# GENERALIZED MIXED-TOPOLOGIES IN DUAL LINEAR SPACES.

By

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**Introduction.** In a previous paper [4] we introduced the definition of the generalized mixed-topology in a locally convex linear topological space and a general method to construct a neighbourhood basis for each of various mixed topologies.

Roughly speaking, the generalized mixed-topology associated with a locally convex linear topology $\mu$ and a family $\mathfrak{A}$ of suitable subsets of a linear space $E$ is the finest locally convex linear topology which is identical with $\mu$ on each member of $\mathfrak{A}$, and in $E$, it is generally finer than $\mu$.

In the special case such that two locally convex linear topologies $\mu$ and $\tau$ are defined in $E$ and $\mathfrak{A}$ is a family of all the $\tau$-bounded subsets of $E$, it coincides with the mixed-topology which was defined by A. Persson [5] by means of abstracting a unique property of the mixed-topology defined by A. Wiweger [6], who is the first to investigate systematically, in a normed space, the mixed topology, however which was called first by A. Alexiewicz and Z. Semadeni.

In this paper, we shall develop at some length the theory of the generalized mixed-topology and we shall show some applications and examples in dual linear spaces.

Henceforth, by the mixed topology we mean the generalized mixed-topology defined in [4].

In the former part of this paper, we shall describe preliminary definitions and summarizations of the results in [4], and in addition to them we shall show some properties of the mixed topology.

In the later, the suitable form of a neighbourhood for the mixed topology in dual linear spaces will be researched, and we shall show examples of mixed topologies in dual linear spaces and investigate properties of them.

## § 1. Preliminary definitions and notations.

A family $\mathfrak{A}$ of subsets of a linear space over the real number field $R$ is called **primitive** if $\mathfrak{A}$ satisfies the following conditions;

($P_1$) *if* $A\in\mathfrak{A}, \lambda\in R, \lambda\neq 0$, *then* $\lambda A\in\mathfrak{A}$,

($P_2$) *if* $A\in\mathfrak{A}, \lambda\in R, |\lambda|\leqslant 1$, *then* $\lambda A\subset A$,

($P_3$) *if* $x\in E$, *then there exists* $A\in\mathfrak{A}$ *such as* $A\ni x$, and a family $\mathfrak{U}$ of subsets of $E$ is called neighbourhood basis at 0 for a linear Hausdorff topology (LHT) if $\mathfrak{U}$ is primitive and satisfies the following conditions;

($0_1$) *if* $x \in E$, *then there exists* $\lambda \in R, \lambda \neq 0$ *such that* $\lambda x \in U$ *for every* $U \in \mathfrak{U}$,

($0_2$) *if* $U_1 \in \mathfrak{U}$ *and* $U_2 \in \mathfrak{U}$, *then there exists* $U_3 \in \mathfrak{U}$ *such that* $U_3 \subset U_1 \cap U_2$,

($0_3$) *if* $U_1 \in \mathfrak{U}$, *then there exists* $U_2 \in \mathfrak{U}$ *such that* $U_2 \supset U_1 + U_1$, *where* $U_1 + U_1$ *denotes the set* $\{x+y; x \in U_1$ *and* $y \in U_1\}$,

($H$) *if* $x \in E, x \neq 0$, *then there exists* $U \in \mathfrak{U}$ *such as* $x \bar{\in} U$.

A linear Hausdorff topology satisfying the condition;

($k$) *each member of* $\mathfrak{U}$ *is convex*,

is called a locally convex linear Hausdorff topology.

We mean this by a locally convex topology (or LCT), and by a locally convex space (LCS) we mean a linear space with a locally convex topology.

We denote the linear space of all continuous linear functionals on $E$ for a locally convex topology $\tau$ by $(E, \tau)'$ or simply by $E'$, and we call $(E, \tau)'$ the dual space of $E$ with $\tau$, or simply the dual of $E$.

In this paper, whenever we speak of the dual space of $E$, we assume that $E$ is a locally convex space.

Then, $E$ and $E'$ are in duality by the natural bilinear functional such that $<x, x'> = x'(x)$ for all $x \in E$ and all $x' \in E'$, that is, $E$ and $E'$ satisfy the dual conditions;

($D_1$) *for any* $x \neq 0$ *in* $E$, *there exists* $x' \in E'$ *such as* $<x, x'> \neq 0$,

($D_2$) *for any* $x' \neq 0$ *in* $E'$, *there exists* $x \in E$ *such as* $<x, x'> \neq 0$.

and $E$ and $E'$ are called the dual linear spaces (for the natural bilinear functional), where the rôles of $E$ and $E'$ are interchagable.

A family $\mathfrak{B}$ of subsets of an *LCS* $E$ (*or* $E'$) is called **admissible** for topologizing the dual space $E'$ of $E$ (or $E$ resp.) if $\mathfrak{B}$ is a primitive family which satisfies the condition ($k$) and the following conditions;

($a$) *if* $B_1 \in \mathfrak{B}$ *and* $B_2 \in \mathfrak{B}$, *then there exists* $B_3 \in \mathfrak{B}$ *such that* $B_3 \supset B_1 \cup B_2$,

($b$) *each member of* $\mathfrak{B}$ *is bounded for the weak topology* $w(E, E')$ (*or* $w'(E', E)$) *which is defined as the weakest topology that makes the bilinear functional continuous in each variable separately*,

($c$) *each member of* $\mathfrak{B}$ *is closed for* $w(E, E')$ (*or* $w'(E', E)$).

A primitive family $\mathfrak{A}$ is called $k$-primitive if $\mathfrak{A}$ satisfies the conditions $(k)$, and is called $K$-primitive if $\mathfrak{A}$ satisfies the conditions $(k)$ and $(c)$, and is called $H$-primitive if $\mathfrak{A}$ satisfies the condition $(H)$, we denote sometimes each of them by $k(\mathfrak{A}), K(\mathfrak{A})$ and $H(\mathfrak{A})$ respectively.

Remarks:

$(R_1)$ *The conditions* $(P_3)$ *and* $(a)$ *imply that each finite set is contained in at least one of members of* $\mathfrak{B}$.

$(R_2)$ *The conditions* $(P_2)$ *and* $(k)$ *are sufficient for the condition* $(0_3)$.

A subfamily $\mathfrak{B}$ of a primitive family $\mathfrak{A}$ is called coarser than $\mathfrak{A}$ (or equivalently $\mathfrak{A}$ is finer than $\mathfrak{B}$) if $\mathfrak{B}$ is also primitive, and this is the case if and only if two primitive families satisfy the condition;

$(Q_1)$ *for every* $B\epsilon\mathfrak{B}$, *there exists* $A\epsilon\mathfrak{A}$ *such that* $A\subset B$.

A subfamily $\mathfrak{B}$ of a family $\mathfrak{A}$ is called a co-base for $\mathfrak{A}$ if $\mathfrak{A}$ and $\mathfrak{B}$ satisfy the conditidn;

$(Q_2)$ *for every* $A\epsilon\mathfrak{A}$, *there exists* $B\epsilon\mathfrak{B}$ *such that* $B\supset A$.

A co-base $\mathfrak{B}$ for $\mathfrak{A}$ is called the circled co-base for $\mathfrak{A}$ if $C(\mathfrak{A})$, the family of all the circled envelopes of members of $\mathfrak{A}$, coincides with $\mathfrak{B}$.

The convex co-base $\mathfrak{B}$ for $\mathfrak{A}$ is similarly defined as the co-base such that $k(\mathfrak{A})$, the family of all convex envelopes of members of $\mathfrak{A}$, coincides with $\mathfrak{B}$.

We denote the topology defined by uniform convergenc on members of an admissible family $\mathfrak{B}$ of $E$ (or $\mathfrak{B}'$ of $E'$) by $\rho(E',\mathfrak{B}^0)$ (or $\rho(E,\mathfrak{B}'^0)$), where $\mathfrak{B}^0(\mathfrak{B}'^0)$ denotes the family of all the polar sets of members of $\mathfrak{B}(\mathfrak{B}')$ and forms a neighbourhood basis at 0 for $\rho(E',\mathfrak{B}^0)$ (or $\rho(E,\mathfrak{B}'^0)$).

It is easily seen that if an admissible family $\mathfrak{B}$ in $E$ is a co-base for a family $\mathfrak{A}$ (which needs not be admissible), then the topology of uniform convergence on $\mathfrak{A}$, $\rho(E',\mathfrak{A}^0)$, is identical with $\rho(E',\mathfrak{B}^0)$.

We say that a topology $\rho$ in $E$ is an admissible topology relative to the dual linear spaces $(E,\tau)$ and $(E,\tau)'$ if it satisfies the three conditions;

$(t_1)$ $\rho$ *is a locally convex topology in* $E$,

$(t_2)$ *each member of* $(E,\tau)'$ *is continuous for* $\rho$ *on* $E$,

$(t_3)$ *there is an admissible family* $\mathfrak{A}'$ *in* $E'$ *such that* $\rho$ *coincides with* $\rho(E,\mathfrak{A}'^0)$.

and this is the case if and only if $w(E,E')\leqslant\rho\leqslant S(E,E')$, where $S(E,E')$, is the strong topology, that is, the uniform convergence topology on every bounded subsets of $E'$, and the weak topology $w(E, E')$ is the uniform convergence topology on every finite subsets of $E'$.

An admissible topology $\rho$ relative to the dual linear spaces $(E, \tau)$ and $(E, \tau)'$ is called compatible with the duality if the following condition is satisfied;

$(t_4)$ $(E, \rho)' = (E, \tau)'$,

and this is the case if and only if $w(E, E') \leqslant \rho \leqslant m(E, E')$, where $m(E, E')$ is the Mackey topology.

$(R_3)$ *It is known that the bounded subsets are the same for any compatible topology.*

§ 2. **(Generalized) mixed-topology in an LCS E.**

In this section, we recall briefly some of the results in [4].

(2.1) Let an LCT $\mu$ be defined in a linear space $E$, and let $\mathfrak{U}$ be a neighbourhood basis at 0 for $\mu$, and let $\mathfrak{A}$ be a primitive family which consists of $\{A_\iota, \iota \in I\}$.

Taking an arbitrary subfamily $\mathfrak{U}_J$ of $\mathfrak{U}$, we set

(1) $U^\alpha = k\{\bigcup_{\iota \in I}(U_\iota \cap A_\iota)\}$

where $U_\iota \in \mathfrak{U}_J \subset \mathfrak{U}$ and $k\{\cdots\}$ denotes the convex envelope of $\{\cdots\}$, then the family $\mathfrak{U}^\alpha$ of all the sets (1) is also a neighbourhood basis at 0 for a new locally convex topology, which is called the (generalized) mixed topology determined by $\mu$ and $\mathfrak{A}$ and is denoted by $\alpha(\mu, \mathfrak{A})$ or simply $\mu^\alpha$.

(2.2) $\mu^\alpha$ is not weaker than $\mu$. i.e. $\mu \leqslant \mu^\alpha$.

Henceforth, we assume that $\mu^\alpha = \alpha(\mu, k(\mathfrak{A}))$, i.e. $\mu^\alpha$ is the mixed topology determined $\mu$ and a $k$-primitive family, and that $\tilde{\mathfrak{A}}$ is the family which contains $k(\mathfrak{A})$ as the circled convex co-base, then:

(2.3) $\mu | \tilde{A} = \mu^\alpha | \tilde{A}$ for every $\tilde{A} \in \tilde{\mathfrak{A}}$

where $\mu | \tilde{A}, \mu^\alpha | \tilde{A}$ denote the topologies induced on $\tilde{A}$ by $\mu$ and $\mu^\alpha$ respectively.

(2.4) Let $\mu$ and $\nu$ be two LCTs in $E$, and let $\mu^\alpha$ and $\nu^\alpha$ be two mixed topologies determined by the same $k$-primitive family, then the following conditions are equivalent;

(i) $\mu | \tilde{A} = \nu | \tilde{A}$ (ii) $\nu \leqslant \mu^\alpha$ and $\mu \leqslant \nu^\alpha$ (iii) $\mu^\alpha = \nu^\alpha$

(2.5) In particular

$\alpha(\mu, k(\mathfrak{A})) = \alpha\{\alpha(\mu, k(\mathfrak{A})), k(\mathfrak{A})\}$.

(2.6) Let $f$ be a linear mapping from an LCS $E$ with LCT $\mu$ into another LCS $F$ with an LCT $\mu'$, then $f$ is $(\mu, \mu')$-continuous on every $\tilde{A}\in\tilde{\mathfrak{A}}$ if and only if $f$ is $(\mu^\alpha, \mu')$-continuous on $E$.

(2.7) It follows from (2.4) that $\alpha(\mu, k(\mathfrak{A}))$ is the finest LCT in $E$ which is identical with $\mu$ on each $\tilde{A}\in\tilde{\mathfrak{A}}$.

(2.8) In particular, if $k(\mathfrak{A})$ is a family of all the $\nu$-bounded circled convex subsets of $E$, then $\alpha(\mu, k(\mathfrak{A}))$ coincides with Persson's mixed-topology, and moreover if $k(\mathfrak{A})$ is a $\nu$-neighbourhood basis at 0 which is locally convex and locally bounded then $\alpha(\mu, k(\mathfrak{A}))$ coincides with Wiweger's mixed-topology.

(2.9) Some important examples of $k$-primitive families in $E$ are;

i) the family of all the bounded (or totally bounded, compact) convex circled subsets for an LHT in $E$,

ii) the family of all the convex envelopes of symmetric finite subsets of $E$,

iii) and, of course, each admissible family in $E$,

iv) a convex neighbourhood basis at 0 for every locally convex topology.

**§ 3.** In this section we shall show, in addition to [4], some properties of the mixed topology.

Let two locally convex topologies $\mu$ and $\nu$ be defined in a linear space $E$, and let $\mathfrak{A}$, $\mathfrak{B}$ be two primitive families in $E$, and let $\mathfrak{U}, \mathfrak{U}^\alpha, \mathfrak{U}^\beta, \mathfrak{V}, \mathfrak{V}^\alpha$ and $\mathfrak{V}^\beta$ be a neighbourhood basis at 0 for $\mu, \alpha(\mu, \mathfrak{A}), \alpha(\mu, \mathfrak{B}), \nu, \alpha(\nu, \mathfrak{A})$ and $\alpha(\nu, \mathfrak{B})$ respectively.

**Proposition 3.1.** *If* $\mu\leqslant\nu$, *then* $\alpha(\mu, \mathfrak{A})\leqslant\alpha(\nu, \mathfrak{A})$.

**Proof.** For every $U\in\mathfrak{U}$, by $\mu\leqslant\nu$, there exists $V\in\mathfrak{V}$ such that $V\subset U$, so for each $\iota\in I$, there exists $V_\iota\in\mathfrak{V}$ such that $V_\iota\cap A_\iota\subset U_\iota\cap A_\iota$, hence for every $U^\alpha\in\mathfrak{U}^\alpha$ there exists $V^\alpha\in\mathfrak{V}^\alpha$ such that

$$V^\alpha=k\{\bigcup_{\iota\in I}(V_\iota\cap A_\iota)\}\subset k\{\bigcup_{\iota\in I}(U_\iota\cap A_\iota)\}=U^\alpha.$$

**Remark 3.1.** In particular, if $\mu\leqslant\nu$ and $\mu|\tilde{A}=\nu|\tilde{A}$ for every $\tilde{A}\in\tilde{\mathfrak{A}}$, then $\alpha(\mu, k(\mathfrak{A}))=\alpha(\nu, k(\mathfrak{A}))$ by (2.4).

**Proposition 3.2.** *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *satisfy the condition* $(Q_2)$, *then* $\alpha(\mu, \mathfrak{B})\leqslant\alpha(\mu, \mathfrak{A})$.

**Proof.** For each $A_\iota\in\mathfrak{A}=\{A_\iota, \iota\in I\}$, let $B_{j_\iota}$ denote one $B_j\in\mathfrak{B}=\{B_j, j\in J\}$ such that $B_j\supset A_\iota$, and take $U_\iota\in\mathfrak{U}$ such that $U_\iota\subset U_{j_\iota}$, then for every $U^\beta\in\mathfrak{U}^\beta$ there exists $U^\alpha\in\mathfrak{U}^\alpha$ such that

$$U^\alpha=k\{\bigcup_{\iota\in I}(U_\iota\cap A_\iota)\}\subset k\{\bigcup_{\iota\in I}(U_{j_\iota}\cap B_{j_\iota})\}\subset k\{\bigcup_{j\in J}(U_j\cap B_j)\}=U^\beta.$$

In the special case such that $\mathfrak{B}$ is a subfamily of a family $\mathfrak{A}$, by rephrasing

Prop. 3.2 we have a corollary.

**Corollary.** *If* $\mathfrak{V}$ *is a co-base for* $\mathfrak{A}$, *then* $\alpha(\mu, \mathfrak{V}) \leqslant \alpha(\mu, \mathfrak{A})$.

**Proposition 3.3.** *If* $\mu \leqslant \nu$, then $\mu \leqslant \alpha(\mu, \mathfrak{V}) \leqslant \nu$.

**Proof.** The former inequality is evident by (2.2), so we shall prove the later. Let $\mathfrak{U}^\nu$ be a neighbourhood basis at 0 for $\alpha(\mu, \mathfrak{V})$. For any $U^\nu \epsilon \mathfrak{U}^\nu$, which is decided correspondingly to a subfamily $\mathfrak{U}_j$ of $\mathfrak{U}$, by hypothesis there exists a subfamily $\mathfrak{V}_j$ of $\mathfrak{V}$ such that

$$\mathfrak{V}_j = \{V_\iota \epsilon \mathfrak{V};\ V_\iota \subset U_\iota \quad \text{for each} \quad U_\iota \epsilon \mathfrak{U}_j\}$$

and moreover there exists $V \epsilon \mathfrak{V}$ such that $V \subset \cap V_\iota$ where $V_\iota \epsilon \mathfrak{V}_j$, consequently there exists $V \epsilon \mathfrak{V}$ such that

$$V = k\{\bigcup_{\iota \epsilon I}(V \cap V_\iota)\} \subset k\{\bigcup_{\iota \epsilon I}(U_\iota \cap V_\iota)\} = U^\nu$$

where $U_\iota \epsilon \mathfrak{U}_j$ and $V_\iota \epsilon \mathfrak{V} = \{V_\iota, \iota \epsilon I\}$.

**Corollary.** $\alpha(\mu, \mathfrak{U}) = \mu$.

In prop. 3.3 setting $\mu = \nu$ we obtain this.

**Proposition 3.4.** *If* $\mu \geqslant \nu$, *then* $\mu = \alpha(\mu, \mathfrak{V})$.

**Proof.** By $\mu \geqslant \nu$, there exists $W_\iota \epsilon \mathfrak{U}$ such that $U_\iota \cap W_\iota \subset U_\iota \cap V_\iota$ for each $\iota \epsilon I$ where $V_\iota \epsilon \mathfrak{V}$, $U_\iota \epsilon \mathfrak{U}_j$ and there exists $U \epsilon \mathfrak{U}$ such that $U \subset U_\iota \cap W_\iota$ for any $\iota \epsilon I$. So, for any subfamily $\mathfrak{U}_j$ of $\mathfrak{U}$, that is, for any $U^\nu \epsilon \mathfrak{U}^\nu$, we have $U \epsilon \mathfrak{U}$ such that

$$U \subset U_\iota \cap W_\iota \subset k\{\bigcup_{\iota \epsilon I}(U_\iota \cap W_\iota)\} \subset k\{\bigcup_{\iota \epsilon I}(U_\iota \cap V_\iota)\} = U^\nu$$

where $U_\iota \epsilon \mathfrak{U}_j$, $V_\iota \epsilon \mathfrak{V}$,
that is, $\alpha(\mu, \mathfrak{V}) \leqslant \mu$, while by (2.2), $\mu \leqslant \alpha(\mu, \mathfrak{V})$.
Hence $\mu = \alpha(\mu, \mathfrak{V})$.

**Corollary 1.** *If* $\mu \leqslant \nu$ *then* $\alpha(\mu, \mathfrak{V}) \leqslant \alpha(\nu, \mathfrak{U})$.

In fact, by Prop. 3.3, $\alpha(\mu, \mathfrak{V}) \leqslant \nu$, and changing the rôles of $\mu$ and $\nu$ in Prop. 3.4 we obtain $\nu = \alpha(\nu, \mathfrak{A})$.
So, we see that $\mu \leqslant \alpha(\mu, \mathfrak{V}) \leqslant \alpha(\nu, \mathfrak{U}) = \nu$ if $\mu \leqslant \nu$.

**Corollary 2.** *If for every* $A \epsilon k(\mathfrak{A})$ *there exists* $U \epsilon \mathfrak{U}$ *such that* $U \subset A$, *or for every* $U \epsilon \mathfrak{U}$ *there exists* $A \epsilon k(\mathfrak{A})$ *such that* $U \subset A$, *then* $\mu = \alpha(\mu, k(\mathfrak{A}))$.

In fact, taking $A_\iota \epsilon k(\mathfrak{A})$ in the place of $V_\iota$ in the proof of Prop. 3.4, we similarly obtain $\mu = \alpha(\mu, k(\mathfrak{A}))$.

In the second case, of course $\mathfrak{U}$ is $k$-primitive family, so, by Prop. 3.2 $\alpha(\mu, k(\mathfrak{A}) \leqslant \alpha(\mu\ \mathfrak{U}) = \mu$, while $\mu \leqslant \alpha(\mu, k(\mathfrak{A}))$, hence $\mu = \alpha(\mu, k(\mathfrak{A}))$.

**Proposition 3.5.** *If* $\mathfrak{P}$ *is the family of all the symmetric line segments,*

$\{\lambda x+(1-\lambda)(-x)\,;0\leqslant\lambda\leqslant 1,\ x\in E\}$, *then for any primitive family* $\mathfrak{B}$

$$\alpha(\mu,\mathfrak{B})\leqslant\alpha(\mu,\mathfrak{P})$$

*that is,* $\mathfrak{P}$ *is the finest primitive family of which mixed topologies associated with* $\mu$, $\alpha(\mu,\mathfrak{P})$, *is the finest of all the mixed topologies determined with the same LCT* $\mu$ *and primitive families.*

**Proof.** It is obvious that $\mathfrak{P}$ satisfies the conditions $(P_1)$, $(P_2)$ and $(P_3)$. Let $\mathfrak{B}$ be an arbitrary primitive family, then for any $P\in\mathfrak{P}$, there exists $B\in\mathfrak{B}$ such that $B\supset P$ by $(P_2)$ and $(P_3)$.

By Prop. 3.2 $\alpha(\mu,\mathfrak{B})\leqslant\alpha(\mu,\mathfrak{P})$.

**Proposition 3.6.** *The family* $\Phi$ *of all the weakly closed convex envelopes of symmetric finite subsets of* $E$ *is the finest admissible family of which mixed topology associated with* $\mu$ *is the finest of all the mixed topologies determined with* $\mu$ *and admissible families.*

**Proof.** It is easily verified that $\Phi$ is an admissible family.

Let $\mathfrak{B}$ be an arbitrary admissible family, for every $\varphi\in\Phi$ there exists $B\in\mathfrak{B}$ such that $B\supset\varphi$ by $(R_1)$ and convexity of $B$. So $\alpha(\mu,\mathfrak{B})\leqslant\alpha(\mu,\Phi)$.

**Theorem 3.1.** *Let* $\mu^{\alpha}$ *be the mixed topology determined by a locally convex topology* $\mu$ *and a primitive family* $\mathfrak{A}$ *in* $E$.

*Then, the bounded subsets in* $E$ *are the same for* $\mu$ *and* $\mu^{\alpha}$.

**Proof.** If a subset $B$ of $E$ is $\mu$-bounded, then for each subfamily $\mathfrak{U}_0$ of $\mathfrak{U}$, there exists $U_0$ such as $U_0\subset\cap U$ where $U\in\mathfrak{U}_0$ and $\lambda_0\in R$, $\lambda_0\neq 0$ such as $\lambda_0 B\subset U_0$, then $\lambda_0 B\cap A\subset U\cap A$ for each $U\in\mathfrak{A}_0$.

So $\bigcup_{\iota\in I}(\lambda_0 B\cap A_\iota)\subset\bigcup_{\iota\in I}(U_\iota\cap A_\iota)$ where $U_\iota\in\mathfrak{U}_0$, $A_\iota\in\mathfrak{A}$.

Since the union of all members of $\mathfrak{A}$ covers $E$, we have a $\lambda_0\in R$, $\lambda_0\neq 0$ such that

$$\lambda_0 B=\lambda_0 B\cap\bigcup_{\iota\in I}A_\iota=\bigcup_{\iota\in I}(\lambda_0 B\cap A_\iota)\subset\bigcup_{\iota\in I}(U_\iota\cap A_\iota)\subset k\{\bigcup_{\iota\in I}(U_\iota\cap A_\iota)\}=U^{\alpha}\quad\text{for each } U^{\alpha}\in\mathfrak{U}^{\alpha}.$$

Hence $B$ is $\mu^{\alpha}$-bounded.

Conversely if $B$ is $\mu^{\alpha}$-bounded, then by (2.2) $B$ is $\mu$-bounded.

§4. A neighbourhood basis at 0 for an admissible mixed-topology in dual linear spaces $E$ and $E'$.

Though a family $\mathfrak{A}^0$ of all polar sets of members of a primitive family $\mathfrak{A}$ is not necessarily primitive, however the following lemmas hold:

**Lemma 4.1.** *If $\mathfrak{A}$ is an H-primitive family in E, then $\mathfrak{A}^0$ is K-H-primitive family in the dual E′ of E.*

**Proof.** For any $\lambda \epsilon R, \lambda \neq 0, \lambda A^0=(A/\lambda)^0 \epsilon \mathfrak{A}^0$ because $A/\lambda \epsilon \mathfrak{A}$, and for $|\lambda| \leqslant 1, \lambda \epsilon R$, $\lambda A^0 \subset A^0$ follows from $A \supset \lambda A$. Hence the conditions $(P_1)$ and $(P_2)$ are satisfied.

Suppose that there exists an $x' \epsilon E'$ which is not contained in any $A^0 \epsilon \mathfrak{A}^0$, that is, $x' \bar{\epsilon} \cup A^0$ where $A \epsilon \mathfrak{A}$.

But the union of all $A^0 \epsilon \mathfrak{A}^0$, $\cup A^0=(\cap A)^0=\{0\}^0 \supset E'$ because that $\mathfrak{A}$ is $H$-primitive, so $x' \bar{\epsilon} E'$. This is a contradiction to $x' \epsilon E'$, hence $\mathfrak{A}^0$ satisfies the condition $(P_3)$.

The intersection of all $A^0 \epsilon \mathfrak{A}^0$, $\cap A^0=(\cup A)^0=E^0=\{0\}$.
This implies that $\mathfrak{A}^0$ satisfies the condition $(H)$.
It is well known that $\mathfrak{A}^0$ satisfies the conditions $(k)$ and $(c)$.

**Lemma 4.2.** *If a K-H-primitive family $\mathfrak{N}$ is the weakly closed convex co-base for an H-primitive family $\mathfrak{M}$, then $\mathfrak{N}^0$ and $\mathfrak{M}^0$ are the same K-H-primitive family.*

**Proof.** Since $\mathfrak{N}$ is a subfamily of a family $\mathfrak{M}, \mathfrak{N}^0 \subset \mathfrak{M}^0$, on the other hand if $M^0 \epsilon \mathfrak{M}^0$ then $M^0=M^{000}=\{K(M \cup 0)\}^0=\{K(M)\}^0 \epsilon \mathfrak{N}^0$, so $\mathfrak{M}^0 \subset \mathfrak{N}^0$. Hence $\mathfrak{N}^0=\mathfrak{M}^0$.

**Lemma 4.3.** *If $\mathfrak{A}$ is a K-H-primitive family in E′, then there exists an H-primitive family $\mathfrak{M}$ in E such that $\mathfrak{M}^0=\mathfrak{A}$.*

**Proof.** By Lemma 4.1, $\mathfrak{A}^0$ in $E$ is a $K$-$H$-primitive family, and $A^{00}=K(A \cup 0)$, since $A \epsilon \mathfrak{A}$ is weakly closed, convex and containing of 0, $K(A \cup 0)=A$, hence $\mathfrak{A}^{00}=\mathfrak{A}$.

Let $\mathfrak{M}$ be the family containing $\mathfrak{A}^0$ as the weakly closed convex co-base. Then $\mathfrak{M}$ is an $H$-primitive family such that $\mathfrak{M}^0=\mathfrak{A}$. In fact, if $M \epsilon \mathfrak{M}, \lambda \epsilon R, \lambda \neq 0$, then there exists $A^0$ such that $K(M)=A^0$ and $K(\lambda M)=\lambda K(M)=\lambda A^0 \epsilon \mathfrak{A}^0$ so $\lambda M \epsilon \mathfrak{M}$, and moreover for $|\lambda| \leqslant 1, K(\lambda M)=\lambda A^0 \subset A^0=K(M)$ so $\lambda M \subset M$, hence $\mathfrak{M}$ satisfies $(P_1)$ and $(P_2)$.

It follows from $\mathfrak{A}^0 \subset \mathfrak{M}$ that $\mathfrak{M}$ satisfies $(P_3)$ and $(H)$.

**Lemma 4.4.** *If $\mathfrak{A}$ is a primitive family of which members are weakly bounded, then $\mathfrak{A}^0$ is a K-H-primitive family satisfying the condition* $(0_1)$

**Proof.** It is similarly verified for $\mathfrak{A}^0$ to satisfy $(P_1), (P_2), (k), (c)$ and $(H)$. We shall show that $(0_1)$ is satisfied, consequently so is $(P_3)$. Since each $A \epsilon \mathfrak{A}$ is weakly bounded, for any $A \in \mathfrak{A}$, there exists $\lambda \in R, \lambda \neq 0$ such that $A \subset \lambda\varphi^0$ for every $\varphi \in \Psi$, the family of all the finite subsets of $E'$, that is, for any $\varphi \in \Psi$, there exists $\lambda \in R, \lambda \neq 0$ such that $\lambda A^0 \supset K(\varphi) \supset \varphi$ for every $A^0 \in \mathfrak{A}^0$.

**Lemma 4. 5.** *Let $\mathfrak{S}$ be a family of subsets of E.*

*If the family $\mathfrak{S}^0$ is a neighbourhood basis at 0 for a locally convex topology which is consistent with the structure of the dual $E'$ of $E$, then the family $\mathfrak{S}$ contains an admissible family $\mathfrak{T}$ relative to $E$ and $E'$ as the weakly closed convex circled co-base such that $\mathfrak{T}^0 = \mathfrak{S}^0$.*

**Proof.** The family $\mathfrak{S}^{00}$ of all bipolar sets of members of $\mathfrak{S}$ is, by Lemma 4.1, a *K-H*-primitive family, and satisfies the conditions (*a*) and (*b*). In fact (*a*) follows from the fact that $S_3^0 \subset S_1^0 \cap S_2^0$ implies $S_3^{00} \supset S_1^{00} \cup S_2^{00}$, and for (*b*), let $\Psi$ be the family of all the finite subsets of $E'$, then for any $\varphi \in \Psi$, there exists $\lambda \neq 0$ such that $\lambda\varphi \subset S^0$ for every $S^0 \in \mathfrak{S}^0$, dually for any $S^{00} \in \mathfrak{S}^{00}$ there exists $\lambda \neq 0$ such that $\varphi^0 \supset \lambda S^{00}$ for every $\varphi^0 \in \Psi^0$, a neighbourhood basis at 0 for weak topology, so $\mathfrak{S}^{00}$ satisfies (*b*).

Since $(S^{00})^0 = S^{000} = S^0$, take $\mathfrak{S}^{00}$ as the family $\mathfrak{T}$, then $\mathfrak{T}^0 = \mathfrak{S}^0$ and for any $S \in \mathfrak{S}$, the weakly closed convex circled envelope $KC(S)$ of $S$ belongs to $\mathfrak{T}$, because that $KC(S) = S^{00}$.

Henceforth, in this section, we assume that whenever we speak of a neighbourhood it is weakly closed.

**Proposition 4. 1.** *Let $\mathfrak{U}^\alpha$ be a neighbourhood basis at 0 for the mixed topology $\alpha(\mu, \mathfrak{A})$ determined by an admissible topology $\mu$ and a K-H-primitive family $\mathfrak{A}$ in the dual $E'$ of E.*

*Then there exists an admissible family $\mathfrak{S}$ in E such that $\mathfrak{U}^\alpha = \mathfrak{S}^0$, aud moreover there exist an admissible family $\mathfrak{B}$ and a K-H-primitve family $\mathfrak{M}$ in E such that for each $S \in \mathfrak{S}$*

$$S = \bigcap_{\iota \in I} (B_\iota \cup M_\iota)$$

*where $M_\iota \in \mathfrak{M} = \{M_\iota\,; \iota \in I\}$ and $B_\iota \in \mathfrak{B}_j \subset \mathfrak{B}$.*

**Proof.** For each $U^\alpha \in \mathfrak{U}^\alpha$, taking a subfamily $\mathfrak{U}_j$ of $\mathfrak{U}$ according to $U^\alpha$ we have

$$U^\alpha = K\{\bigcup_{\iota \in I} (U_\iota \cap A_\iota)\} = \{\bigcup_{\iota \in I} (U_\iota \cap A_\iota)\}^{00} = \{\bigcap_{\iota \in I} (U_\iota^0 \cup A_\iota^0)\}^0$$

where $U_\iota \in \mathfrak{U}_j$, $A_\iota \in \mathfrak{A} = \{A_\iota, \iota \in I\}$.

Let $\mathfrak{S}$ be the family of all the sets such as

$$S = \bigcap_{\iota \in I} (U_\iota^0 \cup A_\iota^0),\ U_\iota^0 \in \mathfrak{U}_j^0 \subset \mathfrak{U}^0,$$

then $\mathfrak{U}^0=\mathfrak{S}^0$ and the family $\mathfrak{S}$ is an admissible family in $E$ because that

$$(U^{\alpha})^0=\{\bigcup_{\iota\in I}(U_{\iota}\cap A_{\iota})\}^{000}=\bigcup_{\iota\in I}(U_{\iota}\cup A_{\iota})\}^0=\bigcap_{\iota\in I}(U_{\iota}\cap A_{\iota})^0=\bigcap_{\iota\in I}(U_{\iota}^0\cup A_{\iota}^0)=S$$

that is, $(\mathfrak{U}^{\alpha})^0=\mathfrak{S}$ and $(\mathfrak{U}^{\alpha})^0$ is an admissible family in $E$ by Lemma 4.5.

Take $\mathfrak{U}^0=\{U^0; U\epsilon\mathfrak{U}\}$ as the admissible family $\mathfrak{B}$, and take $\mathfrak{A}^0$ as the $H$-primitive family $\mathfrak{M}$ by Lemma 4.3, then for $S\epsilon\mathfrak{S}$

$$S=\bigcap_{\iota\in I}(B_{\iota}\cup M_{\iota}), M_{\iota}\epsilon\mathfrak{M}=\{M_{\iota}, \iota\epsilon I\}, B_{\iota}\epsilon\mathfrak{B}_J=\mathfrak{U}_J^0.$$

Conversely the following proposition holds.

**Proposition 4.2.** *Let $\mathfrak{B}$ be a family containing an admissible family in $E$ as a co-base and let $\mathfrak{M}$ be an H-primitive family in E.*

*If $\mathfrak{S}$ is the family of all the sets such as*

$$S=\bigcap_{\iota\in I}(B_{\iota}\cup M_{\iota})$$

*where $M_{\iota}\epsilon\mathfrak{M}=\{M_{\iota}, \iota\epsilon I\}$, $B_{\iota}\epsilon\mathfrak{B}_J$, a subfamily of $\mathfrak{B}$, then the uniform convergence topology on $\mathfrak{S}$, $\rho(E', \mathfrak{S}^0)$ is the mixed topology determined by the uniform convergence topology on $\mathfrak{B}$, $\rho(E', \mathfrak{B}^0)$ and the K-H-primitive family $\mathfrak{M}^0$ in $E'$.*

**Proof.** This follows only from the computation as following; if $S\epsilon\mathfrak{S}$, then

$$S^0=\{\bigcap_{\iota\in I}(B_{\iota}\cup M_{\iota})\}^0=\{\bigcap_{\iota\in I}(B_{\iota}\cup M_{\iota})\}^{000}=K[\{\bigcap_{\iota\in I}(B_{\iota}\cup M_{\iota}))\}^0]=K\{\bigcup_{\iota\in I}(B_{\iota}^0\cup M_{\iota}^0)\}$$

where $B_{\iota}^0\epsilon\mathfrak{B}_J^0\subset\mathfrak{B}^0$, a neighbourhood basis at 0 for $\rho(E', \mathfrak{B}^0)$ and $M_{\iota}^0\epsilon\mathfrak{M}^0$, $\mathfrak{M}^0$ is a $K$-$H$-primitive family by Lemma 4.1.

Noticing that the set $\cap(B_{\iota}\cup M_{\iota})$ in Prop. 4.1 is closed, convex, circled and bounded relative to the weak topology, indeed, in the proof of Prop. 4.1 $\cap(U_{\iota}^0\cup A_{\iota}^0)=\cap(U_{\iota}\cap A_{\iota})^0=\cap(U_{\iota}\cap A_{\iota})^{000}=\cap K(U_{\iota}\cap A_{\iota})^0=\cap K(U_{\iota}^0\cup A_{\iota}^0)$, we may give the following definition.

**Difinitionn.** *We say that the family of all the sets such as $S_J=\cap(B_{\iota}\cap M_{\iota})$ where $B_{\iota}\epsilon\mathfrak{B}_J\subset\mathfrak{B}$, $M_{\iota}\epsilon\mathfrak{M}=\{M_{\iota}, \iota\epsilon I\}$ is the mixed admissible family associated with $\mathfrak{B}$ and $\mathfrak{M}$, if $\mathfrak{B}$ is an admissible family and $\mathfrak{M}$ is a K-H-primitive family in dual linear spaces E and E', and sometimes we denote it* $(\mathfrak{B}\circ\mathfrak{M})$.

## § 5. Properties of mixed topologies in dual linear spaces.

Throughout this section and the next, we assume that a locally convex topology $\tau$ is defined in $E$ as the initial topology, and that whenever we speak of a topology, its neighbourhood is weakly closed.

**Theorem 5.1.** *If a topology $\mu$ is admissible relative to the dual linear spaces E and $(E,\tau)'$ and a family $\mathfrak{A}$ is K-H-primitive, then the mixd topology $\alpha(\mu, \mathfrak{A})$ is also admissible relative to E and $(E,\tau)'$.*

**Proof.** Since $\alpha(\mu, \mathfrak{A})$ is a locally convex topology which is not weaker than $\mu$, so the conditions $(t_1)$ and $(t_2)$ are satisfied and by Prop. 4.1 $(t_3)$ is also satisfied.

**Corollary 1.** *The mixed topology $\alpha(\mu, \mathfrak{A})$ in the preceding theorem is not weaker than the weak topology and is not stronger than the strong topology.*

**Corollary 2.** *Let $\alpha(\varsigma, \mathfrak{A})$ be the mixed topology determined by the strong topology $\varsigma$ and a K-H-primitive family $\mathfrak{A}$.*

*Then $\alpha(\varsigma, \mathfrak{A})=\varsigma$ i.e. the strong topology is invariant to be mixed.*

In fact, by cor. 1.$\alpha(\varsigma. \mathfrak{A})\leqslant\varsigma$, while by (2.2) $\alpha(\varsigma. \mathfrak{A})\geqslant \varsigma$ therefore $\alpha(\varsigma, \mathfrak{A})=\varsigma$.

In the next place, we shall show an approximation theorem concerned with the mixed topology in more general case.

**Theorem 5.2.** *Let $E$ and $F$ be two locally convex spaces in duality, and let $\alpha(w, \mathfrak{A})$ be the mixed topology determined by the weak topology $w(E, F)$ and a K-H-primitive family $\mathfrak{A}$ in $E$. If $g$ is any linear functional which is $\alpha(w, \mathfrak{A})$-continuous on $E$, then for each $\varepsilon\epsilon R, \varepsilon>0$ there exists an $f\epsilon F$ such that*

$$|g(x)-<x,f>| \leqslant\varepsilon$$

*for all $x$ in $E$.*

**Proof.** This is an immediate consequence from (2.6) and a result of Grothendieck's, however we shall give a proof for the convenience of the reader and for developing of arguments.

Let $Q$ be the canonical map of $F$ into the algebraic dual $E^{\#}$, the linear space of all the linear functionals on $E$, defined by, for each $f\epsilon F$, $Q(f)(x)=<x,f>$ for all $x$ in $E$.

Then $Q(F)$ is a linear subspace of $E^{\#}$ and $w(E, Q(F))$ coincides with $w(E, F)$.

By (2.6) $g$ is $\alpha(w, \mathfrak{A})$-continuous on $E$ if and only if $g$ is $w(E, F)$-(equivalently $w(E, Q(F))$-continuous on every $A\ \epsilon\mathfrak{A}$ i.e. for each $\varepsilon>0$, there exists $U\epsilon\mathfrak{U}_{\omega}$, a neighbourhood basis at 0 for $w(E, Q(F))$, such that

$$g/\varepsilon\ \epsilon\ (A\cap U)^{0}\subset E^{\#}$$

for every $A\epsilon\mathfrak{A}$.

Let $\Psi$ be the family of all the finite subsets of $Q(F)$ in $E^{\#}$, then $\mathfrak{U}_{\omega}=\Psi^{0}$ and there is $\phi\epsilon\Psi$ such that

$$(A\cap U)^{0}=(A\cap U)^{000}=K(A^{0}\cup U^{0})=K(A^{0}\cup K(\varphi))\subset A^{0}+K(\varphi)$$

where $K(\varphi)$ is a weakly compact subsets of $Q(F)$ and $A^{0}$ is weakly closed, so $A^{0}+K(\phi)$ is weakly closed.

Hence, for each $\varepsilon>0$, there exist $K(\psi)$ in $Q(F)$ such that $g/\varepsilon \in A^0+K(\psi)$ for every $A^0 \in \mathfrak{A}^0$, that is, for each $\varepsilon>0$, there exists a $Q(f) \in Q(F)$ such that

$Q(f)-g \in \varepsilon A^0$ for every $A^0 \in \mathfrak{A}^0$,

since the union of all members of $\mathfrak{A}$ covers $E$ and $Q(f)(x)=\langle x,f\rangle$ for all $x$ in $E$, for each $\varepsilon>0$, there exists an $f \in F$ such that

$|g(x)-\langle x,f\rangle| \leqslant \varepsilon$ for all $x \in E$.

**Theorem 5.3.** *Let $E$ and $F$ be two locally convex spaces in duality, and let $\omega^\alpha$ be the mixed topology determined by the weak topology $\omega(E,F)$ and an admissible family $\mathfrak{A}$ in $E$ relative to $E$ and $F$. Then;*

i) *the dual $(E,w^\alpha)'$ is complete relative to $\rho$, the topology of uniform convergence on members of $\mathfrak{A}$,*

ii) *the canonical image $Q(F)$ in the algebraic dual $E^\#$ is dense in the dual $(E,\omega^\alpha)'$ relative to $\rho$,*

iii) *$F$ is $\rho$-complete if and only if $Q(F)=(E,\omega)'=(E,w^\alpha)'$.*

**Proof.** At first, we shall verify that the family $\mathfrak{A}$ is admissible relative to $E$ and $(E,w^\alpha)'$, that is, $\mathfrak{A}^0$ in $E^\#$ is a neighbourhood basis at 0 for a locally convex topology in $(E,w^\alpha)'$.

If $x' \in (E,w^\alpha)'$, that is, for $\varepsilon>0$, there exists $U_\varepsilon \in \mathfrak{U}_\omega$ such that $x' \in \varepsilon(A\cap U_\varepsilon)^0$ for every $A \in \mathfrak{A}$, then there exists $\lambda>0$ such that $\lambda/\varepsilon(A\cap U_\varepsilon)\supset A$ because that $A \in \mathfrak{A}$ is circled and $w(E,F)$-bouuded, so $\varepsilon(A\cap U_\varepsilon)^0 \subset \lambda A^0$ for every $A^0 \in \mathfrak{A}^0$, hence $\mathfrak{A}^0$ satisfies the condition $(0_1)$, and the other conditions are obviously satisfied.

Part i) and ii) are consequences from Theorem 5.2 and the fact that $\rho$ is an admissible topology in $(E,w^\alpha)'$ by Lemma 4.5 and the above.

For iii), $F$ is $\rho$-complete if and only if $Q(F)$ is $\rho$-complete. By ii), $Q(F)=(E,w^\alpha)'$. By (2.2), $(E,w)'\subset(E,w^\alpha)'$, while $(E,w^\alpha)'=Q(F)\subset(E,w)'$, hence $Q(F)=(E,w)'=(E,w^\alpha)'$.

**Corollary.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two admissible families in $E$ which satisfy the condition $(Q_2)$; for every $A \in \mathfrak{A}$ there exists $B \in \mathfrak{B}$ such as $B\supset A$. i.e. $\rho(\mathfrak{A}^0)\leqslant\rho(\mathfrak{B}^0)$.*

*If $F$ is $\rho(\mathfrak{A}^0)$-complete then $F$ is $\rho(\mathfrak{B}^0)$-complete.*

In fact, $F$ is $\rho(\mathfrak{A}^0)$-complete if and only if $(E,w')=(E,\alpha(w,\mathfrak{A}))'$, while, by Prop. 3.2 $\alpha(w,\mathfrak{B})\leqslant\alpha(w,\mathfrak{A})$, and by (2.2)

$(E,w)'\subset(E,\alpha(w,\mathfrak{B}))'\subset(E,\alpha(w,\mathfrak{A}))'$

hence $(E,w)'=(E,\alpha(w,\mathfrak{B}))'$, by iii), $F$ is $\rho(\mathfrak{B}^0)$-complete.

**Theorem 5.4.** *Let the dual $E'$ of $E$ with $\tau$ be complete relative to an ad-*

*missible topology $\rho$, and let $\mathfrak{A}$ be the admissible family in E for $\rho$. Then;*

i) *each mixed topology $\alpha(w, \mathfrak{B})$ determined by the weak topology $w(E, E')$ and an admissible family $\mathfrak{B}$ in E satisfying the condition $(Q_2)$ is compatible topology to the duality between E and E'.*

ii) *the mixed admissible family in E' for $\alpha(w, \mathfrak{B})$ is the $w(E', E)$-closed convex circled co-base for the family of all the $\rho(\mathfrak{B}^0)$-compact subsets in E'.*

**Proof.** For i), by the Cor. of Th. 5.3, $F$ is $\rho(\mathfrak{B}^0)$-complete, and by iii) of Th. 5.3 $(E, w)' = (E, \alpha(w, \mathfrak{B}))'$, and it is obvious that $(E, \tau)' = (E, w)'$, thus the condition $(t_4)$ is satisfied, and by theorem 5.1, $\alpha(w, \mathfrak{B})$ is an admissible topology. Hence the part i) holds.

For ii), let $\mathfrak{S}$ be the mixed admissible family in $E'$ for $\alpha(w, \mathfrak{B})$. By Prop. 4.1 if $S \in \mathfrak{S}$, then $S$ is written as following;

$$S = \bigcap_{\iota \in I} (K(\varphi_\iota) \cup U_\iota)$$

where $\varphi_\iota \in \Psi_J$, $\Psi_J$ is a subfamily of the family $\Psi$ of all the symmetric finite subsets of $E'$, $U_\iota \in \mathfrak{U} = \mathfrak{B}^0$.

Since there exists $K(\varphi) \in K(\Psi)$ such that $S \subset K(\varphi) \cup U_\iota$ for every $U_\iota \in \mathfrak{U}$, $S \in \mathfrak{S}$ is $\rho(\mathfrak{B}^0)$-totally bounded, conversely if a subset $M$ of $E'$ is $\rho(\mathfrak{B})^0$-totally bounded, then $M$ has this property, so $M$ is contained in, at least, one $S \in \mathfrak{S}$, and since $\rho$-totally bounded subsets in $\rho$-complete linear space is $\rho$-compact, $\mathfrak{S}$ is a co-base for the family of all the $\rho$-compact subsets of $E'$, moreover each $S$ in $\mathfrak{S}$ is $w(E', E)$-closed, convex and circled, hence ii) holds.

**Theorem 5.5.** *Let E and F be two locally convex spaces in duality, and let $w^\alpha$ be the mixed topology determined by the weak topology $w(E, F)$ and a primitive family $\mathfrak{A}$ in E, then a subset M of E is weakly bounded if and only if M is $w^\alpha$-bounded, i.e. the bounded sets of E are the same for any mixed topology determined by the weak topology.*

**Proof.** This is the particular case such that $\mu$ in Theorem 3.1 is considered as the weak topology.

**Theorem 5.6.** *Let E and F be two locally convex spaces in duality and let $w^\alpha$ be the mixed topology determined by the weak topology $w(E, F)$ and an admissible family $\mathfrak{A}$ in E.*

*If F is $\rho(\mathfrak{A}^0)$-complete, then for any subset M of E, the closed convex envelope of M is the same for $w(E, F)$ and $w^\alpha$.*

**Proof.** Let $w$-$K(M)$ and $w^\alpha$-$K(M)$ denote the closed convex envelopes of $M$ relative to $w(E, F)$ and $w^\alpha$ respectively.

By Eidelheit separation theorem, $w$-$K(M)$ and $w^\alpha$-$K(M)$ are $\cap\{x; f(x)\leqslant sup\{f(y); y\epsilon M\}\}$ for all $f\epsilon(E,w)'$ and for all $f\epsilon(E,w)'$ respectively. By iii) of theorem 5.3 $(E,w)'=(E,w^\alpha)'$ if $F$ is $\rho(\mathfrak{A}^0)$-complete, so we obtain $w$-$K(M)=w^\alpha$-$K(M)$.

**Theorem 5.7.** *Let $w^\alpha$ be the mixed topology determined by the weak topology $w(E,E')$ and a k-primitive family $\mathfrak{A}$ in E.*

*If a locally convex topology $\nu$ in E possesses the property (F);*

*(F) if $f\epsilon E^*$ is $w(E,E')$-continuous on each $A\epsilon\mathfrak{A}$, then f is $\nu$-continuous on E, then $w^\alpha\leqslant\nu$, i.e. $w^\alpha$ is the coarsest topology which possesses the property (F).*

**Proof.** At first, we shall prove a lemma in more general case.

**Lemma.** *Let f be a liner mapping from a locally convex space $(X,\mu)$ into a locally convex space $(Y,\mu')$, and let $\mathfrak{A}$ be a k-primitive family in X. Then, the following properties are equivalent;*

i) *if the restriction of f to each $A\epsilon\mathfrak{A}$ is $(\mu,\mu')$-continuous, then f is $(\nu,\mu')$-continuous on X,*

ii) *$\mu^\alpha\leqslant\nu$, where $\mu^\alpha$ is the mixed topology determined by $\mu$ and $\mathfrak{A}$.*

In fact, since $f$ is $(\mu^\alpha,\mu')$-continuous on $X$ if the restriction of $f$ to each $A\epsilon\mathfrak{A}$ is $(\mu,\mu')$-continuous by (2.6), so $f$ is $(\nu,\mu')$-continuous on $X$ if $\mu^\alpha\leqslant\nu$, hence ii) implies i).

Conversely, if i) holds, then since the restriction to each $A\epsilon\mathfrak{A}$ of the identical mapping $I$ from $(X,\mu)$ onto $(X,\mu^\alpha)$ is $(\mu,\mu^\alpha)$-continuous by (2.3), so $I$ is $(\nu,\mu^\alpha)$-continuous on $X$, that is, for every $U^\alpha\epsilon\mathfrak{U}^\alpha$, there exists $V\epsilon\mathfrak{U}^\nu$ such that $V=I(V)\subset U^\alpha$, this means $\mu^\alpha\leqslant\nu$, hence i) implies ii).

In this lemma, consider the particular case such that $Y=R$, and $\mu=w(E,E')$, then since $w^\alpha$ has certainly the property $(F)$, we obtain immediately the theorem.

## § 6. Various mixed topologies in dual linear spaces.

【1】 Let $w'^\alpha$ be the mixed topology determined by the weak topology $w'(E',E)$ and the family $\mathfrak{N}$ of all the $w'(E',E)$-closed convex circled equicontinuous sets in the dual $E'$ of $E$ with $\tau$.

Then:

1) the admissible family $\mathfrak{S}$ in $E$ for $w'^\alpha$ is the mixed admissible family associated with the family $\Phi$ of all the $w(E,E')$-closed convex envelopes of symmetric finite subsets of $E$ and $\mathfrak{U}_\tau$, a neighbourhood basis at 0 for $\tau$.

2) so, $w^{\alpha\prime}$ is identical with the topology of the uniform convergence on every $\tau$-totally bounded subsets of $E$,

3) therefore, if $E$ is $\tau$-complete, then each member of $\mathfrak{S}$ is $\tau$-compact,

4) the completion of $E$ is isomorphic to $(E', w'^{\alpha})'$,

5) $w'^{\alpha}$ is the finest locally convex topology which is identical with $w'(E', E)$ on every equicontinuous set of $E'$, and on $E', w'(E', E) < w'^{\alpha}$.

6) finally, we conclude that $w'^{\alpha}$ coincides with the almost-weak*-topology in [2] (III. § 1. p. 44.).

In fact, i), by Th. 5.1 $w'^{\alpha}$ is an admissible topology to $E'$, and by Prop. 4.1 and $\mathfrak{U}^0_w = \Phi, \mathfrak{N}^0 = \mathfrak{U}_\tau$ ([1], § 2, IV), i) holds.

2) It is similarly proved as ii) of Th. 5.4 for $\mathfrak{S}$ to consist of $\tau$-totally bounded subsets, because of $\rho(\mathfrak{N}^0) = \tau$, and $\mathfrak{S}$ is the $w(E, E')$-closed convex circled co-base for the family of all the $\tau$-totally bounded subsets of $E$.

3) It follows from 2) and to be complete.

4) Interchanging the rôles of $E$ and $F$ in Theorem 5.3 and noticing that $\rho(\mathfrak{N}^0) = \tau$, we obtain the conclusion. 5) See [4] (th. 3). 6) Let $\alpha^*$ denote the almost-weak* topology in $E'$ described in [2]. It is known that $V$ is $\alpha^*$-neighbourhood at 0 in $E'$ if and only if there exists a finite subset $\varphi$ of $E$ such that $\varphi^0 \cap U^0 \subset V$ for each $U \in \mathfrak{U}_\tau$.

Therefore, there exists a subfamily $\mathfrak{U}_0$ of $\mathfrak{U}_{\omega'}$ such that

$$\bigcup_{\iota \in I} (U'_\iota \cap U^0_\iota) \subset V \text{ where } U'_\iota \in \mathfrak{U}_0,\ U^0_\iota \in \mathfrak{U}^0_\tau,$$

since $V$ is defined to be weakly closed convex and circled, there exists $U^\alpha \in \mathfrak{U}^\alpha$, a $w'^{\alpha}$-neighbourhood basis at 0, such that

$$U^\alpha = K\{\bigcup_{\iota \in I} (U'_\iota \cap U^0_\iota) \subset V$$

so $\alpha^* \leqslant w'^{\alpha}$.

On the other hand, it is known that if $f \in E'^{\#}$ which is $w'(E', E)$-continuous on each $U^0 \in \mathfrak{U}^0_\tau$ is $\alpha^*$-continuous on $E'$.

By Theorem 5.7 (interchanging the rôles $E$ and $E'$), we have the relation $w'^{\alpha} \leqslant \alpha^*$. Hence $w'^{\alpha} = \alpha^*$.

【2】 Let $w^\alpha$ be the mixd topology determined by the weak topology $w(E, E')$ and the family $\Phi$ of all the $w(E, E')$-closed convex envelopes of symmetric finite subsets in $E$.

Then:

1) the admissible family $\mathfrak{S}$ in $E'$ for $w^\alpha$ is the mixed admissible family associated with the family $\Psi$ of all the $w'(E', E)$-closed convex envelopes of symmetric finite subsets in $E'$ and $\mathfrak{U}'_\omega$, a neighbourhood basis at 0 for $w'(E', E)$,

2) so $w^\alpha$ is identical with the topology of the uniform convergence on every $w'(E', E)$-totally bounded subsets of $E'$,

3) in this case, $w^\alpha$ is the finest mixed topology among the mixed topologies determined by the weak topology and every admissible family in $E$.

4) If $(E, w)'=(E, w^\alpha)'$, then $w^\alpha$ coincides with the Mackey topology.

5) $\tau\leqslant w^\alpha$ and on every $K(\varphi)\epsilon\Phi, w(E, E'), \tau$ and $w^\alpha$ are identical.

6) If $\mu$ is a locally convex topology such that $w\leqslant\mu\leqslant w^\alpha$, then the bounded sets of $E$ are the same for $w(E, E')$ and for $\mu$.

In fact, the parts 1) and 2) are similarly verified as 1) and 2) in 【1】,
3) is a particular case of Proposition 3.6.

For 4), let $m(E, E')$ be the Mackey topology. Sinee weakly compact sets are weakly totally bounded, $m(E, E')\leqslant w^\alpha$, while $(E, w)'=(E, w^\alpha)'$ implies that $w^\alpha\leqslant m(E, E)'$, hence $m(E, E')=w^\alpha$.

For 5), $\tau\leqslant w^\alpha$ follows from 4) and by (2.3) the rest holds.

For 6), by Theorem 5.5 and 3) in the above, 6) is true.

【3】 Taking the family $\mathfrak{B}$ of all the $\tau$-bounded subsets of $E$ in the place of $\Phi$ in 【2】, we have the mixed topology $w^\tau$ such that;

1). $w^\tau$ is identical with the uniform convergence topology on every strongly totally-bounded subsets of $E'$,

2). $w^\tau\leqslant w^\alpha$ in 【2】,

3). $(E, \tau)'$ is strongly complete if and only if $(E, \tau)'=(E, w^\tau)'$,

4). $w^\tau$ is the finest locally convex topology which is identical with $w(E, E')$ on every $\tau$-bounded subsets of $E$, i.e. in this case $w^\tau$ coincides with Persson's mixed-iopology associated with $w(E, E')$.

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