ULTRAPOWERS IN CATEGORIES

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Dedicated to Professor Motokiti Kondô in congratulation of his sixtieth birthday.

Introduction.

The concept of ultraproducts and ultrapowers is one of the main methods in the theory of models. The ultraproduct is defined as a quotient image of the direct product of relational structures together with the relations to be induced in it ([8], [17]). Here we give an attempt of an interpretation of it in terms of categories. The advantage of the interpretation is that we can apply the new definition not only to usual structures with relations of finite arguments, but also to any structures, like topological spaces, on which some infinitistic notions may be dealt with, as far as the mappings that preserve the structural relations take place between them. However, as seen later, the application of the new definition to topological spaces causes a curious situation. Indeed, when we consider the ultrapower of a structure, a natural isomorphic injection, which we call the diagonal map in this paper, of the original structure into the ultrapower should be expected, but this expectation does not hold in general. In the last section of this paper, it will be shown that the diagonal map of a structure into the ultrapower would be an isomorphic injection only when the structure satisfies some finitary condition.

To discuss about it, first we have to interpret the meaning of isomorphic injections as a notion in the theory of categories. Injections are a kind of monomaps. It is interesting that many propositions that hold between monomaps also hold between injections. Moreover, under the assumtions of completeness and of a kind of smallness of the treated category, it is seen that any map in it can be expressed as a composition of an epimap and an injection. Thus such a category becomes a bicategory in the sense of Isbell ([3], [4]). After reviewing the general notions about categories and establishing the terminology in § 1, we will give the definition of injections in terms of categories, and investigate the properties of them in § 2. The proposition that the diagonal map of an object into the ultrapower of it is an injection is completely interpreted in the theory of categories. However the finitary condition under which the proposition above holds for an object is no more a notion in general categories, but that in concrete categories, categories whose objects have basic sets. In § 3, we study about concrete categories. As well known, the correspondence between terms in the theory of categories and those in the thory of sets is not strictly literal. In order to make it more literal and natural, it is convenient to introduce some conditions on concrete categories, which are usually satisfied by most of specific substantial concrete categories. Under the condition we investigate about ultrapowers in a concrete category. § 4 is devoted to prove the Theorems about the diagonal maps into ultrapowers and the finitary conditions on objects, as stated before.

§ 1. General notions on categories.

In this section we give the definitions of basic notions about categories, and establish the terminology. Several propositions are stated without proofs, which are to be found in [1], [2], [7], [9], [12], [13] or [15].

A *category* $\mathfrak{A}$ is a class of abstract elements called *maps*, among which associative compositions are partly defined (that is, some pairs f, g in $\mathfrak{A}$ uniquely determine a $h=fg$ in $\mathfrak{A}$) where conditions C. 1), C. 2) and C. 3) stated below are satisfied. The class of all identities in $\mathfrak{A}$ is denoted by $|\mathfrak{A}|$. Maps in $|\mathfrak{A}|$ are called *objects*. Let f, g and h be maps in $\mathfrak{A}$.

C. 1) *Each f uniquely determines an A and a B in $|\mathfrak{A}|$ such that $fA=Bf=f$.*

A and B in C. 1) are called the domain and the range of f respectively, and denoted by $A=\mathrm{Do}(f)$ and $B=\mathrm{Rg}(f)$.

C. 2) *The composition fg is defined in $\mathfrak{A}$ if and only if* $\mathrm{Do}(f)=\mathrm{Rg}(g)$.

C. 3) *If either of $f(gh)$ and $(fg)h$ is defined in $\mathfrak{A}$ then the other is also defined and they are identical.*

Hence we have $\mathrm{Do}(fg)=\mathrm{Do}(g)$ and $\mathrm{Rg}(fg)=\mathrm{Rg}(f)$. The triple composition fgh is defined if and only if both fg and gh are defined. When $\mathrm{Do}(f)=A$ and $\mathrm{Rg}(f)=B$, we write $A\xrightarrow{f}B$ or $f: A\longrightarrow B$, for which we may simply write $A\longrightarrow B$ if there is no need to name the map. Similarly the composition of $A\longrightarrow B$ and $B\longrightarrow C$ is represented by $A\longrightarrow B\longrightarrow C$. However in the diagram of fig. 1, for instance, it is not necessary to have $fg=h$. If the equality holds, we say that the diagram is *commutative*, and write as in fig. 2 when we show the commutativity particularly. Also the diagram in fig. 3 shows $fg=hk$.

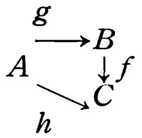


fig. 1

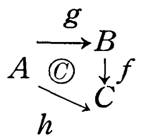


fig. 2

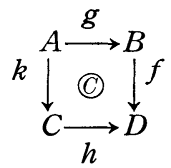


fig. 3

The class of all maps f with $A=\mathrm{Do}(f)$ and $B=\mathrm{Rg}(f)$ is denoted by $\mathrm{Hom}(A, B)$. f is called a *monomap* if $fg=fh$ implies $g=h$. f is called an *epimap* if $gf=hf$ implies $g=h$. When $gf \in |\mathfrak{A}|$, g is called the *reverse* of f, f is called the *coreverse* of g, f is called *reversible* and g is called *coreversible.* A reversible and coreversible map is called an *isomap*. Two maps $f: A \longrightarrow B$ and $g: C \longrightarrow D$ are called *equivalent* if there exist isomaps $a: A \longrightarrow C$ and $b: B \longrightarrow D$ such that $bf=ga$. The reverse of an isomap a, which is unique and is automatically the coreverse of a, is denoted by a^{-1}.

PROPOSITION 1. *A reversible map is a monomap. A coreversible map is an epimap. Reversible epimaps and coreversible monomaps are isomaps.*

(See [9], [15]).

PROPOSITION 2. *If both $A \longrightarrow B$ and $B \longrightarrow C$ are monomaps* (resp. *epimaps*), *then so also is the composite map $A \longrightarrow B \longrightarrow C$. If the composite map $A \longrightarrow B \longrightarrow C$ is a monomap* (resp. *an epimap*), *then so also is $A \longrightarrow B$* (resp. $B \longrightarrow C$).

(See [9], [15]).

In general the class of maps of a category $\mathfrak{A}$ may be very large beyond any cardinality. We say that $\mathfrak{A}$ is *small* if it is a set, and *large* otherwise. However, it is sometimes convenient to require the following smallness for a large category (see [14], [15]). A category $\mathfrak{A}$ is called *locally small* to the right (resp. to the left), if

LS. 1) *for any A and B in $|\mathfrak{A}|$*, $\mathrm{Hom}(A, B)$ *is a set*, and

LS. 2) (resp. LS. 2')) *for each A in $|\mathfrak{A}|$ there exists a set $\mathfrak{E}_A$* (resp. $\mathfrak{M}_A$) *of epimaps* (resp. *monomaps*) *f with* $\mathrm{Do}(f)=A$ (resp. $\mathrm{Rg}(f)=A$) *such that for any epimap* (resp. *monomap*) *g with* $\mathrm{Do}(g)=A$ (resp. $\mathrm{Rg}(g)=A$) *there exists a f in $\mathfrak{E}_A$* (resp. in $\mathfrak{M}_A$) *equivalent to g.*

A category $\mathfrak{A}$ which is locally small both to the right and to the left is simply called *locally small.* The set $\mathfrak{E}_A$ (resp. $\mathfrak{M}_A$) is called the set of *representatives* of epimaps (resp. monomaps) from A (resp. to A).

Example. Let $|\mathfrak{A}|$ bc the class of all ordinal numbers, and let $\mathfrak{A}$ be the category which consists of all maps $f\colon A \longrightarrow B$ uniquely determined by each A and B in $|\mathfrak{A}|$ such that $A \leqq B$. Then every map in $\mathfrak{A}$ is a monomap and an epimap. $\mathfrak{A}$ is locally small to the left, but not to the right.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be categories, and F an application from $\mathfrak{B}$ to $\mathfrak{A}$. F is called a (*covariant*) *functor* if it preserves the compositions. (In this paper we deal with neither contravariant functors nor many-argumented functors, about which refer to [1], [2], [7], [10], [13] or [15].) A functor from a small category $\mathfrak{B}$ to $\mathfrak{A}$ is called a *diagram* in $\mathfrak{A}$, where $\mathfrak{B}$ is called the *index category* and each map a in $\mathfrak{B}$ is called the *index* of the map in the diagram to which a is applied. A diagram is often represented by letters for objects in $|\mathfrak{A}|$ and arrows between them for maps in $\mathfrak{A}$, like fig. 1. However, the same objects or the same maps in $\mathfrak{A}$ may be different things in a diagram, if they are images from different indices. Furthermore, when we say about an object of a diagram, it means the image of an object in the index category. An arrow in a diagram may be incidentally an object in $\mathfrak{A}$, but it is by no means an object in the diagram.

Let $\mathfrak{D}$ be a diagram in $\mathfrak{A}$ with the index category $\mathfrak{B}$, and X an object in $\mathfrak{A}$. Assume that a map $f_A : D_A \longrightarrow X$ (resp. $f_A : X \longrightarrow D_A$) is assigned to each $A \in |\mathfrak{B}|$ where D_A is the object of the diagram indexed with A. Such a family $\{f_A\}_{A\in|\mathfrak{B}|}$ is called *compatible to the right* (resp. *to the left*) for $\mathfrak{D}$ if for each arrow $D_A \longrightarrow D_B$ in $\mathfrak{D}$ we have

$$\begin{array}{ccc} D_A & \xrightarrow{f_A} & \\ \downarrow & \copyright & X \\ D_B & \xrightarrow{f_B} & \end{array} \qquad \left(\text{resp.}\quad \begin{array}{ccc} & \xrightarrow{f_A} & D_A \\ X & \copyright & \downarrow \\ & \xrightarrow{f_B} & D_B \end{array}\right).$$

Further, we say that X is the *direct limit* (resp. the *inverse limi*) of $\mathfrak{D}$ with the *canonical maps* f_A, if $\{f_A\}_{A\in|\mathfrak{B}|}$ is compatible to the right (resp. to the left) for $\mathfrak{D}$, and for any $Y \in |\mathfrak{A}|$ and for any family $\{g_A\}_{A\in|\mathfrak{B}|}$ of maps $g_A : D_A \longrightarrow Y$ (resp. $g_A : Y \longrightarrow D_A$) compatible to the right (resp. to the left) for $\mathfrak{D}$, there exists uniquely a $h : X \longrightarrow Y$ (resp. $h : Y \longrightarrow X$) such that

$$\begin{array}{ccc} & \xrightarrow{f_A} & X \\ D_A & \copyright & \downarrow h \\ & \xrightarrow{g_A} & Y \end{array} \qquad \left(\text{resp.}\quad \begin{array}{ccc} Y & \xrightarrow{g_A} & \\ h \downarrow & \copyright & D_A \\ X & \xrightarrow{f_A} & \end{array}\right)$$

for each $A \in |\mathfrak{D}|$. The map h is called the *characteristic map* induced by the compatible family $\{g_A\}_{A\epsilon|\mathfrak{D}|}$. The direct limit (resp. the inverse limit) is uniquely determined up to the equivalence class, if it exists. The direct limit of a diagram $\mathfrak{D}$ is denoted by $\varinjlim\ [\mathfrak{D}]$ or $\varinjlim_{A\epsilon|\mathfrak{B}|} D_A$ where $\mathfrak{B}$ is the index category of $\mathfrak{D}$. Similarly the inverse limit of $\mathfrak{D}$ is denoted by $\varprojlim\ [\mathfrak{D}]$ or $\varprojlim_{A\epsilon|\mathfrak{B}|} D_A$. Further, for instance, the direct limit of the diagram in fig. 4 is denoted as in fig. 5.

$$A \begin{smallmatrix}\nearrow B\\ \searrow C\end{smallmatrix} \qquad\qquad \varinjlim \left[A \begin{smallmatrix}\nearrow B\\ \searrow C\end{smallmatrix}\right]$$

fig. 4, fig. 5.

Let $\mathfrak{D}$ be a diagram in $\mathfrak{A}$ which consists of an object A, objects B_λ and maps $f_\lambda : A \longrightarrow B_\lambda$ whe ɹe λ is the index that runs through a set Λ. Then the direct limit of $\mathfrak{D}$ is often described as $\varinjlim\ [A \xrightarrow{f_\lambda} B_\lambda]_{\lambda\epsilon\Lambda}$ or as $\varinjlim\ [f_\lambda : A \longrightarrow B_\lambda]_{\lambda\epsilon\Lambda}$. Similar conventions are used for other diagrams as well as for inverse limits.

A category $\mathfrak{A}$ is called *right complete* (resp. *left complete*) if every diagram in $\mathfrak{A}$ has the direct limit (resp. the inverse limit). $\mathfrak{A}$ is called *finitely right conplet* (resp. *finitely left complete*) if every finite diagram in $\mathfrak{A}$ has the direct limit (resp. the inverse limit). $\mathfrak{A}$ is called (*finitely*) *complete* if it is (finitely) both right and left complete.

If the diagram $\mathfrak{D}$ contains no arrows, then $\varprojlim\ [\mathfrak{D}]$ is called the *product* of $A\epsilon\mathfrak{D}$, and denoted by $\Pi\ [\mathfrak{D}]$ or $\prod_{\lambda\epsilon\Lambda} A_\lambda$ where Λ is the index category of $\mathfrak{D}$, and $\varinjlim\ [\mathfrak{D}]$ is called the *coproduct* of $A\epsilon\ \mathfrak{D}$, and denoted by $\Pi^*[\mathfrak{D}]$ or $\Pi^*_{\lambda\epsilon\Lambda} A_\lambda$. If Λ is a finite set $\{1, 2, \cdots, n\}$, then $\prod_{\lambda\epsilon\Lambda} A_\lambda$ and $\Pi^*_{\lambda\epsilon\Lambda} A_\lambda$ are denoted by $A_1 \times A_2 \times \cdots \times A_n$ and $A_1 {*} A_2 {*} \cdots {*} A_n$ respectively. $X = \varprojlim\ [A \xrightarrow{f_\lambda} B]_{\lambda\epsilon\Lambda}$ is called the *equalizer* of maps f_λ, and the canonical map $X \longrightarrow A$ is called the *equalizer map*. $X = \varinjlim\ [A \xrightarrow{f_\lambda} B]_{\lambda\epsilon\Lambda}$ is called the *coequalizer* of maps f_λ, and the canonical map $B \longrightarrow X$ is called the *coequalizer map*. $X = \varprojlim\ [A_\lambda \xrightarrow{f_\lambda} B]_{\lambda\epsilon\Lambda}$ is called the *pullback* of maps f_λ, and the canonical map $X \longrightarrow B$ is called the *pullback map*. $X = \varinjlim\ [A \xrightarrow{f_\lambda} B_\lambda]_{\lambda\epsilon\Lambda}$ *pushout* of maps f_λ, and the canonical map $A \longrightarrow X$ is called is called the *pushout map*. When Λ consists of finite members, the product $\prod_{\lambda\epsilon\Lambda} A_\lambda$ is called the *finite product*. Similarly *finite coproducts*, *finite equalizers*, etc., are defined. A category $\mathfrak{A}$ is said to have (*finite*) *products*, or *closed with* (*finite*) *products*, if every indexed family $\{A_\lambda\}_{\lambda\epsilon\Lambda}$ has the product, where Λ is a (finite) set. $\mathfrak{A}$ is said to *have* (*finite*) *equalizers*, or *closed with* (*finite*) *equalizers*, if every diagram $[A \xrightarrow{f_\lambda} B]_{\lambda\epsilon\Lambda}$ for a (finite) set

A has the equalizer. Similar definitions are given for the closedness with (finite) coproducts, (finite) pullbacks, etc..

PROPOSITION 3. *An equalizer map ıs a monomap. A coequalizer map is an epimap.*

(See [10] or [13]).

PROPOSITION 4. *A category is left* (resp. *right*) *complete if and only if it is closed with products* (resp. *coproducts*) *and with finite eqnalizers* (resp. *finite coequalizers*).

(See [10] or [15]).

PROPOSITION 5. *If* $X=\varprojlim \left[\begin{smallmatrix} A \\ B \end{smallmatrix} \!\!\begin{smallmatrix} \overset{f}{\searrow} \\ \nearrow \end{smallmatrix}\! C \right]$ *and f is a monomap, then the canonical map $X \longrightarrow B$ is a monomap. Also the dual proposition holds for a pushout and epimaps.*

(See [13] or [15]).

Let $\mathfrak{D}$ and $\mathfrak{D}'$ be two diagrams with the same index category $\mathfrak{B}$. A collection $\mathfrak{H}$ of masp $h_A : D_A \longrightarrow D'_A$, where D_A and D'_A are objects in $\mathfrak{D}$ and $\mathfrak{D}'$ respectively indexed by $A \in |\mathfrak{B}|$, is called a *natural transformation from* $\mathfrak{D}$ *to* $\mathfrak{D}'$, if for every a $\in \mathfrak{B}$ with $a : A \longrightarrow B$ we have $f'_a h_A = h_B f_a$ where f_a and f'_a are maps in $\mathfrak{D}$ and $\mathfrak{D}'$ respectively indexed by a. $\mathfrak{H}$ is called a natural *epi-transformation* (resp. a *natnral mono-transformation*), if every h_A is an epimap (resp. a monomap). If $X = \varinjlim [\mathfrak{D}]$, $X' = \varinjlim [\mathfrak{D}']$ and if $g_A : D_A \longrightarrow X$ and $g'_A : D'_A \longrightarrow X'$ are canonical maps, then the set $\{g'_A h_A\}_{A \in |\mathfrak{B}|}$ of maps is compatible to the right for $\mathfrak{D}$, and we have the characteristic map $h : X \longrightarrow X'$, which we call the *direct limit* of $\mathfrak{H}$. Similarly the *inverse limit of* $\mathfrak{H}$ is defined.

PROPOSITION 6. *If $\mathfrak{H}$ is a natural epi-transformation from a diagram $\mathfrak{D}$ to another $\mathfrak{D}'$, then the direct limit of $\mathfrak{H}$ is an epimap. Similar proposiiton holds for a natural mono-transformation and the inverse limit.*

PROOF. We use the same notations as stated above. Assume $u, v : X' \longrightarrow Y$ and $uh = vh$, where $Y \in |\mathfrak{A}|$. Then we have $ug'_A h_A = uhg_A = vhg_A = vg'_A h_A$, and h_A being an epimnp, $ug'_A = vg'_A$. Since the collection $\{ug'_A\}_{A \in |\mathfrak{B}|}$ of maps is compatible to the right for $\mathfrak{D}'$, the uniqueness of the characteristic map $X' \longrightarrow Y$ implies $u = v$.

q. e. d.

Corollary. 1. *If* $f_\lambda : A \longrightarrow B_\lambda$ *is an epimap for every* $\lambda \in \Lambda$, *then the pushout map* $f: A \longrightarrow X$ *is an epimap, provided the pushout* X *exists.*

Proof. Let $\mathfrak{D}'$ be the diagram $[A \xrightarrow{f_\lambda} B_\lambda]_{\lambda \in \Lambda}$, and $\mathfrak{D}$ the constant diagram with the index category $\mathfrak{D}'$ where A is assigned to every member of $\mathfrak{D}'$. Apply the previous proposition to the natural transformation $\mathfrak{H} = \{f_\lambda\}_{\lambda \in \Lambda}$. q. e. d.

Referring to Proposition 2, we can see that every canonical map $B_\lambda \longrightarrow X$ in the corollary above is also an epimap.

Corollary 2. *Let* $\mathfrak{D}$ *be a diagram in* $\mathfrak{A}$ *with the index category* $\mathfrak{B}$, *and* D_A *the object in* $\mathfrak{D}$ *indexed by* $A \in |\mathfrak{B}|$. *We assume that* $X = \varinjlim [\mathfrak{D}]$ *exists. If* $Y \in |\mathfrak{A}|$ *and* $\{g_A : D_A \longrightarrow Y\}_{A \in |\mathfrak{B}|}$ *is a family of epimaps* g_A *compatible to the right for* $\mathfrak{D}$, *then the characteristic map* $X \longrightarrow Y$ *is an epimap.*

Let A and B be two objects in $\mathfrak{A}$. A is called *prime* on B if there is no map $A \longrightarrow B$, and *non-prime* otherwise.

Proposition 7. *Let* $\{A_\lambda\}_{\lambda \in \Lambda}$ *be an indexed family of objects in* $|\mathfrak{A}|$, *and* ξ *a member of* Λ. *If* A_ξ *is non-prime on* A_λ *for every* $\lambda \in \Lambda$, *then the canonical map* $\prod_{\lambda \in \Lambda} A_\lambda \longrightarrow A_\xi$ *is coreversible, and particularly an epimap.*

Proof. Take a map $f_\lambda : A_\xi \longrightarrow A_\lambda$ for each $\lambda \in \Lambda$, and particularly let $f_\xi = A_\xi$. Then the characteristic map $A_\xi \longrightarrow \prod_{\lambda \in \Lambda} A_\lambda$ induced by the family $\{f_\lambda\}_{\lambda \in \Lambda}$ is surely the coreverse of the canonical map $\prod_{\lambda \in \Lambda} A_\lambda \longrightarrow A_\xi$. See [9] for the detail. q. e. d.

Especially if $A_\lambda = A$ for every $\lambda \in \Lambda$, then no A_ξ is prime on another and the canonical map $\prod_{\lambda \in \Lambda} A_\lambda \longrightarrow A_\xi$ is always coreversible. In this case the product $\prod_{\lambda \in \Lambda} A_\lambda$ is called the *power* of A and denoted by A^Λ.

§ 2. Injections.

In this section we will give the definition of injections and study the properties of them. Injections are maps which are to turn out the set-theoretical injections in concrete categories. Injections are monomaps, and it is interesting that many theorems about monomaps are also valid after replacing injections for monomaps. Furthermore, under the assumption of right completeness and local smallness to the right, any map in a category is decomposed into an epimap and an injection. This fact will be seen first. Finally we will give the definition of ultraproducts of objects in terms of the theory of categories. We sometimes omit the dual statement in the definitions or in the theorems if there is no need to state particular terminology or particular notices.

DEFINITION 1. A category is called *right perfect*, if it is locally small to the right and right complete. *Left perfectness* of a category is dually given. A category is *perfect* if it is both right and left perfect.

DEFINITION 2. A map $g: A \longrightarrow C$ is called an *epi-factor* of a map $f: A \longrightarrow B$, if it is an epimap and there exists a map $h: B \longrightarrow C$ with $f=hg$. Dually $g: C \longrightarrow B$ is called a *mono-factor* of $f: A \longrightarrow B$, if it is a monomap and there exists a map $h: A \longrightarrow C$ with $f=gh$.

DEFINITION 3. A factorization $A \xrightarrow{g} C \xrightarrow{h} B$ of $f: A \longrightarrow B$ is called an *epifactorization* if g is an epimap. An epifactorization $A \xrightarrow{g} C \xrightarrow{h} B$ of $f: A \longrightarrow B$ is called *critical* if any epifactor of f is an epifactor of g. When $A \xrightarrow{g} C \xrightarrow{h} B$ is a critical epifactorization of $f: A \longrightarrow B$, g, h and C are called the *epicomponent*, the *injection part* and the *image* of f respectively, and denoted by i_f, j_f and $\mathrm{Im}(f)$ respectively. $f: A \longrightarrow B$ is called an *injection* if it has the domain A itself as the epicomponent.

Let $\mathfrak{A}$ be a right perfect category, and $f: A \longrightarrow B$ a map in it. Since $\mathfrak{A}$ is locally small to the right, there exists a set $\mathfrak{E}_A$ of representatives of epimaps from A. Let $\mathfrak{E}_f$ be the set of all maps in $\mathfrak{E}_A$ each equivalent to an epifactor of f. $\mathfrak{E}_f$ is not void, since at least the domain A itself is an epifactor of f. Let C be the pushout of the family $\mathfrak{E}_f$, and $i: A \longrightarrow C$ the canonical map. We say that a family $\{h_\lambda : A_\lambda \longrightarrow B\}_{\lambda \in \Lambda}$ is *compatible* for the pushout diagram $[A \xrightarrow{f_\lambda} A_\lambda]_{\lambda \in \Lambda}$ if $h_\lambda f_\lambda = h_\xi f_\xi$ for any λ aud ξ in Λ. Since for each $g \in \mathfrak{E}_f$ with $g: A \longrightarrow A_g$, we have $h_g: A_g \longrightarrow B$ with $h_g g=f$, the family $\{h_g\}_{g \in \mathfrak{E}_f}$ is compatible for the pushout diagram $[A \xrightarrow{g} A_g]_{g \in \mathfrak{E}_f}$, and we have the characteristic map $j: C \longrightarrow B$ with $f=ji$. By Corollary 1 of Proposition 6, i is an epimap, and hence an epifactor of f. Obviously i is the epicomponent of f, and $A \xrightarrow{i} C \xrightarrow{j} B$ is the critical epifactorization of f. Thus we have

THEOREM 1. *If a category $\mathfrak{A}$ is right perfect, then any map in it admits the critical epifactotrization of it.*

In general if $f: A \longrightarrow B$ admits the critical epifactorization, it is obvious that the epicomponent of f is uniquely determined up to the equivalence class. Since i_f is an epimap, $f=ki_f$ implies $k=j_f$. Hence the injection part of f is also uniquely determined up to the equivalence class.

Next we shall investigate the properties of injections.

LEMMA 1. *If a map $f: A \longrightarrow B$ admits the critical epifactortzation, then the injection part j_f is an injection.*

PROOF. Let $\mathrm{Im}(f) \xrightarrow{g} C \xrightarrow{h} B$ be an epifactorization of $j_f: \mathrm{Im}(f) \longrightarrow B$. Then since gi_f is an epifactor of f, there exists a $g': C \longrightarrow \mathrm{Im}(f)$ such that $g' g i_f = i_f$. Since i_f is an epimap, this shows $g' g = \mathrm{Im}(f)$. Hence g is a reversible epimap, and is an isomap by Proposition 1. q. e. d.

LEMMA 2. *If the category is closed with coequalizers, then any injection $j: A \longrightarrow B$ is a monomap.*

PROOF. Let $f, g: D \longrightarrow A$ be two maps with $jf = jg$, and $h: A \longrightarrow C$ the coequalizer map of $\{f, g\}$. Since the family $\{j, jf\}$ is compatible to the right for the diagram $[D \overset{f}{\underset{g}{\rightrightarrows}} A]$, we have the characteristic map $k: C \longrightarrow B$ such that $j = kh$. Since h is an epimap by Proposition 3, it is an epifactor of j. Hence it is equivalent to the object A, i. e., an isomap. But then $hf = hg$ implies $f = g$. q. e. d.

Lemma 1 shows that the term "the injection part" for j_f is adequate. Theorem 1, together with Lemma 1 and Lemma 2, shows that a right perfect category is a bicategory in the sense of Isbell (see [3], [6], [19]).

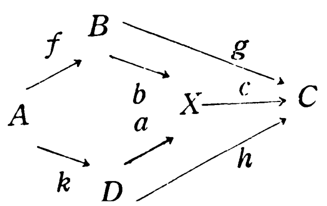
THEOREM 2. *A reversible map is an injection.*

PROOF. Assume that $f: B \longrightarrow A$ is the reverse of $g: A \longrightarrow B$, and $A \xrightarrow{h} C \xrightarrow{k} B$ is an epifactorization of g. Then the epimap h has the reverse $fk: C \longrightarrow A$, and hence is an isomap. q. e. d.

THEOREM 3. *If the composite map $A \xrightarrow{f} B \xrightarrow{g} C$ is an injection, then so also is f. If the category $\mathfrak{A}$ is closed with pushouts, and both maps $f: A \longrightarrow B$ and $g: B \longrightarrow C$ are injections, then the composite map gf is also an injection.*

PROOF. The first statement is obvious, since any epifactor of f is an epifactor of the composite map gf.

Assume that $f: A \longrightarrow B$ and $g: B \longrightarrow C$ are injections and $A \xrightarrow{k} D \xrightarrow{h} C$ is an epifactorization of the composite map gf. Let X be the pushout of $\{f, k\}$ and $a: D \longrightarrow X$, $b: B \longrightarrow X$ the canonical maps. Since the family $\{g, h\}$ of maps is compatible for the pushout diagram $A \begin{smallmatrix} \nearrow^{f} B \\ \searrow_{k} D \end{smallmatrix}$ we have the characteristic map $c: X \longrightarrow C$. b is an epimap by Proposition 5



and hence an epifactor of the injection g. Hence b is an isomap. But then k is an epifactor of the injection f, since $f=b^{-1}ak$. Hence k is an isomap. q. e. d.

THEOREM 4. *An eqalizer map is an injection.*

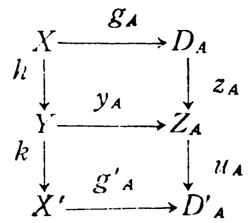
PROOF. Let $g: A \longrightarrow B$ be the equalizer map of the diagram $[B \xrightarrow{f_\lambda} C]_{\lambda \in \Lambda}$ and $A \xrightarrow{h} D \xrightarrow{k} B$ an epifactorization of g. For any $\lambda, \xi \in \Lambda$ we have $f_\lambda kh = f_\lambda g = f_\xi g = f_\xi kh$. But since h is an epimap, we have $f_\lambda k = f_\xi k$. Hence the familv $\{k, f_\lambda k\}_{\lambda \in \Lambda}$ is compatible to the left for the diagram $[B \xrightarrow{f_\lambda} C]_{\lambda \in \Lambda}$ and we have the characteristic map $h': D \longrightarrow A$ such that $gh'=k$. Then $gh'h=kh=g$ which implies $h'h=A$ since g is a monomap by Proposition 3. Hence h is a reversible epimap and hence an isomap by Proposition 1. q. e. d.

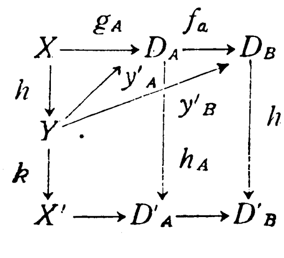
THEOREM 5. *If the category is closed with pushouts,* $X=\varprojlim [{}^{A}_{B} \rightrightarrows^{f} C]$ *and* f *is an injection, then the canonical map* $X \longrightarrow B$ *is an injection.*

This theorem is proved similarly to the next theorem.

THEOREM 6. *Assume that the category is closed with pushout and let* $\mathfrak{D}$ *and* $\mathfrak{D}'$ *be diagrams wiih the same index category* $\mathfrak{B}$. *If* $\mathfrak{H}$ *is a natural transformation of* $\mathfrak{D}$ *into* $\mathfrak{D}'$ *such that every* $h_A \in \mathfrak{H}$ *with* $A \in |\mathfrak{B}|$ *is an injecton, then the inverse limit* x *of* $\mathfrak{H}$ *is an injection of* $X=\varprojlim [\mathfrak{D}]$ *into* $X'=\varprojlim [\mathfrak{D}']$ *(provided they exist).*

PROOF. Let $g_A: X \longrightarrow D_A$ and $g'_A: X' \longrightarrow D'_A$ be the canonical maps of X and X' respectively where D_A and D'_A are objects of diagrams $\mathfrak{D}$ and $\mathfrak{D}'$ respectively indexed by $A \in |\mathfrak{B}|$. Let $X \xrightarrow{h} Y \xrightarrow{k} X'$ be an epifactorization of x. Let Z_A be the pushout of $\{g_A, h\}$ and $z_A: D_A \longrightarrow Z_A$ $y_A: Y \longrightarrow Z_A$ the canonical maps. Since the family $\{h_A, g'_A k\}$ is compatible for the pushout diagram $[X \begin{smallmatrix} \xrightarrow{g_A} D_A \\ \xrightarrow[h]{} Y \end{smallmatrix}]$, there is the characteristic map $u_A: z_A \longrightarrow D'_A$ such that $h_A = u_A z_A$ and $u_A y_A = g'_A k$. Then since z_A is an epimap by Propositian 5, it is an epifactor of the injection h_A. Hence it is an isomap, and putting $y'_A = z_A^{-1} y_A$, we have $y'_A h = z_A^{-1} y_A h = z_A^{-1} z_A g_A = g_A$ and $h_A y'_A = u_A z_A z_A^{-1} y_A = u_A y_A = g'_A k$. But then for the map $f_a: D_A \longrightarrow D_B$





and $f'_a: D'_A \longrightarrow D'_B$ in the diagram $\mathfrak{D}$ and $\mathfrak{D}'$ respectively indexed by $a \in |\mathfrak{B}|$, we have $h_B f_a y'_A = f'_a h_A y'_A = f'_a g'_A k = g'_B k = h_B y'_B$. Since h_B is a monomap by Lemma 2, we have $f_a y'_A = y'_B$ for any $a: A \longrightarrow B$ in the index category $\mathfrak{B}$. Hence the family $\{y'_A\}_{A \in |\mathfrak{B}|}$ is compatible to the left for $\mathfrak{D}$, and we have the characteristic map $h': Y \longrightarrow X$ such that $g_A h' = y'_A$ for every $A \in |\mathfrak{B}|$. Now $g_A h' h = y'_A h = g_A$, and the uniqueness of the characteristic map $X \longrightarrow X$ induced by the family $\{g_A\}_{A \in |\mathfrak{B}|}$ implies $h'h = X$. Hence h is a reversible epimap and hence an isomap by Proposition 1. q. e. d.

One can see that Theorems 2, 3, 4, 5, and 6 for injections correspond to Proposions 1, 2, 3, 5, and 6 for monomaps respectively. Note that we assumed only the closedness of the category with coequalizers and pushouts in the Theorems. Hence the theorems hold for general finitely right complete categories.

Now we shall give the definition of ultraproducts in terms of categories.

Definition 4. Assume that the caregory $\mathfrak{A}$ is closed with products and right complete. Let Λ be a set and A_λ an object in $|\mathfrak{A}|$ assigned to each $\lambda \in \Lambda$. Let Ξ and Ξ' be subsets of Λ. If $\Xi' \subset \Xi$, then the product $\prod_{\lambda \in \Xi'} A_\lambda$ is called a *subproduct* of $\prod_{\lambda \in \Xi} A_\lambda$. Let ξ be an element of Ξ' and $p^{\Xi}_{\xi}: \prod_{\lambda \in \Xi} A \longrightarrow A_\xi$ the canonical map. Then the family $\{p^{\Xi}_{\xi}\}_{\xi \in \Xi'}$ induces a characteristic map $p^{\Xi}_{\Xi'}: \prod_{\lambda \in \Xi} A_\lambda \longrightarrow \prod_{\lambda \in \Xi'} A_\lambda$, which we call the *projection* of $\prod_{\lambda \in \Xi} A_\lambda$ to the subroduct $\prod_{\lambda \in \Xi'} A_\lambda$. Particularly each canonical map $p^{\Xi}_{\xi}: \prod_{\lambda \in \Xi} A_\lambda \longrightarrow A_\xi$ is taken as a projection.

A *filter* Γ of Λ is a family of subsets of Λ such that $\Xi' \subset \Xi$ and $\Xi' \in \Gamma$ imply $\Xi \in \Gamma$ and that $\Xi, \Xi' \in \Gamma$ implies $\Xi \cap \Xi' \in \Gamma$. Γ is called an *ultrafilter* if it is maximal.

The *product system* of objects A_λ with $\lambda \in \Lambda$ over the filter Γ of Λ is the diagram $\mathfrak{D} = [p^{\Xi}_{\Xi'}: \prod_{\lambda \in \Xi} A_\lambda \longrightarrow \prod_{\lambda \in \Xi'} A_\lambda]_{\Xi, \Xi' \in \Gamma, \Xi' \subset \Xi}$ in $\mathfrak{A}$ where $p^{\Xi}_{\Xi'}$ is the projection. We call the direct limit of the product system $\mathfrak{D}$ the *reduced product* of $A_\lambda, \lambda \in \Lambda$, over the filter Γ, and denote it by $\prod_{\lambda \in \Xi} A_\lambda / \Gamma$. If Γ is an ultrafilter, then the reduced product is called an *ultraproduct*. If $A_\lambda = A$ for all $\lambda \in \Lambda$, then the reduced (ultra-) product is called the *reduced* (*ultra-*) *power* of A, and is denoted by A^Λ / Γ. Unless otherwise is stated, we assume that $p^{\Xi}_{\Xi'}$ and p^{Ξ}_{ξ} denote the projections in the senses above, and the canonical map $\prod_{\lambda \in \Xi} A_\lambda \longrightarrow \prod_{\lambda \in \Lambda} A_\lambda / \Gamma$ is denoted by q_Ξ.

When $A_\lambda = A$ for every $\lambda \in \Lambda$, then the identity map $d_\lambda: A \longrightarrow A_\lambda$ $(d_\lambda = A)$ induces the characteristic map $d_\Xi: A \longrightarrow \prod_{\lambda \in \Xi} A_\lambda$ for each $\Xi \subset \Lambda$.

Lemma 3. *If* $\Xi' \subset \Xi$ *where* $\Xi \subset \Lambda$, *then* $p^{\Xi}_{\Xi'} d_\Xi = d_{\Xi'}$.

Proof. Obviously $p^{\Xi'}_{\xi} p^{\Xi}_{\Xi'} d_\Xi = d_\xi$, from which the lemma follows immediately. q. e. d.

Hence we have a map $d: A \longrightarrow A^{\Lambda}/\Gamma$ as the direct limit of the natural transformation $\{d_{\Xi}\}_{\Xi\in\Gamma}$. Note that although each d_{Ξ} is an injection by Theorem 6, its direct limit d is not necessarily an injection.

The maps $d_{\Xi}: A \longrightarrow A^{\Xi}$ and $d: A \longrightarrow A^{\Lambda}/\Gamma$ will be called the diagonal maps. The notations d_{Ξ} and d will be used always in this meaning.

It is easily verified that if $\mathfrak{A}$ is a concrete category of structures in which every relation considered has finite arguments, then the definition above of reduced products agrees with the usual one given in the theory of models. (See [8], [17], [18]). But our definition is slightly wider, since it can be applied to structures with infinitistic relations, like topological spaces. However, the following example shows that the application of our definition to the category of topological spaces leads to a curious conclusion.

Example. Let $\mathfrak{C}$ be the category which consists of all topological spaces as the objects and all continuous functions between them as the maps. Referring to Proposition 4, it is easily seen that $\mathfrak{C}$ is complete. Further, $\mathfrak{C}$ is locally small, and hence it is perfect. The direct product of topological spaces with the weak topology satisfies, as well known, the condition of the product of them in the theory of categories.

Now let $\{A_{\lambda}\}_{\lambda\in\Lambda}$ be an indexed family of topological spaces, and $X=\prod_{\lambda\in\Lambda} A_{\lambda}/\Gamma$ the ultraproduct over a non-trivial ultrafilter Γ of Λ. Let V be a non-void open set of X and Ξ a set in Γ. Then since the canonical map $q_{\Xi}: \prod_{\lambda\in\Xi} A_{\lambda} \longrightarrow X$ is continuous, the inverse image $q_{\Xi}^{-1}(V)$ of V should be open in $\prod_{\lambda\in\Xi} A_{\lambda}$, and hence there exist a finite number of indices $\lambda_1, \lambda_2, \cdots, \lambda_n$ and non-voild open sets $V_1, V_2, \cdots, V_n$ each in $A_{\lambda_1}, A_{\lambda_2}, \cdots A_{\lambda_n}$ respectively such that $x\in\prod_{\lambda\in\Xi} A_{\lambda}$ and $p^{\Xi}_{\lambda_k}(x)\in V_k$ imply $q_{\Xi}(x)\in V$. Let Ξ' be the set obtained by reducing $\lambda_1, \lambda_2, \cdots, \lambda_n$ from Ξ. Since Γ is non-principal Ξ' is also a set in Γ. Since for each $x'\in\prod_{\lambda\in\Xi'} A_{\lambda}$ there exists an $x\in\prod_{\lambda\in\Xi} A_{\lambda}$ such that $p^{\Xi}_{\lambda_k}(x)\in V_k$ for every $k=1, 2, \cdots, n$, x' is mapped into V by the canonical map q_{Ξ}. Since q_{Ξ} is an epimap by Proposition 6, it is continuous function onto whole space X. Therefore $V=X$ and X has only two open sets, the void set and the whole set. Thus X has a very trivial topology regardless of the collection $\{A_{\lambda}\}_{\lambda\in\Lambda}$ of spaces. If we previously assume that the objects in $\mathfrak{C}$ are all Hausdorff spaces, then any non-principal ultraproduct of spaces consists of at most a single point. It is significant that when $A_{\lambda}=A$ for every $\lambda\in\Lambda$, the diagonal map $d: A \longrightarrow A^{\Lambda}/\Gamma$ is no more an injection in this case. In § 4, we shall investigate some conditions for the objects A in a perfect concrete category $\mathfrak{A}$ so that the diagonal map d be an injection for any Λ and Γ.

§ 3. Concrete categories.

In this section we deal with concrete categories, of which first we shall establish the terminology. Originally the terminology of the theory of categories seems to be constructed on the analogy of the theory of sets, but the correspondence of the terms between the both thoeries is not completely literal. We shall introduce some conditions on categories so that the correspondence becomes more natural and literal.

A *concrete category* $\mathfrak{C}$ is a subcategory of the category of all sets, i. e., a category in which with each object A a set $\mu(A)$ called the *basic set* of A is associated in such a way that $\mu(A)=\mu(B)$ implies $A=B$, and each map $f: A\longrightarrow B$ is an application from $\mu(A)$ to $\mu(B)$ where the composition of maps is the usual one of the applications. Each object in a concrete category is called a *structure*, and a structure A whose basic set $\mu(A)$ is a subset of $\mu(B)$ of B is called a *substructure* of B.

DEFINITION 5. A concrete category $\mathfrak{C}$ is called *set-theoretical*, if the following conditions CF 1)$\cdots$CF 5) are satisfied.

CF 1) *If* $A, B\in|\mathfrak{C}|$ *and* $\mu(A)\subset\mu(B)$, *then the application* j *from* $\mu(A)$ *to* $\mu(B)$ *such that* $j(\alpha)=\alpha$ *for any* $\alpha\in A$ *is a map iu* $\mathfrak{C}$, *and for any* $C\in|\mathfrak{C}|$ *and* $f\in$ Hom (C, B) *such that* $f(\mu(C))\subset\mu(A)$ *there exists a* $g\in$ Hom (C, A) *such that* $f=jg$.

CF 2) *Every monomap* $f: A\longrightarrow B$ *is a one-to-one application from* $\mu(A)$ *to* $\mu(B)$, *and every epimap* $f: A\longrightarrow B$ *is an application onto* $\mu(B)$.

CF 3) *The basic set of the product of a family* $\{A_\lambda\}_{\lambda\in\Lambda}$ *of objects is the set-theoretical direct product of the basic sets* $\mu(A_\lambda)$, $\lambda\in\Lambda$.

CF 4) *If* $f: A\longrightarrow B$, *then there exists* a $C\in|\mathfrak{C}|$ *such that* $\mu(C)=f(\mu(A))$.

CF 5) *If* K *is the equalizer of the indexed family* $\{f_\lambda: A\longrightarrow B\}_{\lambda\in\Lambda}$ *of maps* f_λ, *and* $k: K\longrightarrow A$ *is the equalizer map, then* $k(\mu(K))$ *coincides with jnst the set* $\{\alpha\in\mu(A)\,|\,f_\lambda(\alpha)=f_\xi(\alpha)$ *for any* $\lambda, \xi\in\Lambda\}$.

The map j in CF 1) will be called the *identical map*. In CF 5), it is easily seen that $k(\mu(K))$ is included in the latter set, but the coincidence does not seem generally true.

Hereafter we assume that the category $\mathfrak{C}$ is concrete and set-theoretical.

LEMMA 4. *If* $A, B \in |\mathfrak{C}|$ *and* $\mu(A) \subset \mu(B)$, *then the identical map* $j: A \longrightarrow B$ *is an injection. Conversely, if* $A, B \in |\mathfrak{C}|$ *and* $j: A \longrightarrow B$ *is an injection, then there exists a substructure* C *of* B *and an isomap* $a: A \longrightarrow C$ *such that* $j = j'a$ *where* $j': C \longrightarrow B$ *is the identical map.*

PROOF. Let $A \xrightarrow{a} C \xrightarrow{b} B$ be an epifactorization of the identical map $j: A \longrightarrow B$ where $\mu(A) \subset \mu(B)$. Since $\mu(b(C)) = \mu(A)$ by CF 2), there exists a $g: C \longrightarrow A$ such that $b = jg$ by CF 1). Hence we have $j = ba = jga$. Since j is naturally a monomap, $A = ga$. Hence a is a reversible epimap, and hence an isomap by Proposition 1.

Assume that $A, B \in |\mathfrak{C}|$ and $j: A \longrightarrow B$ is an injection. By CF 4), there exists a $C \in |\mathfrak{C}|$ such that $\mu(C) = j(\mu(A))$. Since $\mu(C) \subset \mu(B)$, we have the identical map $j': C \longrightarrow B$ and a map $a: A \longrightarrow C$ such that $j = j'a$ by CF 1). Since $a(\mu(A)) = j(\mu(A))$ $= \mu(C)$, a is an epimap and hence an isomap as an epifactor of the injection j. q. e. d.

Remark. C in CF 4) is a substructure of B. Hence we have the identical map $j: C \longrightarrow B$ and a map $g: A \longrightarrow C$ such that $f = jg$ by CF 1). Since $g(\mu(A)) = f(\mu(A))$ $= \mu(C)$, g is an epimap and $A \xrightarrow{g} C \xrightarrow{j} B$ is a critical epifactorization of f. Hence any map f in a set-theoretical concrete category admits the critical epifactorization. Bisides, it would be compatible to the definition of the image of a map to put $C = \mathrm{Im}(f)$, and we assume that $\mu(\mathrm{Im}(f))$ is a substructure of $\mu(\mathrm{Rg}(f))$ from now on.

In general it often occurs that under some condition an object A and a map $f: A \longrightarrow B$ are determined for $B \in |\mathfrak{C}|$ up to the equivalence class. If further f is an injection, then C in Lemma 4 and the identical map $j: C \longrightarrow B$ are respectively equivalent to A and f by Lemma 4. Hence, if a particular set is not concerned as for the basic set of A, we can assume that A is a substructure of B and the injection f is the identical map from A to B.

LEMMA 5. *If* $\mathfrak{C}$ *is left complete, and* $\{A_\lambda\}_{\lambda \in \Lambda}$ *is a family of substructures of a structure* B, *then there exists a substructure* C *of* B *such that* $\mu(C) = \bigcap_{\lambda \in \Lambda} \mu(A_\lambda)$.

PROOF. Put $B_\lambda = B$ for every $\lambda \in \Lambda$, and let $p_\xi: \prod_{\lambda \in \Lambda} A_\lambda \longrightarrow A_\xi$ be the projection of the product to its component. Letting $j_\xi: A_\xi \longrightarrow B_\xi$ be the identical map for each $\xi \in \Lambda$, which is an injection by Lemma 4, the characteristic map $p: \prod_{\lambda \in \Lambda} A_\lambda \longrightarrow \prod_{\lambda \in \Lambda} B_\lambda$ induced by the family $\{j_\xi p_\xi: \prod_{\lambda \in \Lambda} A_\lambda \longrightarrow B_\xi\}_{\xi \in \Lambda}$ is an injection by Theorem 6. Hence

$\underset{\lambda\epsilon\Lambda}{\Pi} A_\lambda$ can be assumed as a substructure of $\underset{\lambda\epsilon\Lambda}{\Pi} B_\lambda$, by the remark above. Let K be the equalizer of the family $\{j_\xi p_\xi\}_{\xi\epsilon\Lambda}$ of maps (notice that $B_\lambda=B$ for all $\lambda\epsilon\Lambda$) and $k: K\longrightarrow\underset{\lambda\epsilon\Lambda}{\Pi} A_\lambda$ the equalizer map. Then since k is an injection by Theorem 3, it is assumed as a substructure of $\underset{\lambda\epsilon\Lambda}{\Pi} A_\lambda$. Put $C=\mathrm{Im}(j_\xi p_\xi k)$. Since $j_\xi p_\xi k(\alpha)=j_\lambda p_\lambda k(\alpha)$ for any $\alpha\epsilon\mu(K)$ and $\xi,\lambda\epsilon\Lambda$, $\mu(C)$ is determined independently of the index $\xi\epsilon\Lambda$, and so also is the substructure C of B. Since $j_\xi p_\xi(\mu(\underset{\lambda\epsilon\Lambda}{\Pi} A_\lambda))=\mu(A_\xi)$, $\mu(C)\subset\mu(A_\xi)$ for any $\xi\epsilon\Lambda$, and hence $\mu(C)\subset\underset{\lambda\epsilon\Lambda}{\cap}\mu(A_\lambda)$. On the other hand, if $\alpha\epsilon\underset{\lambda\epsilon\Lambda}{\cap}\mu(A_\lambda)$, then $j_\xi(\alpha)=j_\lambda(\alpha)$ for any $\xi,\lambda\epsilon\Lambda$. Since $\mu(\underset{\lambda\epsilon\Lambda}{\Pi} A_\lambda)$ is the direct product $\underset{\lambda\epsilon\Lambda}{\Pi}\mu(A_\lambda)$ by CF 3), α uniquely determines an $\hat{\alpha}\epsilon\mu(\underset{\lambda\epsilon\Lambda}{\Pi} A_\lambda)$ such that $p_\xi(\hat{\alpha})=\alpha$ for any $\xi\epsilon\Lambda$. Hence we have $j_\xi p_\xi(\hat{\alpha})=j_\lambda p_\lambda(\hat{\alpha})=\alpha$ for any $\xi,\lambda\epsilon\Lambda$ and $\hat{\alpha}\epsilon\mu(K)$ by CF 5). Hence $\alpha=j_\xi p_\xi(\hat{\alpha})\epsilon j_\xi p_\xi(\mu(K))=\mu(C)$ which shows $\underset{\lambda\epsilon\Lambda}{\cap}\mu(A_\lambda)\subset\mu(C)$. q. e. d.

COROLLARY. *If $\mathfrak{C}$ is finitely letf complete, and A and A' are substructures of $B\epsilon|\mathfrak{C}|$, then there exists a substructure C of B snch that $\mu(C)=\mu(A)\cap\mu(A')$.*

LEMMA 6. *If $\mathfrak{C}$ is finitely left complete and $f: A\longrightarrow B$ is an epimap, then for any substructure B' of B there exists a substructure A' of A such that $\mu(A')=f^{-1}(\mu(B'))$.*

PROOF. Let $A\times B$ be the product of $\{A,B\}$ and $p_A: A\times B\longrightarrow A$ the projection. The family $\{A,f\}$ of maps induces a characteristic map $c: A\longrightarrow A\times B$, and there exists a substructure C of $A\times B$ such that $\mu(C)=c(\mu(A))$ by CF 4). Letting j be the identical map frou B' to B, $A\times B'$ can be assumed as a substructure of $A\times B$ with the injection which is the characteristic map induced by the family $\{A,j\}$ of injections. By the corollary of Lemma 5, there exists a substructure C' of $A\times B$ such that $\mu(C')=\mu(C)\cap\mu(A\times B')$. Let $c': C'\longrightarrow A\times B$ be the identical map. Then there exists a substructure A' of A such that $\mu(A')=p_A c'(\mu(C'))$ by CF 4). It is easy to see that A' satisfies the condition of the lemma. q. e. d.

COROLLARY. *If $\mathfrak{C}$ is finitely left complete, $f: A\longrightarrow B$ is a map in $\mathfrak{C}$, and C is a substructure of B, then there exists a substructure A' of A such that $\mu(A')=f^{-1}(f(\mu(A))\cap\mu(C))$.*

Since the correspondence between a structure and its basic set $\mu(A)$ is one-to-one, hereafter we will rather identify them, and the letters A, B, etc. for structures are assumed to denote the basic sets $\mu(A), \mu(B)$ etc. themselves. (For instance we write $A\subset B$ in place of $\mu(A)\subset\mu(B)$). Accordingly, the structure C in CF 4) and A' in

Lemma 6 are denoted by $f(A)$ and $f^{-1}(A)$ respectively. According to the original difinition, an object A means the identical map of the basic set of a structure to itself. Such an idntical map, the object, will be denoted by $\vec{A}$ from now on.

Henceforth, we often deal with subsets of the basic set of a structure, say A, which are not necessary the basic sets of substrucutres of A, and with functions from the basic set of a structure, or from its subset, to another, which are not necessarily maps in the category. We will write $f: A \cdots\rightarrow B$ with the broken arrow to show that f is a function from A to B, which is not necessarily a map in the category, where A and B are sets or structures, and in the latter case they are automatically taken as the basic sets of them. When $f: X \cdots\rightarrow Z$ and $Z \subset Y$, where X, Y and Z are sets, the restriction of f on X is denoded by f / X. Particularly if X, Y and Z are structures, and f is a map in $\mathfrak{C}$, then letting j be the identical map from X to Y we have $f / X = fj$.

Definition 6. Let $\mathfrak{C}$ be a concrete category, A and B structures in $\mathfrak{C}$, and $\varphi : A \cdots\rightarrow B$ a function from A to B (not necessarily a map in $\mathfrak{C}$). φ is called *finitely compatible* if for any finite subset X of A there exists a map $f: A \longrightarrow B$ in $\mathfrak{C}$ such that $\varphi / X = f / X$. A structure B is called *finitary* if every finitely comptible function from any structure A to B is a map in $\mathfrak{C}$. A structure B is called *finite* if its basic set is finite. B is called *strongly finitary*, if it is equivalent to a substructure of the product of a family of finitary finite structures. The category $\mathfrak{C}$ itself is called (*strongly*) *finitary*, if every structure in it is (strongly) finitary.

Concerning to the nature of finitary categories, refer to [16].

Lemma 7. *In a set-theoretical concrete category, a substructure of a (strongly) finitary structure is (strongly) finitary.*

Proof is easy.

Lemma 8. *Let $\mathfrak{C}$ be a set-theoretical concrete category, $\{B_\lambda\}_{\lambda\in\Lambda}$ a indexed family of objects and $B = \prod_{\lambda\in\Lambda} B_\lambda$. If B_λ for every $\lambda\in\Lambda$ is finitary, then B is also finitary.*

Proof. Let A be a structure and $\varphi : A \cdots\rightarrow B$ a finitely compatible function. Then for any finite set X of A there exists a map $f: A \longrightarrow B$ such that $f / X = \varphi / X$. Hence, letting $p_\lambda : B \longrightarrow B_\lambda$ be the projection, $p_\lambda f / X = p_\lambda \varphi / X$ for every $\lambda\in\Lambda$ where

$p_\lambda f: A \longrightarrow B_\lambda$ is a map in $\mathfrak{C}$. Thus the function $p_\lambda \varphi: A \cdots\rightarrow B_\lambda$ is finitaly compatible and hence it is a map in $\mathfrak{C}$, since B_λ is finitary. Let $g: A \longrightarrow B$ be the characteristic map induced by the family $\{p_\lambda \varphi\}_{\lambda \in \Lambda}$ of maps. Then for each $\lambda \in \Lambda$ and $\alpha \in A$, we have $p_\lambda g(\alpha) = p_\lambda \varphi(\alpha)$, which shows $g(\alpha) = \varphi(\alpha)$ by CF 3). Therefore $\varphi = g$ and $\varphi \in \mathfrak{C}$. q. e. d.

COROLLARY 1. *In a set-theoretical concrete category, a strongly finitary structure is finitary.*

COROLLARY 2. *If $\mathfrak{C}$ is a set-theoretical concrete category and B is the inverse limit of a diagram $\mathfrak{D}$ in $\mathfrak{C}$, in which every object is finitary, then B is also finitary.*

We omit the detail of the proof, but referring to the proof in [10] of Proposition 4 of this paper, we can know that the inverse limit B is obtained as an equalizer of maps from a product of objects in $\mathfrak{D}$ to another. Hence B is taken as a substructure of a suitable product of objects in $\mathfrak{D}$ by Lemma 4. From this and Lemmas 7, 8, the corollary follows immediately.

In the proof of Lemma 8, assumption CF 3) is essential and the corollary above follows from Lemma 4. Lemma 8 and the corollaries do not seem valid if we omit the assumption that $\mathfrak{C}$ is set-theoretical.

§ 4. Ultrapowers in a concrete category.

As seen in the example at the end of § 2, the ultrapower of a structure is sometimes reduced to trivial one. As stated there we will investigate here the condition for a structure, say A, not to make the ultrapower of it so trivial, and particularly the condition so that the diagonal map $d: A \longrightarrow A^\Lambda/\Gamma$ be an injection for any Λ and Γ. To spare the notice in each occurence, we agree that when $\mathfrak{D}$ is the product system of objects A_λ, $\lambda \in \Lambda$, over the filter Γ of Λ, $p^{\Xi}_{\Xi'}$ or p^{Ξ}_{ξ} denotes the projections from $\prod_{\lambda \in \Xi} A_\lambda$ to $\prod_{\lambda \in \Xi'} A_\lambda$ or to A_ξ respectively where $\Xi, \Xi' \in \Gamma$, and $\xi \in \Xi$, q_Ξ denotes the canonical map from $\prod_{\lambda \in \Xi} A_\lambda$ to $\prod_{\lambda \in \Lambda} A^\lambda/\Gamma$, and when $A_\lambda = A$ for every $\lambda \in \Lambda$, $d_\Xi: A \longrightarrow A^\Xi$ and $d: A \longrightarrow A^\Lambda/\Gamma$ denote the diagonal maps. For convenience sake we sometimes identify the maps f and g in CF 1), though they should be distinguished from each other in the strict sense. Further, throughout this section, the category $\mathfrak{C}$ is assumed concrete, set-theoretical and perfect to both sides.

THEOREM 7. *Let A be an structure in $\mathfrak{C}$. If the diagonal map $d: A \longrightarrow A^\Lambda/\Gamma$ is an injection for any set Λ and its ultrafiltr Γ, theu A is finitary.*

PROOF. Let B be a structure, φ a finitely compatible function from B to A and

$\mathfrak{X}$ the collection of all finite subsets of B. Put $\Lambda=\mathrm{Hom}\ (B, A)$, and for each $X\in\mathfrak{X}$, let Ξ_X be the set $\{f\in\Lambda\,|\,f(\alpha)=\varphi(\alpha)$ for any $\alpha\in X\}$. Since φ is finitely compatible, Ξ_X is not void for any $X\in\mathfrak{X}$. Put $\Gamma'=\{\Xi_X\,|\,X\in\mathfrak{X}\}$. Since $\Xi_{X''}\subset\Xi_X\cap\Xi_{X'}$ for any $X, X'\in\mathfrak{X}$ and $X''=X\cup X'$, there is an ultrafilter Γ of Λ which includes Γ'. Let $d: A\longrightarrow A^\Lambda/\Gamma$ be the diagonal map. Since it is an injection, it is an isomap from A to a substrucjure $d(A)$ of A^Λ/Γ by Lemma 4 (here the convention stated above is used). Let $A_f(=A)$ be the f-component of the product A^Λ and A^Ξ for $\Xi\in\Gamma$ and we name the maps as in fig. 6 where $p^\Xi_{\Xi'}, p^\Xi_f$ and p^Λ_f are the projections, q_Ξ (and similarly for q_Λ) is the canonical map and g_Ξ (similarly for g_Λ) is the characteristic map induced by the family $\{f: B\longrightarrow A_f\}_{f\in\Xi}$. The commutativity indicated in the diagram follows from the definition of the product system and that of the characteristic map g_Ξ. Further, we have $d_\Xi=p^\Lambda_\Xi d_\Lambda$ and $d=q_\Xi d_\Xi$ for the diagonal maps d_Λ, d_Ξ and d. Now we shall show that for any $\beta\in B$ we have $q_\Lambda g_\Lambda(\beta)=d\varphi(\beta)$. Indeed, let X be any finite subset of B containing β, and put $\Xi=\Xi_X$. Then by the definition of Ξ_X, $p^\Xi_f g_\Xi(\beta)=f(\beta)=\varphi(\beta)$ for any $f\in\Xi$. Hence $g_\Xi(\beta)=d_\Xi\varphi(\beta)$ and $q_\Lambda g_\Lambda(\beta)=q_\Xi p^\Lambda_\Xi g_\Lambda(\beta)=q_\Xi g_\Xi(\beta)=q_\Xi d_\Xi\varphi(\beta)=d\varphi(\beta)$. This shows $\varphi=d^{-1}q_\Lambda g_\Lambda$ where $d^{-1}: d(A)\longrightarrow A$ is the isomap. Hance $\varphi\in\mathfrak{C}$. q. e. d.

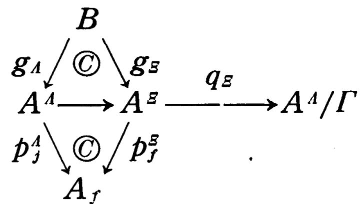


fig. 6

However, it is doubtful that we have the converse of this theorem.

Example. Let N be the additive semi-group of all natural numbers excluding 0 with the addition $+$. We generate a category $\mathfrak{C}'$ from $\{N\}$ so that it is closed with products, coproducts, equalizers and coequalizers, which are to be defined in the sense of the theory of general algebric systems, where the maps are addition preserving functions. Then $\mathfrak{C}'$ is a perfect set-theoretical concrete category. Let Γ be a non-principal ultrafilter of the basic set Λ of N and put $M=N^\Lambda/\Gamma$. M is prime on N. At thie stage, the diagonal map $d: N\longrightarrow M$ would be an injection. Omit all structures from $\mathfrak{C}'$ which are prime on N, except for the unit structure $U=\{0\}$, obtaing a full subcategory $\mathfrak{C}$ of $\mathfrak{C}'$. $N^\Lambda/\Gamma=U$ in $\mathfrak{C}$ and $\mathfrak{C}$ would be complete (after an adequate modification if necessary). Further N will remain finitary in $\mathfrak{C}$, but the diagonal map $N\longrightarrow U$ is no more an injection.

Observe that the essential point of this example is that N^Λ/Γ is prime on N

Now we shall investigate the condition of a structure A which infer that the diagonal map $A\longrightarrow A^{\Lambda}/\Gamma$ be an injection for any Λ and Γ.

LEMMA 9. *If the structure A consists of a finite number, say n, of elements, then the number of elements in A^{Λ}/Γ is at most n.*

PROOF. Let the basic set of A be $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ and $\mathfrak{D}$ the product system of $A_\lambda(=A)$, $\lambda\in\Lambda$, over an ultrafilter Γ of Λ. Let $\hat{\delta}_k$ be an element in A^{Ξ}, $\Xi\in\Gamma$, for each $k=1,2,\cdots,n$ such that $p_\lambda^{\Xi}(\hat{\delta}_k)=\alpha_k$ for every $\lambda\in\Xi$. We will show that for any element $\hat{\alpha}$ in A^{Ξ}, there exists a $\hat{\delta}_k$ such that $q_\Xi(\hat{\alpha})=q_\Xi(\hat{\delta}_k)$. Indeed, put $\Xi_k(\hat{\alpha})=\{\lambda\in\Xi \mid p_\lambda^{\Xi}(\hat{\alpha})=\alpha_k\}$. Then $\Xi=\bigcup_{k=1}^{n}\Xi_k(\hat{\alpha})$ and there exists a k such that $\Xi_k(\hat{\alpha})\in\Gamma$ since Γ is maximal. Put $\Xi'=\Xi_k(\hat{\alpha})$ and $\hat{\alpha}'=p_{\Xi'}^{\Xi}(\hat{\alpha})$. Then $p_\lambda^{\Xi'}(\hat{\alpha}')=p_\lambda^{\Xi}(\hat{\alpha})=\alpha_k$ for any $\lambda\in\Xi'$ which shows $p_{\Xi'}^{\Xi}(\hat{\alpha})=p_{\Xi'}^{\Xi}(\hat{\delta}_k)$ by CF 3). Hence $q_\Xi(\hat{\alpha})=q_{\Xi'}p_{\Xi'}^{\Xi}(\hat{\alpha})=q_{\Xi'}p_{\Xi'}^{\Xi}(\hat{\delta}_k)=q_\Xi(\hat{\delta}_k)$ and the image $q_\Xi(A^{\Xi})$ consists of at most n elements $q_\Xi(\hat{\delta}_k)$, $k=1,2,\cdots,n$. Since q_Ξ is an epimap by Proposition 4, it maps A^{Ξ} onto A^{Λ}/Γ by CF 2). Hence A^{Λ}/Γ consists of at most n elements.

q. e. d.

LEMMA 10. *If A is a finitary finite structure, then the diagonal map $d: A\longrightarrow A^{\Lambda}/\Gamma$ is an isomap, where Γ is an ultrafilter of Λ.*

PROOF. We use the same notations as in the previous lemma. Put $\varphi_\Xi(\hat{\alpha})=\alpha_k$ where k is such that $\Xi_k(\hat{\alpha})\in\Gamma$, we shall show that $\varphi_\Xi: A^{\Xi}\cdots\to A$ is finitely compatible. Indeed, let $X=\{\hat{\alpha}_1, \hat{\alpha}_2, \cdots, \hat{\alpha}_m\}$ be a finite subset of A^{Ξ}, and the number k_i is such that $\Xi_{k_i}(\hat{\alpha}_i)\in\Gamma$ for $i=1,2\cdots,m$. Then $\bigcap_{i=1}^{m}\Xi_{k_i}(\hat{\alpha}_i)$ is not void. Let ξ be an index in the intersection. Then $\xi\in\Xi_{k_i}(\hat{\alpha}_i)$ implies $p_\xi^{\Xi}(\hat{\alpha}_i)=\alpha_{k_i}=\varphi_\Xi(\hat{\alpha}_i)$. Hence $\varphi_\Xi/X=p_\xi^{\Xi}/X$ while p_ξ^{Ξ} is a map in $\mathfrak{C}$. Hence φ_Ξ is finitely compatible and so a map in $\mathfrak{C}$, since A is finitary. Assume $\Xi'\in\Gamma$ and $\Xi'\subset\Xi$. Put $\hat{\alpha}'=p_{\Xi'}^{\Xi}(\hat{\alpha})$. Then $p_\lambda^{\Xi'}(\hat{\alpha}')=p_\lambda^{\Xi}(\hat{\alpha})$ and hence $\Xi_k'(\hat{\alpha}')=\Xi'\cap\Xi_k(\hat{\alpha})$. Hence $\Xi_k(\hat{\alpha})\in\Gamma$ implies $\Xi_k'(\hat{\alpha}')\in\Gamma$. This shows $\varphi_\Xi(\hat{\alpha})=\varphi_{\Xi'}(\hat{\alpha}')=\varphi_{\Xi'}p_{\Xi'}^{\Xi}(\hat{\alpha})$ and the family $\{\varphi_\Xi \mid A^{\Xi}\longrightarrow A\}_{\Xi\in\Gamma}$ is compatible to the right for the product system. Hence we have the characteristic map $c: A^{\Lambda}/\Gamma\longrightarrow A$ such that $\varphi_\Xi=cq_\Xi$. Particularly $\varphi_\Xi(\hat{\delta}_k)=\alpha_k$, and φ_Ξ is an epimap. Hence c is an epimap by Proposition 2 and c maps A^{Λ}/Γ onto A by CF 2). But since A^{Λ}/Γ contains at most n elements by Lemma 9, it has just n elements and c is a monomap. Now since $d_\Xi(\alpha_k)=\hat{\delta}_k$ for the diagonal map $d_\Xi: A\longrightarrow A^{\Xi}$, we have $cq_\Xi d_\Xi=\overrightarrow{A}$. Hence c is a coreversible monomap, and hence an isomap by Proposition 2. Therefore $d=q_\Xi d_\Xi=c^{-1}$ is an isomap. q. e. d.

LEMMA 11. *Let B be a finitary finite structure. If there exists a map*

$f: A \longrightarrow B$ from a structure A, then for any ultrapower A^{Λ}/Γ, there exists a map $g: A^{\Lambda}/\Gamma \longrightarrow B$ such that $f=gd$ where $d: A \longrightarrow A^{\Lambda}/\Gamma$ is the diagonal map.

PROOF. Let $\mathfrak{D}$ and $\mathfrak{D}'$ be the product system of $A_\lambda (=A)$ and of $B_\lambda (=B)$ respectively over the same ultrafilter Γ of Λ. The notations for projections, the canonical maps and diagonal maps for $\mathfrak{D}$ are as they were, while let $p'^{\Xi}_{\Xi'}, p'^{\Xi}_{\lambda}, q'_{\Xi}$, and d'_{Ξ}, d' be the corresponding projections, canonical maps and diagonal maps for $\mathfrak{D}'$. The family $\{A_{\Xi} \xrightarrow{p^{\Xi}_{\lambda}} A_\lambda \xrightarrow{f} B_\lambda\}_{\lambda \in \Xi}$ of maps induces the characteristic map $r_{\Xi}: A^{\Xi} \longrightarrow B^{\Xi}$, and we have $p'^{\Xi}_{\Xi'} r_{\Xi} = r_{\Xi'} p^{\Xi}_{\Xi'}$. Hence the family of maps $\{r_{\Xi}\}_{\Xi \in \Gamma}$ is a natural transformation from $\mathfrak{D}$ to $\mathfrak{D}'$, and we have the direct limit $g' = \varinjlim_{\Xi \in \Gamma} r_{\Xi}: A^{\Lambda}/\Gamma \longrightarrow B^{\Lambda}/\Gamma$. Since $p'^{\Xi}_{\lambda} r_{\Xi} d_{\Xi} = f$ for any $\lambda \in \Xi$, we have $r_{\Xi} d_{\Xi} = d'_{\Xi} f$, and hence $g'd = g'q_{\Xi} d_{\Xi} = q'_{\Xi} r_{\Xi} d_{\Xi} = q'_{\Xi} d'_{\Xi} f = d'f$. But $d': B \longrightarrow B^{\Lambda}/\Gamma$ is an isomap by Lemma 10. Hence putting $g = d'^{-1} g' d$, we have $f=gd$. q. e. d.

THEOREM. 8. *If A is strongly finitary, then the diagonal map $d: A \longrightarrow A^{\Lambda}/\Gamma$ is an injection for any ultrapower A^{Λ}/Γ.*

PROOF. Since A is strongly finitary, there exists a family $\{B_\nu\}_{\nu \in \Delta}$ of finitary finite structures such that A is equivalent to a substructure of $\prod_{\nu \in \Delta} B_\nu$. Let $j: A \longrightarrow \prod_{\nu \in \Delta} B_\nu$ be the injection and $r_\nu: \prod_{\nu \in \Delta} B_\nu \longrightarrow B_\nu$ the projection for $\nu \in \Delta$. Then there exists a $g_\nu: A^{\Lambda}/\Gamma \longrightarrow B_\nu$ for each $\nu \in \Delta$ such that $r_\nu j = g_\nu d$ by Lemma 11. Now let $g: A^{\Lambda}/\Gamma \longrightarrow \prod_{\nu \in \Delta} B_\nu$ be the characteristic map induced by the family $\{g_\nu\}_{\nu \in \Delta}$, then by the uniqueness of the characteristic map, we have $j=gd$. Since j is an injection, so also is d by Theorem 1. q. e. d.

However, in order that the diagonal map $d: A \longrightarrow A^{\Lambda}/\Gamma$ be an injection, it is not necessary that A is strongly finitary. For example, let $\mathfrak{C}$ be the category of all torsion-free abelian groups. Then $\mathfrak{C}$ is a perfect set-thoretical concrete category. Let A be the additive group of all integers, then it can be seen that the diagonal map $A \longrightarrow A^{\Lambda}/\Gamma$ is an injection for any ultrapower, but $\mathfrak{C}$ contains no finite group except for $U = \{0\}$.

REFERENCES

[1] S. Eilenberg and S. MacLane, *General theory of natural equivalences,* Trans. Amer. Math. Soc. **58** (1945) p. 231–294.

[2] P. Freyd, *Abelian categories,* 1964, New York.

[3] J. R. Isbell, *Some remark concerning categories and subspaces,* Canadian J. Math. **9** (1957), p. 563–577.

[4] ...., *Algebras of uniformly continuous functions,* Ann. of Math. **68** (1960) p. 96–125.

[5] ...., *Natural sums and direct decompositions,* Duke Math. J. **27** (1960) p. 507–512.

[6] J. R. Isbell and Z. Semadeni, *Projection constants and spaces of continuous functions,* Trans. Amer. Math. Soc. **107** (1963) p. 38–48.

[7] D. M. Kan, *Adjoint functors,* Trans. Amer. Math. Soc. **87** (1958) p. 294–329.

[8] Simon Kochen, *Ultraproducts in the theory of models,* Ann. Math. **74** (1961) p. 221–261.

[9] A. G. Kurosh, A. H. Livshic and E. G. Shulgeifer, *Foundations of the theory of categories,* English traslation in Russian Math. Surveys from Uspiekhy Mat. Nauk. **15** (1960) (6) p. 1–46.

[10] F. W. Lawvere, *Functorial sematics of algebraic theories,* Dessertation, Columbia University, New York, 1963.

[11] L. Łos, *Quelques Remarques, Theoremes et Problemes sur les Classes Definissables d'Algebres* in *Mathematical Interpretation of Formal Systems,* Studies in Logic. (1955) 98–113.

[12] S. MacLane, *Duality of groups,* Bull. Amer. Math. Soc. **56** (1950), p. 485–516.

[13] ...., *Categorical Algebra,* Bull. Amer. Math. Soc. **71** (1965), p. 40–106.

[14] ...., *Locally small category and the foundation of set theory,* Proc. Symp. Foundation Math. Warsaw (1959).

[15] B. Michell, *Theory of categories,* (1965), New York.

[16] T. Ohkuma, *Closed classes of models,* Comment. Math. Univ. Sancti Pauli, **XIII**, p. 63–80.

[17] A. Robinson, *Complete Theories,* Studia in Logic, (1956).

[18] ...., *Introduction to model theory and to the metamathematics of algebra,* Studia in Logic (1963).

[19] Z. Semadeni, *Projectivity, injectivity and duality,* Rozprawy Math. **XXXV** (1963), p. 3–47.

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