frac{(a)_r}{r!} (xy)^r \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a+r)_{m+n}}{m!\,n!} x^m y^n,$$

we first change x to ux and operate by

$$\Omega^{(1,\frac{1}{2})}_{(\alpha,\beta,-1)},$$

and then change y to vy and operate by

$$\Omega^{(1,\frac{1}{2})}_{(\gamma,\delta,-1)},$$

we finally get

$$(8.3)\qquad {}_2F_0\left[\begin{matrix}a,\delta\,;\\ -\,;\end{matrix}u\right]{}_2F_0\left[\begin{matrix}a,\alpha\,;\\ -\,;\end{matrix}v\right]\simeq\sum_{r=0}^{\infty}(-)^r\frac{(a)_r(\alpha)_r(\delta)_r}{r!}(uv)^r.$$

$$\cdot F\left[\begin{matrix}a+r:\alpha+r\,;\delta+r\,;\\ -\;:\;-\;;\;-\;;\end{matrix}u,v\right],$$

in view of (2.9) and (2.16); provided a is a non-positive integer. This result is a confluent case of the known formula (27) in [4].

Next we consider the formula [16, p. 210]

$$L_n^{(\alpha)}(x)=\binom{\alpha+n}{n}\sum_{k=0}^{n}\frac{(1+\alpha-c)_k}{(1+\alpha)_k}L_k^{(\alpha)}(-x)\,L_{n-k}^{(2c-\alpha-2)}(x)$$

which in view of (2.22) gives us

$$(8.5)\qquad \frac{(c)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n,\lambda\,;\\ 1+\alpha\,;\end{matrix}u\right]=\sum_{k=0}^{n}\frac{(1+\alpha-c)_k(2c-\alpha-1)_{n-k}}{k!(n-k)!}$$

$$\cdot F_2[\lambda,-n+k,-k\,;2c-\alpha-1,1+\alpha\,;u,-u]$$

Put $u=1$, and this reduces to

$$(8.6)\qquad \frac{(c)_n}{n!}\,{}_2F_1\left[\begin{matrix}-n,\lambda\,;\\ 1+\alpha\,;\end{matrix}1\right]=\sum_{k=0}^{n}\frac{(1+\alpha-c)_k(2c-\alpha-1)_{n-k}}{k!(n-k)!}\sum_{s=0}^{\infty}(-)^s\frac{(\lambda)_s(-k)_s}{s!(1+\alpha)_s}$$

$$\cdot\,{}_2F_1\left[\begin{matrix}-n+k,\lambda+s\,;\\ 2c-\alpha-1\,;\end{matrix}1\right].$$

Applying *Vandermonde's* theorem to sum the ${}_2F_1$'s we get

$$(8.7)\qquad \frac{(1+\alpha-\lambda)_n(c)_n}{n!(1+\alpha)_n}=\sum_{k=0}^{n}\frac{(1+\alpha-c)_k(2c-\alpha-\lambda-1)_{n-k}}{k!(n-k)!}\,.$$

$$\cdot\,{}_3F_2\left[\begin{matrix}-k,\lambda,2+\alpha+\lambda-2c\,;\\ 2+\alpha+\lambda-2c-n+k,1+\alpha\,;\end{matrix}-1\right]$$

which, by analytic continuation, holds for such values of α and λ for which the hypergeometric series on the right-hand side has a meaning.

Take $\lambda=0$ and make use of *Vandermonde's* theorem; the right-hand side of (8.7) reduces to $\frac{(c)_n}{n!}$.

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