A NOTE ON THE QUEUE WITH MULTIPLE POISSON INPUTS

TAKEHISA FUJISAWA

§ 1. INTRODUCTION

The queueing system studied is the one in which (i) two different types of customers arrive at a service mechanism in independent Poisson streams with mean rates λ_1 and λ_2; (ii) the queue-discipline is "first come, first served"; and (iii) the service-time distributions of both types are general with probability densities $dD_1(s)$ and $dD_2(s)$ respectively. In a previous paper [1] we discussed the distributions of the number of customers in the system and the waiting time. In the present paper the distribution of the time taken for a waiting time w to reduce to zero for the first time is obtained, and from this are deduced the explicit expressions for the distributions of (1) the length of the busy period, (2) the number of customers served during a busy period and (3) the time taken for a queue of given length to disappear. N. U. Prabhu [2] has studied these quantities in the queueing system $M/G/1$, i. e., the system in which the customers arrive in a Poisson process with mean rate λ, and the service-time distribution is $dB(t)$ $(0<t<\infty)$. The waiting-time process was investigated by L. Takács [3], who also made a systematic study of the distributions of the busy period and the number of customers served during a busy period. The purpose of this paper is to study some aspects of the extended queueing system $M(\lambda_1; \lambda_2)/G(D_1, D_2)/1$ by using the technique introduced in N. U. Prabhu [2].

§ 2. THE TIME TAKEN FOR THE WAITING-TIME TO REDUCE TO ZERO

There is a queue in front of a single server, and the waiting customers are served in order of arrival, with no defections from the queue. We are interested in the virtual waiting-time $W(t)$, which can be defined as the time a customer would have to wait for service if he arrived at time t. $W(t)$ is continuous from the left; at epochs of arrival of customer, $W(t)$ jumps upwards discontinuously by an amount equal to the service-time of the arriving customer; otherwise $W(t)$ has slope -1 while it is positive. If it reaches zero, it remains at zero until the next arrival epoch. The proposed definition of the time, T, required for the waiting time to reduce to zero is as follows. Let w be the waiting time $W(t_0)$, and let the waiting time reduce to zero for the first time at t_0+T; then a random variable T, which is the first passage time from w to zero of the process $W(t)$, is the desired one and we wish to obtain its distribution. In order to do this, we introduce the notation

$$dB_{m,n}(t\,|\,w)=P\{t<T<t+dt, M=m, N=n\,|\,W(t_0)=w\},$$

i.e., conditionally on $W(t_0)=w$, the joint distribution of T and the numbers M and N of two types of customers served during the interval (t_0, t_0+T), excluding the customers waiting in the queue or being served at the instant t_0. The random variables M and N are called the numbers of "new" type 1 and 2 customers respectively. Denoting the conditional probability density of T by $dB(t\,|\,w)$ we have the relation

$$dB(t\,|\,w)=\sum_{m,n} dB_{m,n}(t\,|\,w). \tag{1}$$

Let us derive a recurrence relation for $dB_{m,n}(t\,|\,w)$. For $m=n=0$ we have

$$dB_{0,0}(t\,|\,w)=e^{-(\lambda_1+\lambda_2)t}\,\delta(t-w) \tag{2}$$

where $\delta(t)$ is the Dirac delta function.

For if no new customers arrive during (t_0, t_0+w) then the waiting time w reduces to zero at time t_0+w. For $m+n\geqq 1$ the joint distribution $dB_{m,n}(t\,|\,w)$ of T, M and N satisfies the integral equations;

$$dB_{m,0}(t\,|\,w)=\int_{\tau=0}^{w}\int_{s=0}^{t-w}\lambda_1\,e^{-(\lambda_1+\lambda_2)\tau}dB_{m-1,0}(t-\tau\,|\,w-\tau+s)\,d\tau\,dD_1(s),\ m\geqq 1, \tag{3}$$

$$dB_{m,n}(t\,|\,w)=\int_{\tau=0}^{w}\int_{s=0}^{t-w}\{\lambda_1\,e^{-(\lambda_1+\lambda_2)\tau}dB_{m-1,n}(t-\tau\,|\,w-\tau+s)\,d\tau\,dD_1(s)$$
$$+\lambda_2\,e^{-(\lambda_1+\lambda_2)\tau}dB_{m,n-1}(t-\tau\,|\,w-\tau+s)\,d\tau\,dD_2(s)\},\ m, n\geqq 1 \tag{4}$$

and

$$dB_{0,n}(t\,|\,w)=\int_{\tau=0}^{w}\int_{s=0}^{t-w}\lambda_2\,e^{-(\lambda_1+\lambda_2)\tau}dB_{0,n-1}(t-\tau\,|\,w-\tau+s)\,d\tau\,dD_2(s),\ n\geqq 1. \tag{5}$$

Notice that, for $m\geqq 1$ and $n=0$ (or $m=0$ and $n\geqq 1$) at least one new type 1 (or type 2) customer must arrive during (t_0, t_0+w), as otherwise the waiting time w will reduce to zero at $t_0+w<t_0+t$; let the first new customer arrive at the instant $t_0+\tau$ $(0<\tau<w)$.

If $s\,(0<s\leq t-w)$ is the service time of this customer, then the waiting time of a customer who arrives at time $t_0+\tau+0$ becomes $w-\tau+s$. During the residual interval $(t_0+\tau, t_0+t)$ $m-1$ type 1 (or $n-1$ type 2) customers must be served. Thus we have the recurrence relation (3) (or (5)). The expression (4) can easily be derived by noticing if the first new arrival be type 1 or type 2.

From (3), (4) and (5) we obtain, for $t\geqq w$,

$$dB_{m,0}(t\,|\,w)=e^{-(\lambda_1+\lambda_2)t}\,\lambda_1\,w\,\frac{(\lambda_1\,t)^{m-1}}{m\,!}\,dD_1^{*m}(t-w), \qquad m\geqq 1, \tag{6}$$

$$dB_{m,n}(t\,|\,w)=e^{-\lambda t}\lambda w\frac{(\lambda t)^{m+n-1}}{m!\;n!}\alpha^m\beta^n\,dD_1^{*m}*dD_2^{*n}(t-w),\qquad m,n\geqq 1 \tag{7}$$

and

$$dB_{0,n}(t\,|\,w)=e^{-(\lambda_1+\lambda_2)t}\lambda_2 w\frac{(\lambda_2 t)^{n-1}}{n!}\,dD_2^{*n}(t-w),\qquad n\geqq 1 \tag{8}$$

where $\lambda_1=\alpha\lambda$, $\lambda_2=\beta\lambda$, $\alpha+\beta=1$ and $D_i^{*k}(t)$ is the k-fold convolution of $D_i(t)$ with itself and $D_i^{*0}(t)$ is zero if $t<0$ and unity if $t\geqq 0$.

Using the identity

$$\int_{s=0}^{z} s\,dD_i^{*n}(z-s)\,dD_i^{*m}(s)=\frac{mz}{m+n}\,dD_i^{*m+n}(z) \tag{9}$$

the expressions (6) and (8) can be easily proved by mathematical induction (see Appendix). As a check we shall prove that the solution (7) satisfies (4). To do this, substituting for $dB_{m-1,n}(t\,|\,w)$ and $dB_{m,n-1}(t\,|\,w)$ in the right-hand side of (4) we obtain

$$\begin{aligned}
&\int_{\tau=0}^{w}\int_{s=0}^{t-w}\Big[\lambda\alpha e^{-\lambda\tau}\cdot e^{-\lambda(t-\tau)}\lambda(w-\tau+s)\frac{\{\lambda(t-\tau)\}^{m+n-2}}{(m-1)!\,n!}\alpha^{m-1}\beta^n dD_1^{*m-1}*dD_2^{*n}(t-w-s)dD_1(s)\\
&+\lambda\beta e^{-\lambda\tau}\cdot e^{-\lambda(t-\tau)}\lambda(w-\tau+s)\frac{\{\lambda(t-\tau)\}^{m+n-2}}{m!\,(n-1)!}\alpha^m\beta^{n-1}dD_1^{*m}*dD_2^{*n-1}(t-w-s)dD_2(s)\Big]d\tau\\
&=\frac{e^{-\lambda t}\lambda^{m+n}\alpha^m\beta^n}{(m-1)!\,(n-1)!}\int_{\tau=0}^{w}(t-\tau)^{m+n-2}\Big[(w-\tau)\Big(\frac{1}{m}+\frac{1}{n}\Big)dD_1^{*m}*dD_2^{*n}(t-w)\\
&+\int_{s=0}^{t-w}s\Big\{\frac{dD_1^{*m-1}*dD_2^{*n}(t-w-s)\,dD_1(s)}{n}+\frac{dD_1^{*m}*dD_2^{*n-1}(t-w-s)\,dD_2(s)}{m}\Big\}\Big]d\tau\\
&=e^{-\lambda t}\lambda w\frac{(\lambda t)^{m+n-1}}{m!\,n!}\alpha^m\beta^n\,dD_1^{*m}*dD_2^{*n}(t-w)=dB_{m,n}(t\,|\,w).
\end{aligned}$$

In above expression we have used the identity

$$\int_{s=0}^{z}s\Big\{\frac{dD_1^{*m-1}*dD_2^{*n}(z-s)\,dD_1(s)}{n}+\frac{dD_1^{*m}*dD_2^{*n-1}(z-s)\,dD_2(s)}{m}\Big\}=\frac{z}{mn}\,dD_1^{*m}*dD_2^{*n}(z) \tag{10}$$

which can easily be proved by taking Laplace transforms of both sides. Notice that the right-hand side of (7) can be written as follows:

$$\frac{w}{t}\cdot\frac{e^{-\lambda_1 t}(\lambda_1 t)^m}{m!}\cdot\frac{e^{-\lambda_2 t}(\lambda_2 t)^n}{n!}\,dD_1^{*m}*dD_2^{*n}(t-w),\qquad m,n\geqq 1.$$

It is of some interest to point out the meaning which is given by the right-hand side of (6). Suppose that m type 1 customers arrive in the period (t_0, t_0+t). If all our m type 1 customers happened to arrive after the time t_0+w then our modified busy period T would end at $t_0+w<t_0+t$. Furthermore there are other ways in which the customers could arrive and produce an end to the modified busy period earlier than t_0+t, even though their service times have the required total. Fortunately, the probability that the customers arriving at random, would arrive in such a way that t_0+t is the first instant when the server becomes free is given by w/t. The details of the argument are found in D. R. Cox and W. L. Smith [4]. Here the details are omitted.

Notice that the probability that no type 2 customers arrive for time interval (t_0, t_0+t) is $e^{-\lambda_2 t}$ and the probability that exactly m type 1 customers arrive in (t_0, t_0+t) is $e^{-\lambda_1 t}(\lambda_1 t)^m/m!$. Since $dD_1^{*m}(t)$ is the probability density of the sum of m independent service times of type 1 customers the probability that no type 2 and exactly m type 1 customers arrive in (t_0, t_0+t) and that their service times sum to within a differential of $t-w$ is

$$e^{-\lambda_2 t}\cdot\frac{e^{-\lambda_1 t}(\lambda_1 t)^m}{m!}dD_1^{*m}(t-w).$$

Thus, the expression (6) gives the probability that no type 2 customers and exactly m type 1 customers arrive in (t_0, t_0+t) and the modified busy period, T, ends within a differential of t_0+t. Similarly the meaning of the right-hand side of (8) can easily be given by symmetry.

It is easy to see that the meaning of the alternating form for (7) is given by the same sort of argument.

Now it follows from (6), (7) and (8) that

$$\sum_{i=0}^{n} dB_{i,n-i}(t\mid w)=e^{-\lambda t}\lambda w\frac{(\lambda t)^{n-1}}{n!}\sum_{i=0}^{n}\binom{n}{i}\alpha^i\beta^{n-i}dD_1^{*i}dD_2^{*n-i}(t-w)$$

$$=e^{-\lambda t}\lambda w\frac{(\lambda t)^{n-1}}{n!}\{\alpha dD_1^*+\beta dD_2^*\}^n(t-w). \tag{11}$$

Hence the joint distribution, $dB_n(t\mid w)$, of T and the total number of both types of customers served during the interval (t_0, t_0+t), excluding the customers in the system at time t_0 is given by (11).

Using (1) and (11) we find that the conditional distribution, $dB(t\mid w)$, of T is given by

$$dB(t\mid w)=\sum_{n=0}^{\infty}\frac{w}{t}e^{-\lambda t}\frac{(\lambda t)^n}{n!}\{\alpha dD_1^*+\beta dD_2^*\}^n(t-w), \qquad t\geqq w. \tag{12}$$

We notice that the conditional joint distribution $dB_n(t\mid w)$ given by (11) may be derived by substituting $\alpha D_1(s)+\beta D_2(s)$ into the service-time distribution in the results derived by Prabhu [2]. In this case we can easily extend above arguments to the case of $k(>2)$ types of customers.

§3. THE BUSY PERIOD

A busy period begins when a (type 1 or 2) customer arrives to find the server free to deal with him at once (i. e., there is a 'zero queue'). It ends when the server completes the service of a customer and finds that there are no customers presently demanding service (i. e., there is a 'zero' queue again). Let the server be idle at time t_0-0 and a type 1 or 2 customer arrive at t_0; if then the server becomes idle for the first time at t_0+T, the random variable T is the length of the busy period. Denote the numbers of type 1 and 2 customers (including the one who arrived at time t_0) who are served during the busy period (t_0, t_0+T) by M and N respectively. From (7) the joint distribution of T, M and N is given by

$$dB_{m,n}(t)=\int_{s=0}^{t}\{\alpha\, dB_{m-1,n}(t\mid s)\,dD_1(s)+\beta\, dB_{m,n-1}(t\mid s)\,dD_2(s)\}$$

$$=e^{-\lambda t}\frac{(\lambda t)^{m+n-1}}{m!\;n!}\alpha^m\beta^n\, dD_1^{*m}\, dD_2^{*n}(t) \tag{13}$$

since the service time of the type i customer who arrived at time t_0 has the distribution function $D_i(t)$. In (13) we have again applied (10).

We call s in (13) the initial service-time of the busy period that will be derived later and say that the busy period is generated by s.

The distribution of the length of the busy period is given by

$$dB(t)=\sum_{m,n} dB_{m,n}(t)=\sum_{j=1}^{\infty}\sum_{\substack{m+n=j\\ m,n\leq 0}} dB_{m,n}(t)$$

$$=\sum_{j=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^{j-1}}{j!}\{\alpha dD_1+\beta dD_2\}^{j}(t),\qquad 0<t<\infty. \tag{14}$$

Similarly, the distribution of the length of the busy period started by type i customer can be obtained. Let $f_{m,n}$ be the probability of exactly m type 1 and n type 2 customers receiving service during a busy period. From (13) we have

$$f_{m,n}=\int_{t=0}^{\infty} dB_{m,n}(t)=\frac{\alpha^m\beta^n}{m!\,n!}\int_{t=0}^{\infty} e^{-\lambda t}(\lambda t)^{m+n-1}\, dD_1^{*m}\, dD_2^{*n}(t),\qquad m+n\geq 1. \tag{15}$$

In the ordinary queue $M/G/1$ we shall write $\varphi_{i,n}$ for the probability that, if we were to 'start' a busy period with the simultaneous arrival of i customers, then exactly n customers will have been served by the time the busy period ends. This number n includes the initial i customers who started the busy period. This probability parallels the distribution (12) with $D_1(t)=D_2(t)$. A 'normal' busy period will, of course, start with the arrival of just one customer, and $\psi_{1,n}$ is the probability that exactly n customers are served in the course of such a busy period. Thus the probability $\psi_{1,n}$ parallels the distribution (14) with $D_1(t)=D_2(t)$. The asymptotic formulae of $\psi_{m,n}$ (and $\psi_{1,n}$) for large values of n have been given by D. R. Cox and W. L. Smith.

§ 4. The Time Taken for the Queue of Given Length to Disappear

Let t_0 be an epoch just before the commencement of service of type 1 or type 2 customer, and let the numbers of type 1 and type 2 customers in the queue at that instant be m and n respectively.

We are interested in the time taken for the queue to disappear.

If this event occurs at time t_0+T, then the result of § 2 can be applied to obtain the conditional distribution, $dF(t\,|\,m, n)$, of T as follows.

$$dF(t\,|\,m, n)\equiv P\{t<T<t+dt\,|\,(m, n) \text{ at time } t_0\}$$
$$=\int_{w=0}^{t} dD_1^{*m}\cdot dD_2^{*n}(w)\, dB(t\,|\,w)$$
$$=\sum_{j=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^{j-1}}{j\,!}\int_{w=0}^{t}\lambda w\,\{\alpha\, dD_1^{*}+\beta\, dD_2^{*}\}^{j}(t-w)\cdot dD_1^{*m}\cdot dD_2^{*n}(w).$$

For the waiting time of a customer who arrives at t_0 is equal to the total service time w of m type 1 and n type 2 customers waiting in the queue, and this should reduce to zero at t_0+T.

If there are k customers (i. e., the sum of numbers of both types of customers) at instant t_0, the corresponding distribution $dF(t\,|\,m+n=k)$ of T is given by

$$dF(t\,|\,m+n=k)=\int_{w=0}^{t}\sum_{m=0}^{k}\binom{k}{m}\alpha^{m}\beta^{k-m}\, dD_1^{*m}\cdot dD_2^{*k-m}(w)\, dB(t\,|\,w)$$
$$=\sum_{j=0}^{\infty} e^{-\lambda t}\lambda\,\frac{(\lambda t)^{j-1}}{j\,!}\int_{w=0}^{t} w\,\{\alpha\, dD_1^{*}+\beta\, dD_2^{*}\}^{k}(w)\cdot\{\alpha\, dD_1^{*}+\beta\, dD_2^{*}\}^{j}(t-w)$$
$$=\sum_{j=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^{j}}{j\,!}\cdot\frac{k}{k+j}\{\alpha\, dD_1^{*}+\beta\, dD_2^{*}\}^{k+j}(t).$$

In above algebra we have again applied (9).

APPENDIX

Proof for (6): It is easily seen that (6) is true for $m=1$. We shall now suppose (6) has been proved for $1, 2, \cdots, m-1$ and attempt to show that it is then necessarily for m. Once we have accomplished this the truth of (6) for all m is established.

Let us assume that (6) holds for $1, 2, \cdots, m-1$; then substituting for $dB_{m-1,0}(t\,|\,w)$ in the right-hand side of (3) we have

$$dB_{m,0}(t\,|\,w)=e^{-(\lambda_1+\lambda_2)t}\frac{\lambda_1^m}{(m-1)!}\int_{\tau=0}^{w}(t-\tau)^{m-2}d\tau\int_{s=0}^{t-w}(w-\tau+s)\,dD_1^{*m-1}(t-w-s)\,dD_1(s)$$

$$=e^{-\lambda t}\frac{\lambda_1^m}{(m-1)!}\int_{\tau=0}^{w}(t-\tau)^{m-2}\left\{w-\tau+\frac{t-w}{m}\right\}dD_1^{*m}(t-w)\,d\tau$$

$$=e^{-\lambda t}\lambda_1 w\,\frac{(\lambda_1 t)^{m-1}}{m!}\,dD_1^{*m}(t-w).$$

This completes the proof of (6). In above expression we have used the identity (9) with $m=1$.

ACKNOWLEDGMENT

The author wishes to thank Prof. T. Homma and Dr. H. Morimura for their valuable assistance.

REFERENCES

[1] T. Homma and T. Fujisawa, "Some notes on the queues with multiple inputs", The Yokohama Math. J., Vol. 12 (1964), pp. 1—15.

[2] N. U. Prabhu, "Some results for the queue with Poisson arrivals", J. Roy. Stat. Soc., Vol. 22 (1960), pp. 104—107.

[3] L. Takács, "Investigation of waiting time problems by reduction to Markov processes", Acta Math. Acad. Sci. Hung., Vol. 6 (1955) pp. 101—129.

[4] D. R. Cox and W. L. Smith, QUEUES, Methuen's monographs on statistical subjects, London (1961).

Waseda University,
College of Science & Technology.

(Received Oct. 1, 1965)