ON THE HAUPTVERMUTUNG FOR SMOOTHABLE MANIFOLDS

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In [3], Homma gave a conditition which was sufficient for proving that every closed n-manifold has a combinatorial triangulation and for proving the Hauptvermutung for closed combinatorial manifolds. This condition was shown by Mazur [5] not to be necessary in dimensions greater than or equal to 23. We give here a condition which is both necessary and sufficient for the Hauptvermutung to hold for an unbounded smoothable triangulated manifold K. We do not assume that K is either closed or combinatorial and the smoothing is not assumed to be a compatible one. We do assume that the reader is familiar with the notation and terminology of Whitehead's paper on transverse fields [7].

Theorem. *Let K be a smoothable triangulated manifold. Then the Hauptvermutung holds for K if and only if for any smooth embedding $f: K \to R^n$ of K into a Euclidean space with codimension greater than zero, any transverse field φ of class C^r $(r \geq 1)$ on $f(K)$ in R^n, any φ-neighborhood N of $f(K)$, and any triangulated manifold L homeomorphic with K there is a piecewise-linear embedding $g: L \to N$ such that $\pi g: L \to f(K)$ is a homeomorphism (onto) where π is the φ-projection.*

Proof. Suppose the Hauptvermutung holds for K. By the Whitney Embedding Theorem, there is a smooth embedding $f: K \to R^n$ which we may assume to have codimension greater than zero. By (1.10) of [7], $f(K)$ admits a transverse field φ of class C^∞ and by (1.5) of [7] there is a φ-neighborhood N of $f(K)$. It follows by (1.9) and (13.3) of [7] that there is a combinatorial manifold L' in N such that $\pi | L' : L' \to f(K)$ is a homeomorphism (onto). Now let L be a triangulated manifold which is homeomorphic with K. By hypothesis, there is a piecewise-linear homeomorphism $g: L \to L'$. It follows that $\pi g: L \to f(K)$ is a homeomorphism.

Conversely, suppose K, f, g satisfy the conditions of the theorem. If $g: L \to N$ is a piecewise-linear mapping such that $\pi | g(L) : g(L) \to f(K)$ is a homeomorphism, it follows that $\pi | g(L)$ is a C^1-triangulation of $f(K)$ since $\pi : N \to f(K)$ is a regular C^1-mapping

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by remarks following (1.7) of [7]. By hypothesis, there is a piecewise-linear embedding $g': K \to N$ such that $\pi g': K \to f(K)$ is a homeomorphism. It follows that $\pi | g'(K)$ is a C^1-triangulation of $f(K)$. By theorem 8 of [6], two triangulated manifolds which C^1-triangulate the same differentiable manifold are piecewise-linearly homeomorphic. Thus $g(L)$ and $g'(K)$ are piecewise-linearly homeomorphic and hence L and K are also.

We have thus reduced the Hauptvermutung for smoothable triangulated manifolds to a problem of nicely approximating differentiable embeddings of such manifolds by piecewise-linear ones. Actually, it is only the niceness condition that gives difficulty as any locally tame embedding of a locally finite simplicial n-complex as a closed subset of a Euclidean space can be approximated by a piecewise-linear embedding provided the codimension is greater than $n+1$ [1, 2]. We may replace $n+1$ by 2 and delete the local tameness condition in the case of embeddings of closed combinatorial manifolds [4].

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