A THEOREM OF PIECEWISE LINEAR APPROXIMATIONS

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1. In this paper we shall assume that *a complex is finite.* If K is a complex, $P=\bigcup_{\xi\epsilon K}\xi$ is called a *polyhedron* and denoted by $|K|$, and K is called a *simplical division* of P. Let K, H be complexes and $f\colon |K|\to|H|$ be a continuous mapping such that for any $\xi\epsilon K$ there is a $\sigma\epsilon H$ satisfying i) $f(\xi)=\sigma$ and ii) $f|\xi\to\sigma$ is linear. Then f is called a *simplicial mapping* of K into H and also a *piecewise linear* (*p. w. l.*) *mapping* of $|K|$ into $|H|$. If K is a complex and $f\colon|K|\to E^n$ is a continuous mapping of $|K|$ into E^n such that for any $\xi\epsilon K$ $f|\xi\to E^n$ is linear, f is called a *semi-simplicial mapping* of K into E^n and also a *p. w. l. mapping* of $|K|$ into E^n. A p. w. l. mapping $f\colon|K|\to|H|$ (or E^n) is said to be *non-degenerate*, if for any $\xi\epsilon K$ $\dim\xi=\dim f(\xi)$. If $f\colon P\to Q$ is a p. w. l. mapping, a point $p\epsilon P$, such that $f^{-1}f(p)\neq p$, is called a *singular point* of f and the closure of the set of singular points of f is denoted by S_f. If K is a complex and ξ is a simplex of K, we denote by $St_k(\xi)$ the polyhedron which is the union of all simplexes of K having ξ as a face. If K is a complex and for any $\xi\epsilon K$ $St_k(\xi)$ is p. w. l. homeomorphic to an n-simplex, the polyhedron $|K|$ is called a (*combinatorial*) n-manifold. If M is an n-manifold, we denote by $\mathring{M}$ and $\dot{M}$ the interior of M and the boundary of M respectively. Throughout this paper we shall assume that *any complex K is contained in some euclidean space E^k* and *any simplex ξ of K is linearly imbedded in E^k.* If $\Gamma=\{P_1,\cdots,P_l\}$ is a set of polyhedra such that $P_1\cup\cdots\cup P_l$ is connected and $P_i\cap P_j$ is a point or ϕ for $i\neq j$, Γ is called a *chain of polyhedra* and l is called a *length* of Γ. If

$$P_i\cap P_j=\begin{cases}\text{one point} & \text{for } |i-j|=1\\ \phi & \text{for } |i-j|>1,\end{cases}$$

Γ is called a *simple chair*. If

$$P_i\cap P_j=\begin{cases}\text{one point} & \text{for } |i-j|=1,\, l-1\\ \phi & \text{for } 1<|i-j|<l-1,\end{cases}$$

Γ is called a *cyclic chain.* The main thorem of this paper is following;

Theorem 1. *Let M be an n-manifold. Then there is a positive number $\eta(M)$ such that if $f; P\to Q$ is a p. w. l. mapping of a polyhedron P onto a polyhedron Q, $g\colon P\to\mathring{M}$ is a p. w. l. mapping and R is a subpolyhedron of P satisfying*

$g|R$ *is a homeomorphism*

$n > \dim Q + 2 \operatorname*{Max}_{q \in Q} \dim f^{-1}(q)$

$\eta(M) > \operatorname{dia} f^{-1}(q)$, *for any* $q \in Q$,

then for any $\varepsilon > 0$ *there is a p. w. l. mapping* $h : P \to \overset{\circ}{M}$ *satisfying*

1) $d(h, g) (= \sup_{p \in P} d(h(p), g(p)) < \varepsilon$

2) $h|R = g|R$

3) h *is non-degenerate*

4) $S_h \cap f^{-1}(q)$ *is finite*

5) $\dim S_h < \dim Q$

α_1) $h|f^{-1}(q)$ *is a homeomorphism*

α_2) $h f^{-1}(q_1) \cap h f^{-1}(q_2) =$ *one point or* ϕ

for any $q_1 \neq q_2 \in Q$

α) *there is no cyclic chain in* $\Gamma = \{hf^{-1}(q) | q \in Q\}$

β) *there is no simple chain of length* $\geq n+2$ *in* Γ.

2. In this section we shall prove the following:

Lemma 1 *Let* K *be a complex consisting of two simplexes* ξ_1, ξ_2 *and their faces. Let* $f : K \to H$ *be a simplicial mapping of* K *onto a complex* H *and* $g : K \to E^n$ *a semi-simplicial mapping such that*

$$n > \dim H + 2 \operatorname*{Max}_{q \in |H|} \dim f^{-1}(q).$$

Let $\lambda : D \to f(\xi_2)$ *be a linear homeomorphism of a convex polyhedron* D *of* $f(\xi_1)$ *into* $f(\xi_2)$. *Then for any* $\varepsilon > 0$ *there is a semi-simplicial mapping* $h : K \to E^n$ *such that*

i) $d(h, g) < \varepsilon$

ii) $h(f^{-1}(q) \cap \xi_1) \cap h(f^{-1}\lambda(q) \cap \xi_2)$

$\subset h(\xi_1 \cap \xi_2)$, for any $q \in D$.

If P is a polyhedron, we denote by $\boldsymbol{E}(\boldsymbol{P})$ the minimal euclidean space containing P. We shall consider any euclidean space E^l as a *vector space* and use *vector notation*. Points $p_0, p_1, \cdots, p_i$ are said to be *linearly independent* if the vectors $p_1 - p_0, \cdots, p_i - p_0$ are linearly independent. If $v_1, v_2, \cdots, v_i$ are vectors, we denote by $\boldsymbol{\rho}(\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_i)$ the maximal number of linearly independent vectors in $v_1, v_2, \cdots, v_i$. Vectors $v_1, \cdots, v_i, v_{i+1}, \cdots, v_j$ are said to be *linearly independent with respect to* $v_{i+1}, \cdots, v_j$ if

$$\rho(v_1, v_2, \cdots, v_i, \cdots, v_j) - \rho(v_{i+1}, \cdots, v_j) = i.$$

Proof of Lemma 1. Since λ is a linear mapping of D into $f(\xi)$. We may consider λ as a linear mapping of $E(D)$ into $E(f(\xi))$. Furthermore we denote by f_1 and f_2 the linear mappings of $E(f^{-1}(D) \cap \xi_1)$ onto $E(D)$ and $E(f^{-1}\lambda(D) \cap \xi_2)$ onto $E(\lambda(D))$ satisfying

$$f_1 | f^{-1}(D) \cap \xi_1 = f | f^{-1}(D) \cap \xi_1 \qquad \text{and}$$
$$f_2 | f^{-1}\lambda(D) \cap \xi_2 = f | f^{-1}\lambda(D) \cap \xi_2 \qquad \text{respectively.}$$

We choose linearly independent points of $E(f^{-1}(D) \cap \xi_1)$

$$\{a_0, a_1, \cdots, a_i, \cdots, a_j, \cdots, a_k, \cdots, a_l\}$$

and linearly independent points of $E(f^{-1}\lambda(D) \cap \xi_2)$

$$\{a'_0, a'_1, \cdots, a'_i, \cdots, a'_j, \cdots, a'_{k'}, \cdots, a'_{l'}\}$$

such that

$\alpha)$ $\lambda f_1(a_0) = f_2(a'_0), \ \lambda f_1(a_1) = f_2(a'_1), \cdots, \ \lambda f_1(a_j) = f_2(a'_j)$

$\beta)$ $f_1(a_0), \cdots, f_1(a_j)$ are linearly independent points of $E(D)$ and $j = dim\ E(D)$

$\gamma)$ $a_0, a_1, \cdots, a_i, a_{j+1}, \cdots, a_k \in E(\xi_1 \cap \xi_2)$

$a'_0, a'_1, \cdots, a'_i, a'_{j+1}, \cdots a'_{k'} \in E(\xi_1 \cap \xi_2)$

$a_{i+1} \cup a'_{i+1}, \cdots, a_j \cup a'_j \quad \not\subset E(\xi_1 \cap \xi_2)$

$a_{k+1}, \cdots, a_l, a'_{k'+1}, \cdots, a'_{l'} \notin E(\xi_1 \cap \xi_2)$.

Since we have

$$dim\ K \leqq dim\ H + \underset{q \in |H|}{Max}\ dim\ f^{-1}(q) < n \qquad \text{and}$$
$$1 + j + (l - j) + (l' - j) \leqq 1 + dim\ H + 2 \underset{q \in |H|}{Max}\ dim\ f^{-1}(q) \leqq n.$$

We can consturct a semi-simplicial mapping h of K into E^n so that

$\alpha')$ $d(h, g) < \varepsilon$

$\beta')$ h is non-degenerate

$\gamma')$ vectors $h_1(a_0) - h_2(a'_0), \cdots, h_1(a_j) - h_2(a'_j), h_1(a_{j+1}), \cdots, h_1(a_l), h_2(a'_{j+1}), \cdots, h_2(a'_{l'})$ are linearly independent with respect to $h_1(a_0) - h_2(a'_0), \cdots, h_1(a_i) - h_2(a'_i), h_1(a_{j+1}), \cdots, h_1(a_k), h_2(a'_{j+1}), \cdots, h_2(a'_{k'})$, where h_1 and h_2 are the linear mappings of $E(f^{-1}(D) \cap \xi_1)$ into E^n and $E(f^{-1}\lambda(D) \cap \xi_2)$ into E^n satisfying

$$h_1 | f^{-1}(D) \cap \xi_1 = h | f^{-1}(D) \cap \xi_1 \qquad \text{and}$$
$$h_2 | f^{-1}\lambda(D) \cap \xi_2 = h | f^{-1}\lambda(D) \cap \xi_2 \qquad \text{respectively.}$$

We shall prove that h is the required semi-simplicial mapping. If $p\in f^{-1}(q)\cap\xi_1, p'\in f^{-1}\lambda(q)\cap\xi_2, q\in D$ and $h(p)=h(p')$, we can write

$$p=\sum_{s=0}^{l}\mu_s a_s,\quad p'=\sum_{s=0}^{l'}\mu'_s a'_s,$$

$$\sum_{s=0}^{l}\mu_s=\sum_{s=0}^{l'}\mu'_s=1 \qquad \text{and}$$

$$\mu'_0=\mu_0,\ \mu'_1=\mu_1,\ \cdots,\ \mu'_j=\mu_j.$$

Then $h(p)-h(p')=\mu_0(h_1(a_0)-h_2(a'_0))+\cdots+\mu_j(h_1(a_j)-h_2(a'_j))$

$$+\mu_{j+1}h_1(a_{j+1})+\cdots+\mu_l h_1(a_l)$$
$$-\mu'_{j+1}h_2(a'_{j+1})-\cdots-\mu'_{l'}h_2(a'_{l'})$$
$$=0$$

From the linear independentness of vectors, we have

$$\mu_{i+1}=\cdots=\mu_j=\mu_{k+1}=\cdots=\mu_l=\mu'_{k'+1}=\cdots=\mu'_{l'}=0$$

Then $p, p'\in\xi_1\cap\xi_2$ and therefore $h(p)=h(p')\in h(\xi_1\cap\xi_2)$.

Hence we have proved

$$h(f^{-1}(q)\cap\xi_1)\cap h(f^{-1}\lambda(q)\cap\xi_2)\subset h(\xi_1\cap\xi_2),\ \text{for any}\ q\in D$$

and completed the proof of Lemma 1.

3. In this section we shall assume that $f\colon P\to Q$ and $g\colon P\to E^n$ are p. w. l. mappings and R is a subpolyhedron of P such that

$g:R$ *is a homeomorphism*

$n>\dim Q+2\ \underset{q\in Q}{Max}\ \dim f^{-1}(q).$

We shall prove the following:

Theorem 2. *For any* $\varepsilon>0$ *there is a p. w. l. mapping* $h\colon P\to E^n$ *satisfying the conditions* 1), 2), 3), 4), 5), α_1), α_2), α), β) *of Theorem* 1.

At first we shall prove the following:

Lemma (2, 1) *For any* $\varepsilon>0$ *there is a p. w. l. mapping* $h_1\colon P\to E^n$ *satisfying the conditions* 1), 2), 3), 4), 5), *and* α_1).

Proof of Lemma (2, 1) We choose simplicial divisions $K\supset J$ and H of $P\supset R$ and G such that $f\colon K\to H$ is simplicial and $g\colon K\to E^n$ is semi-simplicial. If we construct a semi-simplicial mapping $h'_1\colon K\to E^n$ sufficiently close to g and satisfying all conditions of Lemma (2. 1) except 2), $h'_1|J$ is an isomorphissm. Since $g|R$ is a homeomorphism and then $g|J$ is an isomorphism, Furthermore we may assume that $h'_1|J$ is sufficiently close to $g|J$ so that there is a p. w. l. homeomorphism $\pi: E^n \rightarrow E^n$ sufficiently close to identity mapping 1 and satisfying $\pi h'_1|R=g|R$. Put $\pi h'_1=h_1$. Then it is clear that h_1 is the required p. w. l. mapping. Therefore we shall construct a semi-simplicial mapping which is sufficiently close to g and satisfies 3), 4), 5), α). The conditions 3), 4), 5) follow respectively from following formulas:

$$n>dim\ Q+\underset{q\epsilon Q}{Max}\ dim\ f^{-1}(q)\geqq dim\ P$$

$$n>dim\ Q+2\ \underset{q\epsilon Q}{Max}\ dim\ f^{-1}(q)\geqq dim\ P+\underset{q\epsilon Q}{Max}\ dim\ f^{-1}(q)$$

$$dim\ Q>2\ dtm\ Q+2\ \underset{q\epsilon Q}{Max}\ dim\ f^{-1}(q)-n\geqq 2\ dim\ P-n.$$

We denote $\triangle=\{(\xi_1, \xi_2)|f(\xi_1)=f(\xi_2), \xi_1, \xi_2\ K\}$. Let $(\xi_1, \xi_2)\ \epsilon\triangle$. Then put $f(\xi_1)=D, \lambda=1$ and let K' be the subcomplex of K consisting of ξ_1, ξ_2 and their faces. We apply Lemma 1 to $f|K', g|k'$, $D=f(\xi)$ and $\lambda=1$. Then we get a semi-simplical mapping $h': K'\rightarrow E^n$ satisfying ii) of Lemma 1 and sufficiently close to g. It is clear that any semi-simplicial mapping $h'': K'\rightarrow E^n$ sufficiently close to h' satisfies ii). By induction with respect to elements of $\triangle$ we get a semi-simplicial mapping $h_1: K\rightarrow E^n$ satisfying the condition ii) for all elements of $\triangle$. We shall prove that h_1 satisfies α_1). Let $p_1\neq p_2\ \epsilon f^{-1}(q)$. Then there is a $(\xi_1, \xi_2)\ \epsilon\ \triangle$ such that $\xi_1 \ni p_1$ and $\xi_2\ni p_2$. If $p_1, p_2\ \epsilon\ \xi_1\cap\xi_2$, from 3) $h_1(p_1)\neq h_1(p_2)$. If $p_1 \notin \xi_1\cap\xi_2$, from 3) $h_1(p_1)\notin h_1(\xi_1\cap\xi_2)$. By ii) we have $h_1(p_1)\neq h_1(p_2)$. Therefore $h|f^{-1}(q)$ is a homeomorphism and then Lemma (2, 1) has been proved.

Lemma (2, 2) *There is a p. w. l. mapping $h_2: P\rightarrow E^n$ satisfying* 1), 2), 3), 4), 5), α_1), α_2).

Proof of Lemma (2, 2) Let h_1 be the p. w. l. mapping of P into E^n sufficiently close to g and satisfymg 2 , 3), 4), 5), α_1). Let $K\supset J$ and H be the simplicial division of $P\supset R$ and Q respectively such that $f: K\rightarrow H$ is simplicial and $h_1: K\rightarrow E^n$ is semi-simplicial. If $h_2: K\rightarrow E^n$ is sufficiently close to h_1 and satisfies all conditions of Lemma (2, 2) except 2), similarly as the proof of Lemma (2, 1) we can modify the p. w. l. mapping $h_2: P\rightarrow E^n$ so that it satisfies the condition 2) too. Furthermore if the semi-simplicial mapping $h_2: K\rightarrow E^n$ is sufficiently close to h_1, it is clear that h_2 satisfies the conditions 3), 4), 5), α_1). Therefore we shall construct a semi-simplicial mapping $h_2: K\rightarrow E^n$ which is sufficiently close to h_1 and satisfies α_2). Assume that $(\xi_{11}, \xi_{12}), (\xi_{21}, \xi_{22})\ \epsilon\ \triangle$.

and $h_1(\xi_{12})\cap h_1(\xi_{21})=C\neq\phi$. Then C is a convex polyhedron and $h_1^{-1}((C)\cap\xi_1, h_1^{-1}(C)\cap\xi_2$ are also convex polyhedra. The condition 3) implies that $\tilde{h}_1=h_1\,|\,h_1^{-1}(C)\cap\xi_{1\,2}\xrightarrow{onto}C$ and $\tilde{h}_2=h_1\,|\,h_1^{-1}(C)\cap\xi_{21}\xrightarrow{onto}C$ are linear homeomorphisms. Furthermore the condition 4) implies that $f(h_1^{-1}(C)\cap\xi_{1\,2})=D_1$ and $f(h_1^{-1}(C)\cap\xi_{21})=D_2$ are convex polyhedra and $\tilde{f}_1=f\,|\,h_1^{-1}(D)\cap\xi_{12}\xrightarrow{onto}D_1, \tilde{f}_2=f\,|\,h_1^{-1}(C)\cap\xi_{21}\xrightarrow{onto}D_2$ are linear homeomorphisms. Therefore $\lambda=\tilde{f}_2\,\tilde{h}_2^{-1}\,\tilde{h}_1\tilde{f}_1^{-1}: D_1\xrightarrow{onto}D_2$ is a linear homeomorphism. We apply Lemma 1 to $f\,|\,\xi_{11}\cup\xi_{22}$, $h_1\,|\,\xi_{11}\cup\xi_{12}, \lambda, D=D_1$. Then by induction we get a semi-simplicial mapping $h_2\,|\,K\rightarrow E^n$ satisfies the condition ii) of Lemma 1 for all such pairs $\{(\xi_{11},\xi_{22})\}$. We shall prove that h_2 satisfies α_2). If $q_1\neq q_2\in Q$ and $h_2f^{-1}(q_1)\cap h_2f^{-1}(q_2)\ni p$, from α_1) $h_2^{-1}(p)\cap f^{-1}(q_i)=p_i$, $i=1,2$, is a point. *If* $p_1'\neq p_1\in f^{-1}(q_1)$ and $p_2'\neq p_2\in f^{-1}(q_2)$, there are four simplexes $\xi_{21}\ni p_1', \xi_{12}\in p_1, \xi_{21}\ni p_2, \xi_{22}\ni p_2'$ such that $(\xi_{11},\xi_{12}),(\xi_{21},\xi_{22})\in\triangle$. If $p_1', p_2'\in\xi_{11}\cap\xi_{22}$, from 3) we have $h_2(p_1')\neq h_2(p_2')$. If $p_1'\notin\xi_{11}\cap\xi_{22}$, from 3 we have $h_2(p_1')\notin h_2(\xi_{11}\cap\xi_{22})$ and from ii) $h_2(p_1')\neq h_2(p_2')$. Therefore $h_2f^{-1}(q_1)\cap h_2f^{-1}(q_2)=p$ and we have proved Lemma (2. 2)

By induction with respect to i we shall prove the following:

Lemma (2, i), i $\geqq 2$. *There is a p. w. l. mapping* $h_i: P\rightarrow E^n$ *satisfying* 1), 2), 3), 4), 5), α_1), α_2) *and*

α_i) *There is no cyclic chain of length* $\leqq i$ *in* $\Gamma_i=\{h_if^{-1}(q)\,|\,q\in Q\}$,

β_i) *dim* $X_i\leqq$ *dim* $Q+1-i$,

where X_i=*closure of* $\{q\,|\,q\in Q, h_if^{-1}(q)$ *is an element of a simple chain of length*$\geqq i$ *in* $\Gamma_i\}$.

Proof of Lemma (2, i). We have already proved Lemma (2, 2). In fact it is clear that $X_2=f(S_{h2})$ and then the condition 5) is equivalent to the condition β_2). Therefore we assume that Lemma (2, i-1) is true. Let h_{i-1} be the p. w. l. mapping satisfying all conditions of Lemma (2, i-1) and sufficiently close to g. Let $K\supset J$ and H be simplicial subdivision of $P\supset R$ and Q respectively such that $f: K\rightarrow H$ is simplicial and $h_{i-1}\,|\,K\rightarrow E^n$ is semi-simplicial. *If* $(\xi_{11},\xi_{12}),\cdots,(\xi_{i1},\xi_{i2})\in\triangle$ and $h_{i-1}(\xi_{j2})\cap h_{i-1}(\xi_{j+11})=C_i\neq\phi$. We put $\tilde{h}_{j2}=h_{i-1}^{-1}(C_i)\cap\xi_{j2}\xrightarrow{onto}C_j$ and $\tilde{h}_{j+11}=h_{i-1}\,|\,h_{i-1}^{-1}(C_j)\cap\xi_{j+11}\xrightarrow{onto}C_j$, from 3) $\tilde{h}_{j2}$ and $\tilde{h}_{j+11}$ are linear homeomorphisms. Put $\tilde{f}_{j2}=f\,|\,h_{i-1}^{-1}(C_j)\cap\xi_{j2}$ and $\tilde{f}_{j+11}=f\,|\,h_{i-1}^{-1}(C_j)\cap\xi_{j+11}$. Then from 4) $\tilde{f}_{j2}$ and $\tilde{f}_{j+11}$ are linear homeomorphisms. $D_{j+1}=\tilde{f}_{j+11}\,\tilde{h}_{j+11}^{-1}(C_j)$ is a convex polyhedron in $\sigma_{j+1}=f(\xi_{j+11})=f(\xi_{j+12})$ and $\lambda_j=\tilde{f}_{j+11}\,\tilde{h}_{j+11}^{-1}\,\tilde{h}_{j2}\tilde{f}_{j2}^{-1}$ is a linear homeomorphism of the convex polyhedron $\tilde{f}_{j2}\tilde{h}_{j2}^{-1}(C_j)$ onto D_{j+1}. If $D=\lambda_1^{-1}(D_2 \cap \lambda_2^{-1}(D_3 \cap \cdots \cap \lambda_{i-2}^{-1}(D_{i-1} \cap \lambda_{i-1}^{-1}(D_i))\cdots) \neq \phi$. We apply Lemma 1 to $f|\xi_{11} \cup \xi_{i2}, h_{i-1}|\xi_{11} \cup \xi_{i2}, D, \lambda=\lambda_{i-1}\cdots\lambda_2\lambda_1|D$. Then by induction we get a semi-simplicial mapping $h_i|K\to E^n$, which is sufficiently close to h_{i-1} and satisfying ii) of Lemma 1 for all such pairs $\{(\xi_{11}, \xi_{i2})\}$. If $q_1, q_2, \cdots, q_i \epsilon Q, q_j \neq q_k$, and $h_i f^{-1}(q_j) \cap h_i f^{-1}(q_{j+1})=r_j, j=1, \cdots, i-1$, there are $p_j \epsilon f^{-1}(q_j)$ and $p'_{j+1} \epsilon f^{-1}(q_{j+1})$ such that $h_i(p_j)=h_i(p'_{j+1})=r_j$. If $p_1 \epsilon f^{-1}(q_1), p_i' \epsilon f^{-1}(q_i)$ and $p_1 \neq p_1', p_i' \neq p_i$. We can choose $(\xi_{11}, \xi_{12}), \cdots, (\xi_{i1}, \xi_{i2}) \epsilon \triangle$ and λ which are similar as the above ones and satisfies that $p_j \epsilon \xi_{j1}, p_j' \epsilon \xi_{j2}$ and $\lambda(q_1)=q_i$. From ii) and 3) we have $h_i(p_1) \neq h_i(p_i')$. Therefore we have $h_i f^{-1}(q_1) \cap h_i f^{-1}(q_i)=\phi$ and then we have proved that α_i) there is no cyclic chain of length $\geqq i$ in $\Gamma_i=\{h_i f^{-1}(q) | q \epsilon Q\}$. Since

$$
\begin{aligned}
& dim\ f^{-1}(X_{i-1})+dim P-n \\
& \leqq dim\ X_{i-1}+\underset{q \epsilon X_{i-1}}{Max}\ dim\ f^{-1}(q)+dim\ Q+\underset{q \epsilon Q}{Max}\ dim\ f^{-1}(q)-n \\
& \geqq dim\ Q+1-(i-1)+dim\ Q+2\underset{q \epsilon Q}{Max}\ dim\ f^{-1}(q)-n \\
& \leqq dim\ Q+(1-i).
\end{aligned}
$$

We can modify h_i so that β_i) $dim\ X_i \leqq dim\ Q+1-i$. Then we have proved Lemma (2, i).

Proof of Theorem 2. Since $dim\ X_{n+2} \leqq dim\ Q+1-(n+2)<0$. $X_{n+2}=\phi$ and then there is no simple chain of length $\geqq n+2$ in Γ_{n+2}. It is clear that α_{n+2}) and β_{n+2}) implies that there is no cyclic chain in Γ_{n+2}. Therefore $h=h_{n+2}$ is the required p. w. l. mapping.

4. **Proof of Theorem 1.** Let $\{C_1'', \cdots, C_l''\}, \{C_1', \cdots, C_l'\}, \{C_1, \cdots, C_l\}$ be families of (combinatorial) n-cell of M such that

$$
\begin{aligned}
& C_1'' \cup C_2'' \cup \cdots \cup C_l''=M \\
& C_i'' \subset \mathring{C}_i', C_i' \subset \mathring{C}_i, \qquad i=1, \cdots, l.
\end{aligned}
$$

Let η be a positive number such that

$$\eta<d(C_i'', \dot{C}_i')/n+2, \quad d(C_i', \dot{C}_i).$$

Then η is the requied number. In fact if f, g, P, Q, R satisfy the assumption of Theorem 1, put $P_i=\cup\{f^{-1}(q) | q \epsilon Q, g f^{-1}(q) \cap C_i' \neq \phi\}, R_i=P_i \cap R, f(P_i)=Q_i$. Then $\mathring{C}_i \supset g(P_i)$ and $\mathring{C}_i$ is p. w. l. homeomorphic to E^n. From Theorem 1 and by induction with respect to i we have a p. w. l. mapping $h: P\to\mathring{M}$ such that $h_i=h|P_i$

satisfies the conditions 1), 2), 3), 4), 5), α_1), α_2), α), β) for P_i, R_i, Q_i and $g_i=g|P_i$, $f_i=f|P_i$. If $\{hf^{-1}(q_1), \cdots, hf^{-1}(q_{n+2})\}$ is a chain, from $\cup \dot{C}_i''=M$ there is an i such that $C_i''\cap gf^{-1}(a_1)\neq\phi$. Since $dia \bigcup_{j=1}^{n+2} gf^{-1}(q_j) < (n+2)\eta < d(C_i'', \dot{C}_i')$. We have $\bigcup_{j=1}^{n+2} gf^{-1}(q_j)\subset C_i'$ and then $\bigcup_{j=1}^{n+2} f^{-1}(q_j)\subset P_i$. From the condition β) for h_i $\{h_i f^{-1}(q_1), \cdots, h_i f^{-1}(q_{n+2})\}$ is not a simple chain. Hence we have proved that β) Γ has no simple chain of length $\geqq n+2$. From β) it is clear that Γ has no cyclic chain of length $\geqq n+3$. Furthermore the condition α) for h_i implies that Γ has no chain of length $\leqq n+2$. Therefore we have proved Theorem 1.

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