A THEORY OF DOUBLY EXTENDED LIE TRANSFORMATION GROUPS.

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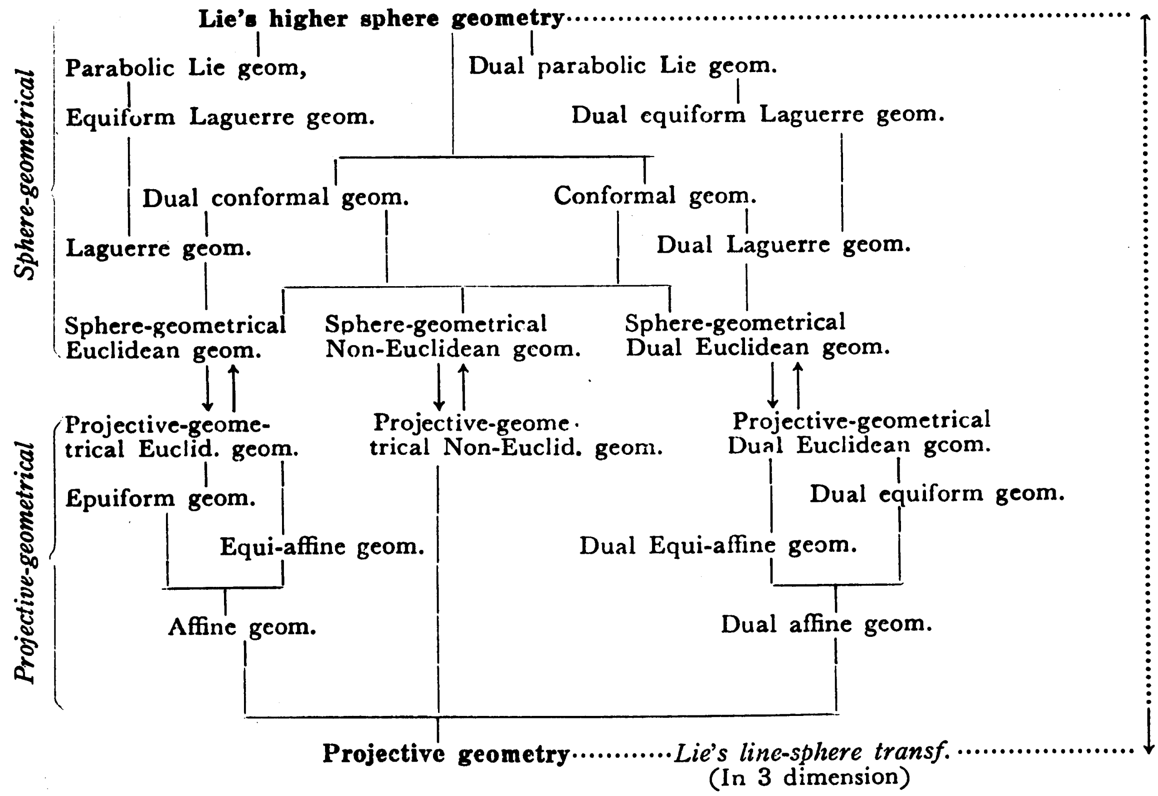
(Received November 20, 1965)

RÉSUMÉ. I. The theory of Lie transformation groups was extended to a theory of ***extended*** Lie transformation groups by extending the group parameters (a^t) to functions $(a^t(x))$ of coordinates of the base manifolds ([15]). The Lie's fundamental theorems were thereby simplified.

II. In this paper, ***a theory of further extended Lie transformation groups*** will be established by extending the extended group parameters $(a^t(x))$ further to the case $(a^t(x, \dot{x}, \ddot{x}, \cdots, \overset{(m)}{x}))$, where $\dot{x}=dx/dt$, etc., $t=$the canonical parameter.

INTRODUCTION. I. The transformation parameters hitherto considered had been exclusively of the nature of the variable constants untill the present author succeeded in extending all the branches enlisted below by extending respective group parameters further to appropriate functions* of coordinates ([1]–[13]), ***respective invariants being retained***:



*) A glimpse of an embryo of this idea is found in [45].

Thereby the combined manifold (e.g.) $\{\{x\}, \{a^l_m(x), a^l_o\}\}$, $(|a^l_m(x)| \neq 0; l, m, p=1, 2, \cdots, n)$ of the base manifold $\{x\}$ and the extended group manifold $\{a^l_m(x), a^l_o\}$ was considered. Hereby x are the local coordinates in the

base differentiable manifold	classical space

and the global II-geodesic curves

$$\frac{d}{dt}\frac{\omega^l}{dt}=0, \quad (\omega^l \overset{\mathrm{def}}{=} \omega^l_m(x)\,dx^m = a^l_m(x)\,dx^m),$$

which exist in the

base differentiable manifold	classical space

owing to the fact that ω^l are written in invariant forms and behave as for meet and join like straight lines, were considered. Further, the global II-*geodesic parallel coordinates* ξ^l such that $d\xi^l=\omega^l=a^l dt$ were introduced adopting at least one system of $\omega^l_m(x) \in C^v$, (v=positive integer or ∞ or ω[1]) such that $|\omega^l_m(x)| \neq 0$.

Thereafter the present author was in the situation *to extend his extension of group parameters to functions of coordinates of the base manifolds to the general case* and led *to extend the theory of Lie transformation groups by extending the group parameters to functions of coordinates. The abstract theory itself of the Lie groups remained however thereby unaltered, although the domain of validity is thus enlarged.* Thereby the combined manifolds (M, G) were considered, where M is the base manifold $\{x\}$ and G the extended Lie transformation group manifold $\{a^t(x)\}$, $(t=1.2, \cdots, r)$.

The famous Fundamental Theorems of Otto Schreier ([17], [18]) had till that time enabled us *to reduce the global theory of Lie groups to the case of the vicinity of uuit element. The present author introduced the global* II-*geodesic parallel coordinates* ξ^l, *etc. not only in the base manifold* M[2] *but also in the transformation group space* ($\{\alpha^l\}$, say). Thus they enabled us to establish *the theory of the extended Lie*

groups	*transformation groups*

in the large without taking the Otto Schreier's Fundamental Theorems into account.

The resulting theory of extended Lie transformation groups includes the various extended geometries hitherto considered by the present author ([1] $\cdots$ [13]) as special cases, the parameter t spoken of above being a special *canonical*

1) The cases of analytic functions.

2) Usually the Euclidean space E^n only is treated as the base manifold.

parameter.

Dual to that the present author has obtained $d\xi^l=\omega^l_m(x)\,dx^m$, he rendered usual notation

$$X_i=\xi^j_i(x)\frac{\partial}{\partial x^j}$$ in the differentiable manifold $\{x\}$	$$Z_i=\alpha^k_i(a)\frac{\partial}{\partial a^k}$$ in the group manifold $\{a\}$

into the form

$$\frac{\partial}{\partial \xi^i},$$ where (ξ^i)	$$\frac{\partial}{\partial \alpha^l},$$ where (α^l)

are the II-geodesic parallel coordinates corresponding to

$$\xi^j_i(x).$$	$$\alpha^k_i(a).$$

Thus the fundamental theorems of the extened Lie transformation groups were made extremely simple as the following underlying formulas suggest:

$$X_i=\frac{\partial}{\partial\xi^i},\ (X_i, X_j)=0,$$ the structure constants $C^i_{jk}=0$ $$d\,(\omega^l_m(x)\,dx^m)=0,$$	$$Z_i=\frac{\partial}{\partial\alpha^i},\ (Z_i, Z_j)=0,$$ the structure constants $\bar{C}^i_{jk}=0,$ $$d\,(b^i_j(a)\,da^j)=0,$$

$$\alpha^k_i\frac{\partial f}{\partial a^k}=\xi^j_i\frac{\partial f}{\partial x^j}\quad\longrightarrow\quad\frac{\partial f}{\partial\alpha^l}=\frac{\partial f}{\partial\xi^l}.$$

In Art. 19 of [15], *E. Cartan's theories in his "géomêtrie des groupes"* ([18]) *concerning "equipollence des vecteurs", "parallêlisme des vecteurs" and "géodesique" were extended to the case, where the groups are the extended ones in the present author's sense, the fact that E. Cartan's geodesics are* II-*geodesics in the present author's sense* being shown.

II. In the present paper, *a theory of further extended Lie transformation groups will be established by extending the extended group parameters* $(a^t(x))$ further *to the case* $(a^t(x, \dot{x}, \ddot{x}, \cdots, \overset{(m)}{x}))$. *The results are mostly parallel and similar to those, which were recapitulated under* I *and contain the latter.*

§ 1. Otto Schreier's Two Fundamental Theorems.

1. Recapitulation of the Otto Schreier's Two Fundamental Theorems. The study of the global Lie groups has hitherto been based on the following principles.

First Fundamental Theorem of Otto Schreier ([17], [18]). If U be an arbitrary vicinity of the unit element of a connected topological space G, then every element of G is expressible * as the product of a finite number of elements $a^1, a^2, \cdots, a^m$ belonging to U.

COR. Connected r-dimensional continuous group G may be covered by at most enumerable open sets of the forms $a^r U, (r=1, 2, \cdots, m)$, where U is an arbitrary vicinity of the unit element of G.

SECOND FUNDAMENTAL THEOREM OF OTTO SCHREIER ([17], [18]). If we divide a connected r-dimensional continuous group into subsets by the equivalence relations of locally continuous isomorphism, then each subset contains only one simply connected group, provided that we do not distinguish the subsets, which are continuously isomorphic to one another. Every continuous coset group of the simply connected group (belonging to the subset) formed with its isolated invariant subgroup as modulus. And conversely, such a coset group is a continuous group belonging to one and the same subset as its simply connected group.

In the First Fundamental Theorem of Otto Schreier, the expressibility * holds only but for local continuous isomorphism and by the continuous group, locally continuously isomorphic subset only come into our consideration. Hence we see that *the study of connected continuous groups is admissible to that of the*

vicinity of the unit element	*group germ (local group)*

only.

§2. The Theory of Lie Groups in the Large by Extending the Group Parameters to Appropriate Functions of Coordinates.

2. Differentiable Manifolds. In order to fix our notion, we will recapitulate a number of definitions of terms etc. under consideration.

Let R^n be an n-dimensional Cartesian space with the real coordinates (x). We call the topological representation of an open subset U_α of an n-dimensional manifold $M=V^n$ on an open subset $x(U_\alpha)$ of R^n a *system of local coordinates* (or a *local chart*) of M. U_α is called the *domain of the chart* (or the *domain of coordinate system*). To each point P of $U_\alpha \subset M$, there correspond a point of R^n, which is represented by (x) called the *coordinates of* P *in the chart* under consideration.

DEFINITION. *A differentiable manifold* M *of the class* C^v

(v=positive integer	$v=\infty$	$v=\omega$ (analyticity!))

is an n-dimensional manifold [3], to which a system A (atlas) of charts satisfying the following conditions are associated:

A_1. $M = \cup U_\alpha$.

A_2. $P \in U_1 \cap U_2$, $(U_1, U_2$: two domains of charts of $A)$, and (x) and (y) are the local coordinates having U_1 and U_2 as the domains respectively, then

$y^\lambda = y^\lambda(x)$	$x^\lambda = x^\lambda(y)$

are functions of class C^v such that

$\dfrac{\partial(y^1, \cdots, y^n)}{\partial(x^1, \cdots, x^n)} \neq 0.$	$\dfrac{\partial(x^1, \cdots, x^n)}{\partial(y^1, \cdots, y^n)} \neq 0.$

DEFINITION. Two atlas A and B are said to be ***equivalent***, when their reunion is also an atlas of class C^v.

THEOREM. In order that two atlas A and B of one and the same differentiable manifold M may be equivalent, it is necessary and sufficient that A, B satisfy the axiom A_2.

DEFINITION. Two equivalent atlas are said to define one and the same ***structure of differentiable manifold of class*** C^v ***on*** M.

DEFINITION. A system of local coordinates of M is said to be ***compatible*** with the structure of differentiable manifold (or to be ***admissible***), when the reunion with an atlas defining M as differentiable manifold is also an atlas of the same class.

THEOREM. Every compact differentiable manifold can be covered by a finite number of domains of the charts.

3. The Lie Groups as r-Dimensional Differentiable Manifolds of class C^3
At the end of Art. 1, we have seen that the ***study of connected continuous group is reducible to that of the***

vicinity of the unit element	***group germ (local group)***

only.

Now we have succeeded ***in introducing global*** II-***geodesic parallel coordinates*** (ξ^l) ***into differentiable manifolds and any point of a differentiable manifold may be considered as the origin by virtue of the extended affine transformation group.***

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3) A topological space is said to be *locally Euclidean* at a point, if there exists a chart A on a vicinity of P. A Hausdorff space, which is locally Euclidean at each point, is called a *manifold.*

Thus we are led to the

THEOREM. *The Lie group is a differentiable manifold of class* C^3.

In order to prove this theorem, we begin with the definition of the r-dimensional Lie group germ.

DEFINITION. A set G of elements $S_a = S(a^1, a^2, \cdots, a^r)$ having points $a = (a^1, a^2, \cdots, a^r)$ belonging to a vicinity U_0 of the origin (0) of the r-dimensional Euclidean space as parameters, is called an ***r*-dimensional Lie group germ**, when it is characterized by the following conditions:

(i) If we take a vicinity $U_1 \subset U_0$ of the origin appropriately, then for $a = (a^1, a^2, \cdots, a^r) \in U_0$, and $b = (b^1, b^2, \cdots, b^r) \in U_1$, the product

$$S_a \cdot S_b = S_c, \qquad (c = (c^1, c^2, \cdots, c^r) \in U_0)$$

is defined, where the composition function

$$c^i = \varphi^i (a^1, a^2, \cdots, a^r ; b^1, b^2, \cdots, b^r), \qquad (i = 1, 2, \cdots, r)$$

are of class C^3.

(ii) For arbitrary $a \in U_0$, the relation

$$S_a \cdot S_0 = S_0 \cdot S_a = S_a.$$

i. e.

$$(3.1) \qquad \varphi^i (a^1, a^2, \cdots, a^r ; 0, \cdots, 0) = \varphi^i (0, \cdots, 0 ; a^1, \cdots, a^r) = a^i, \qquad (i = 1, 2, \cdots, r)$$

holds.

(iii) If $a, b, c \in U_2$ for sufficiently small vicinity U_2 of the origin, then the associative law

$$S_a \cdot (S_b \cdot S_c) = (S_a \cdot S_b) \cdot S_c$$

i. e.

$$(3.2) \qquad \varphi^i (a ; \varphi (b ; c)) = \varphi^i (\varphi (a ; b) ; c), \qquad (i = 1, 2, \cdots, r)$$

holds.

LEMMA. If a and b be sufficiently near the origin, then

$$\frac{\partial (\varphi^1 (a ; b), \cdots, \varphi^r (a ; b))}{\partial (a^1, a^2, \cdots, a^r)} \neq 0, \quad \frac{\partial (\varphi^1 (a ; b), \cdots, \varphi^r (a ; b))}{\partial (b^1, b^2, \cdots, b^r)} \neq 0,$$

so that we can solve

$$c^i = \varphi^i (a ; b), \qquad (i = 1, 2, \cdots, r)$$

with respect to a or b. In particular, $S_x = S_a^{-1}$ such that

$$S_x \cdot S_a = S_a \cdot S_x = S_0$$

is determined for arbitrary S_a.

PROOF. $\frac{\partial \varphi^i(a\,;b)}{\partial a^j}$ and $\frac{\partial \varphi^i(a\,;b)}{\partial b^j}$ and thus the fundamental determinants $\frac{\partial(\varphi)}{\partial(a)}$ aud $\frac{\partial(\varphi)}{\partial(b)}$ are continuous functions in the vicinity of the origin. If we set $b=0$ resp. $a=0$, then by (3.1), we have

$$\left(\frac{\partial(\varphi)}{\partial(a)}\right)_{b=0}=\left(\frac{\partial(\varphi)}{\partial(b)}\right)_{a=0}=(\delta_{ij})=1$$

and thus

$$\frac{\partial(\varphi)}{\partial(a)}\neq 0,\quad \frac{\partial(\varphi)}{\partial(b)}\neq 0$$

in the vicinity of the origin

If, in particular, we solve $S_x\cdot S_a=S_0$, we have

$$S_x=(S_x\cdot S_a)\cdot S_x=S_x\cdot(S_a\cdot S_x)$$

by the associative law. Comparing this with $S_x=S_x\cdot S_0$, we obtain $S_a\cdot S_x=S_0$. Thus $S_x=S_a^{-1}$ exists.

PROOF OF THE THEOREM. I. When a vicinity of the unit element of a topological group G in a r-dimensional Lie group germ, the topological group G is called an ***r*-dimensional Lie group.**

II. A topological group G is an ***r*-dimensional continuous group**, when G is provided with a vicinity of the unit element of G, which is homeomorphic to an open hypersphere of the r-dimensional Euclidean space.

From I and II, we see that ***the r-dimensional Lie group G is an r-dimensional continuous group***, since for the Lie group germ, the existence of the vicinity of the unit element of G, which is homeomorphic to an open hypersphere of the r-dimensional Euclidean space, is preassumed.

Now

III. ***an r-dimensional continuous group is a topological group, whose group space is an r-dimensional manifold.***

Hence ***the r-dimensional Lie group G is an r-dimensional manifold.***

By the Cor. above, this r-dimensional manifold is a differentiable manifold of class C^3, since, by the Cor. of the First Fundamental Theorem of Otto Schreier, Axiom A_1 of Art. 2 is satisfied and by the Theorem above, the Axiom A_2 of Art. 2 is satisfied.

Hence the r-dimensional Lie group is an r-dimensional differentiable manifold of class C^3.

§ 3. Doubly Extended Affine Geometry.

4. II - Geodesic Curves. Take

(4.1) $\omega^l \overset{\text{def}}{=} \omega^l_\mu (x, \dot{x}, \cdots, \overset{(m)}{x})\, dx^\mu,$ $(l, \cdots; \lambda, \mu, \cdots = 1, 2, \cdots, n),$	$\alpha^\lambda \overset{\text{def}}{=} \alpha^\lambda_l (a, \dot{a} \cdots, \overset{\bar{m}}{a})\, da^l,$ $(\lambda, \cdots; l, h, \cdots = 1, 2, \cdots, r), (r \geqq n),$

where $\dot{x} = dx/dt$, etc., t being the so-called affine parameter ([38], [39]) and

ω^l	α^λ

are assumed to be linearly independent, so that the condition that

(4.2) $\lVert \omega^l_\mu (x, \dot{x}, \cdots, \overset{(m)}{x}) \rVert^2 \neq 0$ in M.	$\lVert \alpha^\lambda_l (a, \dot{a}, \cdots, \overset{(\bar{m})}{a}) \rVert^2 \neq 0$ in G

is satisfied. Since (4.1) is written in an ***invariant*** from,

ω^l	α^λ

are global.

For the given

$\omega^l_\mu (x, \dot{x}, \cdots, \overset{(m)}{x}),$	$\alpha^\lambda_l (a, \dot{a}, \cdots, \overset{(\bar{m})}{a}),$

we introduce

$\Omega^\lambda_l (x, \dot{x}, \cdots, \overset{(m)}{x})$	$\beta^l_\lambda (a, \dot{a}, \cdots, \overset{(\bar{m})}{a})$

by the condition:

(4.3) $\Omega^\lambda_l \omega^l_\mu = \delta^\lambda_\mu \Longleftrightarrow \Omega^\lambda_h \omega^l_\lambda = \delta^l_h,$	$\beta^l_\lambda \alpha^\lambda_h = \delta^l_h \Longleftrightarrow \beta^l_\mu \alpha^\lambda_l = \delta^\lambda_\mu,$

where δ's are Kronecker deltas.

We define the connection parameters ([43], p. 11)

$\Lambda^\lambda_{\mu\nu} (x, \dot{x}, \cdots, \overset{(m)}{x})$	$\Lambda^l_{hk} (a, \dot{a}, \cdots, \overset{(\bar{m})}{a})$

by the conditions of teleparallelism for

ω^l_μ and Ω^λ_l:	α^λ_h and β^l_λ:
(4.4) $d\omega^l_\mu - \Lambda^\sigma_{\mu\nu} \omega^l_\sigma dx^\nu = 0,$	$d\alpha^\lambda_h - \Lambda^s_{hk} \alpha^\lambda_s da^k = 0,$
(4.4)′ $d\Omega^\lambda_l + \Lambda^\lambda_{\sigma\nu} \Omega^\sigma_l dx^\nu = 0,$	$d\beta^l_\lambda + \Lambda^l_{sk} \beta^s_\lambda da^k = 0,$

which become

(4.5) $\Lambda^\lambda_{\rho\nu} dx^\nu = \Omega^\lambda_l d\omega^l_\rho,$	$\Lambda^l_{hk} da^k = \beta^l_\lambda d\alpha^\lambda_h,$
(4.5)′ $\Lambda^\lambda_{\mu\nu} dx^\nu = -\omega^l_\mu d\Omega^\lambda_l,$	$\Lambda^l_{hk} da^k = -\alpha^\lambda_h d\beta^l_\lambda,$

by multipltication of (4.4), (4.4)′ with

Ω^λ_l, | β^l_λ,

where

(4.6) $d\omega^l_\mu$ | $d\Omega^\lambda_l$

$$=dx^\nu\left(\frac{\partial}{\partial x^\nu}+\frac{\ddot x^\sigma}{\dot x^\nu}\frac{\partial}{\partial \dot x^\sigma}+\cdots+\frac{\overset{(m+1)}{x^\sigma}}{\dot x^\nu}\frac{\partial}{\partial \overset{(m)}{x^\sigma}}\right)$$

$$\cdot\,\omega^l_\mu(x,\dot x,\cdots,\overset{(m)}{x}),\qquad \cdot\,\Omega^\lambda_l(x,\dot x,\cdots,\overset{(m)}{x}),$$

the M

$d\Omega^\lambda_l$ | $d\beta^l_\lambda$

$$=da^k\left(\frac{\partial}{\partial a^k}+\frac{\ddot a^s}{\dot a^k}\frac{\partial}{\partial \dot a^s}+\cdots+\frac{\overset{(\bar m+1)}{a^s}}{\dot a^k}\frac{\partial}{\partial \overset{(\bar m)}{a^s}}\right)$$

$$\cdot\,\alpha^\lambda_l(a,\dot a,\cdots,\overset{(\bar m)}{a}),\qquad \cdot\,\beta^l_\lambda(a,\dot a,\cdots,\overset{(\bar m)}{a}),$$

the G

being of class C^v, so that

$$(4.7)\quad \Lambda^\lambda_{\mu\nu}\overset{\text{def}}{=\!=}\Omega^\lambda_l\left(\frac{\partial}{\partial x^\nu}+\frac{\ddot x^\sigma}{\dot x^\nu}\frac{\partial}{\partial \dot x^\sigma}+\cdots+\frac{\overset{(m+1)}{x^\sigma}}{\dot x^\nu}\frac{\partial}{\partial \overset{(m)}{x^\sigma}}\right)\omega^l_\mu(x,\dot x,\cdots,\overset{(m)}{x})\,\dot x^\mu$$

$$\equiv-\omega^l_\mu\left(\frac{\partial}{\partial x^\nu}+\frac{\ddot x^\sigma}{\dot x^\nu}\frac{\partial}{\partial \dot x^\sigma}+\cdots+\frac{\overset{(m+1)}{x^\sigma}}{\dot x^\nu}\frac{\partial}{\partial \overset{(m)}{x^\sigma}}\right)\Omega^\lambda_l(x,\dot x,\cdots,\overset{(m)}{x}),$$

$$\Lambda^l_{hk}\overset{\text{def}}{=\!=}\beta^l_\lambda\left(\frac{\partial}{\partial a^k}+\frac{\ddot a^s}{\dot a^k}\frac{\partial}{\partial \dot a^s}+\cdots+\frac{\overset{(\bar m+1)}{a^s}}{\dot a^k}\frac{\partial}{\partial \overset{(\bar m)}{a^s}}\right)\alpha^\lambda_l(a,\dot a,\cdots,\overset{(\bar m)}{a})\,\dot a^k$$

$$\equiv-\alpha^\lambda_k\left(\frac{\partial}{\partial a^k}+\frac{\ddot a^s}{\dot a^k}\frac{\partial}{\partial \dot a^s}+\cdots+\frac{\overset{(\bar m)}{a^s}}{\dot a^k}\frac{\partial}{\partial \overset{(\bar m)}{a^s}}\right)\beta^l_\lambda(a,\dot a,\cdots,\overset{(\bar m)}{a}),$$

the last identity arising from (4.3).

We have further

$$\frac{d}{dt}\frac{\omega^l}{dt}=\frac{d}{dt}\{\omega^l_\sigma(x,\dot x,\cdots,\overset{(m)}{x})\,\dot x^\sigma\}$$

$$=\frac{d}{dt}\left(\dot x^\sigma\frac{\partial}{\partial x^\sigma}+\ddot x^\sigma\frac{\partial}{\partial \dot x^\sigma}+\cdots+\overset{(m+1)}{x^\sigma}\frac{\partial}{\partial \overset{(m)}{x^\sigma}}\right)\omega^l_\mu(x,\dot x,\cdots\overset{(m)}{x})$$

$$+\omega^l_\sigma(x,\dot x\cdots,\overset{(m)}{x})\,\ddot x^\sigma,$$

$$(4.8)\quad \frac{d}{dt}\frac{\omega^l}{dt}\equiv\omega^l_\lambda(x,\dot x,\cdots,\overset{(m)}{x})\,\{\ddot x^\lambda+\Lambda^\lambda_{\mu\nu}(x,\dot x,\cdots,\overset{(m)}{x})\,\dot x^\mu\dot x^\nu\},$$

$$\frac{d}{dt}\frac{\alpha^\lambda}{dt}=\frac{d}{dt}\{\alpha^\lambda_s(a,\dot a,\cdots,\overset{(\bar m)}{a})\,\dot a^s\}$$

$$=\frac{d}{dt}\left(\dot a^s\frac{\partial}{\partial a^s}+\ddot a^s\frac{\partial}{\partial \dot a^s}+\cdots+\overset{(\bar m+1)}{a^s}\frac{\partial}{\partial \overset{(\bar m)}{a^s}}\right)\alpha^\lambda_k(a,\dot a,\cdots,\overset{(\bar m)}{a})$$

$$+\alpha^\lambda_s(a,\dot a,\cdots,\overset{(\bar m)}{a})\,\ddot a^s,$$

$$\frac{d}{dt}\frac{\alpha^\lambda}{dt}\equiv\alpha^\lambda_l(a,\dot a,\cdots,\overset{(\bar m)}{a})\,\{\ddot a^l+\Lambda^l_{hk}(a,\dot a,\cdots,\overset{(\bar m)}{a})\,\dot a^h\dot a^k\},$$

or

$$(4.9)\quad \Omega^\lambda_l\frac{d}{dt}\frac{\omega^l}{dt}\equiv\frac{d^2x^\lambda}{dt^2}+\Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}.\qquad \beta^l_\lambda\frac{d}{dt}\frac{\alpha^\lambda}{dt}=\frac{d^2a^l}{dt^2}+\Lambda^l_{hk}\frac{da^h}{dt}\frac{da^k}{dt}.$$

From (4.8) and (4.9), we see that

$$(4.10)\quad \text{(i)}\ \frac{d}{dt}\frac{\omega^l}{dt}=0 \Longleftrightarrow \qquad\Big|\qquad \text{(i)}\ \frac{d}{dt}\frac{\alpha^\lambda}{dt}=0 \Longleftrightarrow$$

$$\text{(ii)}\ \frac{d^2x^\lambda}{dt^2}+\Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}=0. \qquad\Big|\qquad \text{(ii)}\ \frac{d^2a^l}{dt^2}+\Lambda^l_{hk}\frac{da^h}{dt}\frac{da^k}{dt}=0.$$

(4.8), (4.9) and (4.10) tell us that the global path (i) is transformed piecewise onto the local path (ii) by the inverse transformation of (4.1):

$$(4.11)\qquad dx^\lambda=\Omega^\lambda_l(x, \dot{x}, \cdots, \overset{(m)}{x})\,\omega^l. \qquad\Big|\qquad da^l=\beta^l_\lambda(a, \dot{a}, \cdots, \overset{(\overline{m})}{a})\,\alpha^\lambda.$$

The differential equation (i) are integrated readily:

$$(4.12)\qquad \omega^l=a^l\,dt=d\xi^l, \text{ say, } (a^l=\text{const.}), \qquad\Big|\qquad \alpha^\lambda=e^\lambda\,dt=d\eta^\lambda, \text{ say, } (e^\lambda=\text{const.}),$$

$$(4.13)\qquad \int\frac{\omega^l}{dt}dt=a^lt+c^l=\xi^l, \text{ say, } (c^l=\text{const.}), \qquad\Big|\qquad \int\frac{\alpha^\lambda}{dt}dt=e^\lambda t+c^\lambda=\eta^\lambda, \text{ say, } (c^\lambda=\text{const.}),$$

the integration (4.13) being guided by the simple clear form of

$$a^l\,dt. \qquad\Big|\qquad e^\lambda\,dt.$$

Thus we obtain

$$(4.14)\qquad \xi^l=\int\frac{\omega^l}{dt}dt=a^lt+c^l. \qquad\Big|\qquad \eta^\lambda=\int\frac{\alpha^\lambda}{dt}dt=e^\lambda t+c^\lambda.$$

From (4.14), we see that ***the curves represented*** by (4.10), (i) or (4.14) ***behave as for meet and join like straight lines.***

We will call these curves ***non-locally line-elemented*** II-***geodesic*** [4] ***curves.***

N. B. A glimpse of (4.10), (i); ($\overline{m}=0$) for the group manifolds is found in ([19], p. 62).

The (4.1) may be rewritten as follows:

$$(4.15)\qquad d\xi^l=a^l_\mu(x, \dot{x}, \cdots, \overset{(m)}{x})\,dx^\mu, \qquad\Big|\qquad d\eta^\lambda=\alpha^\lambda_h(a, \dot{a}, \cdots, \overset{(\overline{m})}{a})\,da^h,$$

and (4.10), (i) as follows:

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4) In the group manifolds, such curves of the case $\overline{m}=0$ have been called *geodesic curves* (E. Cartan, [19], p. 62). The present author has found that the II-geodesic curves of the case $\overline{m}=0$ are geodesics for α^λ.

(4.16) $$\frac{d^2\xi^l}{dt^2}=0. \qquad\qquad \frac{d^2\eta^\lambda}{dt^2}=0.$$

Multiplying (4.12) with

$$\Omega^\lambda_l, \qquad\qquad \beta^l_\lambda,$$

we see that the relations

(4.17) $$\frac{dx^\lambda}{dt}=a^k\,\Omega^\lambda_k(x,\dot{x},\cdots,\overset{(m)}{x}) \qquad\qquad \frac{da^l}{dt}=e^\lambda\,\beta^l_\lambda(a,\dot{a},\cdots,\overset{(\bar{m})}{a})$$

holds along the non-locally line-elemented II-*geodesic line-elements.*

We will call the

$$(\xi^l) \qquad\qquad (\eta^\lambda)$$

the non-local II-*geodesic parallel coordinates corresponding to*

$$a^l_\mu(x,\dot{x},\cdots,\overset{(m)}{x}) \qquad\qquad \alpha^\lambda_k(a,\dot{a},\cdots,\overset{(\bar{m})}{a})$$

referred to the non-local II-*geodesic coordinates*

ξ^l*-axes.* The (ξ^l) are *global.* | η^λ*-axes.* The (η^λ)

5. **Double Extension of the Affine Transformation Group by Doubly Extending the Group Parameters to Functions of Coordinates.** In particular, the

(ξ^l) may stand for (x), | (η^λ) may stand for (a),

so that we come to consider

(5.1) $$d\bar{\xi}^l=a^l_h(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\,d\xi^h, \qquad\qquad d\bar{\eta}^\lambda=\alpha^\lambda_\mu(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})\,d\eta^\mu,$$

$$(\,|\,a^l_h(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\,|^2\neq0 \text{ in } M) \qquad\qquad (\,\|\,\alpha^\lambda_\mu(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})\,|^2\neq0 \text{ in } G)$$

in place of (4.15) for the non-locally line-elemented II-geodesic line-elements corresponding to

$$a^l_h(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi}). \qquad\qquad \alpha^\lambda_\mu(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta}).$$

In order that the non-locally line-elemented II-geodesic curves

$$\xi^l(t),\quad \left(\frac{d^2\xi^l}{dt^2}=0\right) \qquad\qquad \eta^\lambda(t),\quad \left(\frac{d^2\eta^\lambda}{dt^2}=0\right)$$

may be transformed by (5.1) into the non-locally line-elemented II-geodesic curves

$$\bar{\xi}^l(t),\quad \left(\frac{d^2\bar{\xi}^l}{dt^2}=0\right) \qquad\qquad \bar{\eta}^\lambda(t),\quad \left(\frac{d^2\bar{\eta}^\lambda}{dt^2}=0\right)$$

corresponding to

$$a_h^l(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi}), \qquad\Big|\qquad \alpha_\mu^\lambda(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta}),$$

we must have

$$da_h^l(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\,d\xi^h=0 \qquad\Big|\qquad d\alpha_\mu^\lambda(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})\,d\eta^\mu=0 \tag{5.2}$$

along the non-locally line-elemented II-***geodesic line-elements.*** For, from (5.1), we obtain

$$\frac{d^2\bar{\xi}^l}{dt^2}=\frac{d}{dt}a_h^l(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\frac{d\xi^h}{dt}+a_h^l(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\frac{d^2\xi^h}{dt^2}. \qquad\Big|\qquad \frac{d^2\bar{\eta}^\lambda}{dt^2}=\frac{d}{dt}\alpha_\mu^\lambda(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})\frac{d\eta^\mu}{dt}+\alpha_\mu^\lambda(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})\frac{d^2\eta^\mu}{dt^2}. \tag{5.3}$$

Integrating (5.1) along the

$\bar{\xi}^l$ - axis, | $\bar{\eta}^\lambda$ - axis,

which is a non-locally line-elemented II-geodesic curve, we obtain

$$\bar{\xi}^l=a_h^l(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\,\xi^h-\int\xi^h\,(da_h^l(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\,/\,dt)\,dt. \qquad\Big|\qquad \bar{\eta}^\lambda=\alpha_\mu^\lambda(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})\,\eta^\mu-\int\eta^\mu\,(d\alpha_\mu^\lambda(\eta,\eta,\cdots,\overset{(\bar{m})}{\eta})\,/\,dt)\,dt.$$

Now

$$\int\xi^h\frac{da_h^l}{dt}dt=\int\frac{da_h^l}{dt}dt\int d\xi^h=\int\int\left(\frac{da_h^l}{dt}\,dtd\xi^h\right)=\text{const.}\ (=-a_o^l,\ \text{say}) \qquad\Big|\qquad \int\eta^\mu\frac{d\alpha_\mu^\lambda}{dt}dt=\int\frac{d\alpha_\mu^\lambda}{dt}dt\int d\eta^\mu\ \ \int\int\left(\frac{d\alpha_\mu^\lambda}{dt}\,dtd\eta^\mu\right)=\text{const.}\ (=-\alpha_o^\lambda,\ \text{say})$$

by (5.2), the condition for that the repeated integral may be converted into the double integral (i.e. that the integrand 1 is continuous) being evidently satisfied. Thus we have

$$\bar{\xi}^l=a_h^l(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\,\xi^h+a_o^l,\quad (a_o^l=\text{const.},\ |a_h^l|\neq0\text{ in }M). \qquad\Big|\qquad \bar{\eta}^\lambda=\alpha_\mu^\lambda(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})\,\eta^\mu+\alpha_o^\lambda,\quad (\alpha_o^\lambda=\text{const.},\ |\alpha_\mu^\lambda|\neq0\text{ in }G). \tag{5.4}$$

We will call the transformation (5.4) *doubly extended affine transformation*.

From (5.1) and (5.4), we see that

$$da_h^l(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\,\xi^h=0 \qquad\Big|\qquad d\alpha_\mu^\lambda(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})\,\eta^\mu=0 \tag{5.5}$$

along the non-locally line-elemented II-*geodesic line-elements.*

The totality of the doubly extended affine transformations (5.4), *whose inverse transformation is*

$$(5.6)\qquad \xi^k=\Omega^k_h(\bar{\xi},\dot{\bar{\xi}},\cdots,\overset{(m)}{\bar{\xi}})\bar{\xi}^h+\Omega^k_o, \qquad\qquad \eta^\mu=\beta^\mu_\lambda(\bar{\eta},\dot{\bar{\eta}},\cdots,\overset{(\bar{m})}{\bar{\eta}})\bar{\eta}^\lambda+\beta^\mu_o,$$

$$(\Omega^k_o=\text{const.},\ |\Omega^k_h|\neq 0), \qquad\qquad (\beta^\mu_o=\text{const.},\ |\beta^\mu_\lambda|\neq 0),$$

$$(5.7)\qquad a^k_h\Omega^l_k=\delta^l_h \iff a^l_h\,\Omega^h_k=\delta^l_k, \qquad\qquad \alpha^\mu_\nu\,\beta^\lambda_\mu=\delta^\lambda_\nu \iff \alpha^\lambda_\nu\,\beta^\nu_\mu=\delta^\lambda_\mu,$$

forms a group ($\bar{\mathfrak{G}}$, say) | *forms a group* ($\bar{\mathfrak{F}}$, say)

as will be proved as follows:

The combination of (5.4):

$$\bar{\xi}^h=a^h_k(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\xi^k+a^h_o, \qquad\qquad \bar{\eta}^\mu=\alpha^\mu_\nu(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})\eta^\nu+\alpha^\mu_o,$$

$$(a^h_o=\text{const.},\ |a^h_k(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})|\neq 0) \qquad\qquad (\alpha^\mu_o=\text{const.},\ |\alpha^\mu_\nu(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})|\neq 0)$$

with

$$(5.8)\qquad \tilde{\xi}^l=\bar{a}^l_h(\bar{\xi},\dot{\bar{\xi}},\cdots,\overset{(m)}{\bar{\xi}})\bar{\xi}^h+\bar{a}^l_o, \qquad\qquad \tilde{\eta}^\lambda=\bar{\alpha}^\lambda_\mu(\bar{\eta},\dot{\bar{\eta}},\cdots,\overset{(\bar{m})}{\bar{\eta}})\bar{\eta}^\mu+\bar{\alpha}^\lambda_o,$$

$$(\bar{a}^l_o=\text{const.},\ |\bar{a}^l_h(\bar{\xi},\dot{\bar{\xi}},\cdots\overset{(m)}{\bar{\xi}})|\neq 0) \qquad\qquad (\bar{\alpha}^\lambda_o=\text{const.},\ |\bar{\alpha}^\lambda_\mu(\bar{\eta},\dot{\bar{\eta}},\cdots,\overset{(\bar{m})}{\bar{\eta}})|\neq 0)$$

is of the form:

$$(5.9)\qquad \tilde{\xi}^l=\tilde{b}^l_k(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\xi^k+\tilde{b}^l_o, \qquad\qquad \tilde{\eta}^\lambda=\tilde{\beta}^\lambda_\nu(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})\eta^\nu+\tilde{\beta}^\lambda_o,$$

$$(\tilde{b}^l_o=\text{const.},\ |\tilde{b}^l_k(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})|\neq 0 \text{ in } M), \qquad\qquad (\tilde{\beta}^\lambda_o=\text{const.},\ |\tilde{\beta}^\lambda_\nu(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})|\neq 0 \text{ in } G),$$

where

$$(5.10)\qquad \tilde{b}^l_k(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi}) =\bar{a}^l_h(a^q_s(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\xi^s+a^q_o,\cdots)\,a^h_k(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi}),$$

$$\tilde{\beta}^\lambda_\mu(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta}) =\bar{\alpha}^\lambda_\mu(\alpha^\xi_\sigma(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})\eta^\sigma+\alpha^\xi_o,\cdots)\,\alpha^\mu_\nu(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta}),$$

$$(5.11)\qquad \tilde{b}^l_o=\bar{b}^l_h(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\,a^h_o+\bar{a}^l_o, \qquad\qquad \tilde{\beta}^\lambda_o=\bar{\beta}^\lambda_\mu(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})\,\alpha^\mu_o+\bar{\alpha}^\lambda_o,$$

$$(5.12)\qquad \bar{b}^l_h(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi}) =\bar{a}^l_h(a^q_k(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\xi^k+a^q_o,\cdots).$$

$$\bar{\beta}^\lambda_\mu(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta}) =\bar{\alpha}^\lambda_\mu(\alpha^\xi_\nu(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})\eta^\nu+\alpha^\xi_o,\cdots).$$

We shall see that

$$(5.13)\qquad \tilde{b}^l_o=\bar{b}^l_h(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\,a^h_o+\bar{a}^l_o =\text{const.} \qquad\qquad \tilde{\beta}^\lambda_o=\bar{\beta}^\lambda_\mu(\eta,\dot{\eta},\cdots,\overset{(\bar{m})}{\eta})\,\alpha^\mu_o+\bar{\alpha}^\lambda_o =\text{const.}$$

owing to the summation with respect to

h, | μ.

for which it suffices to prove that

(5.14) $a^h_o\, d\bar b^l_h(\xi,\dot\xi,\cdots,\overset{(m)}{\xi})=0$	$\alpha^\mu_o\, d\bar\beta^\lambda_\mu(\eta,\dot\eta,\cdots,\overset{(\bar m)}{\eta})=0$

on summation with respect to

$h.$	$\mu.$

For (5.8), the condition (5.5) for that the

$\bar\xi^l$ - axes	$\bar\eta^\lambda$ - axes

may be non-locally line-elemented II-geodesic curves corresponding to

$\bar a^l_h(\bar\xi,\dot{\bar\xi},\cdots,\overset{(m)}{\bar\xi})$	$\bar\alpha^\lambda_\mu(\bar\eta,\dot{\bar\eta},\cdots,\overset{(\bar m)}{\bar\eta})$

becomes

(5.15) $\bar\xi^h d\bar a^l_h(\bar\xi,\dot{\bar\xi},\cdots,\overset{(m)}{\bar\xi})=0.$	$\bar\eta^\mu d\bar\alpha^\lambda_\mu(\bar\eta,\dot{\bar\eta},\cdots,\overset{(\bar m)}{\bar\eta})=0.$

We shall show that (5.14) follcws from (5.15). Indeed, (5.15) becomes

$\{a^l_h(\xi,\dot\xi,\cdots,\overset{(m)}{\xi})\xi^k+a^k_o\}\, d\bar a^l_h(\bar\xi,\dot{\bar\xi},\cdots,\overset{(m)}{\bar\xi})$ $=\{a^h_k(\xi,\dot\xi,\cdots,\overset{(m)}{\xi})\xi^k+a^h_o\}\, d\bar b^l_h(\xi,\dot\xi,\cdots,\overset{(m)}{\xi})$ $=0,$	$\{\alpha^\mu_\mu(\eta,\dot\eta,\cdots,\overset{(\bar m)}{\eta})\eta^\pi+\alpha^\mu_o\{\, d\bar\alpha^\lambda_\mu(\bar\eta,\dot{\bar\eta},\cdots,\overset{(\bar m)}{\bar\eta})$ $=\{\alpha^\mu_\kappa(\eta,\dot\eta,\cdots,\overset{(\bar m)}{\eta})\eta^\kappa+\alpha^\mu_o\}\, d\bar\beta^\lambda_\mu(\eta,\dot\eta,\cdots,\overset{(\bar m)}{\eta})$ $=0,$

so that

(5.16) $a^h_o\, d\bar a^l_h(\bar\xi,\dot{\bar\xi},\cdots,\overset{(m)}{\bar\xi})$ $=a^h_o\, d\bar b^l_h(\xi,\dot\xi,\cdots,\overset{(m)}{\xi})$ $=-a^h_k(\xi,\dot\xi,\cdots,\overset{(m)}{\xi})\, d\bar b^l_h(\xi,\dot\xi,\cdots,\overset{(m)}{\xi})\,\xi^k$ $=-a^h_k(\xi,\dot\xi,\cdots,\overset{(m)}{\xi})\, d\bar b^l_h(\xi,\dot\xi,\cdots,\overset{(m)}{\xi})\,\xi^k$ $-\{\xi^k\, da^h_k(\xi,\dot\xi,\cdots,\overset{(m)}{\xi})\}\,\bar b^l_h(\xi,\dot\xi,\cdots,\overset{(m)}{\xi})$	$\alpha^\mu_o\,\bar\alpha^\lambda_\mu(\bar\eta,\dot{\bar\eta},\cdots,\overset{(\bar m)}{\bar\eta})$ $=\alpha^\mu_o\, d\bar\beta^\lambda_\mu(\eta,\dot\eta,\cdots,\overset{(\bar m)}{\eta})$ $=-\alpha^\mu_\kappa(\eta,\dot\eta,\cdots,\overset{(\bar m)}{\eta})\, d\bar\beta^\lambda_\mu(\eta,\dot\eta,\cdots,\overset{(\bar m)}{\eta})\,\eta^\kappa$ $=-\alpha^\mu_\kappa(\eta,\dot\eta,\cdots;\overset{(\bar m)}{\eta})\, d\bar\beta^\lambda_\mu(\eta,\dot\eta\cdots;\overset{(\bar m)}{\eta})\,\eta^\kappa$ $-\{\eta^\kappa\, d\alpha^\mu_\kappa(\eta,\dot\eta,\cdots,\overset{(\bar m)}{\eta})\}\,\bar\beta^\lambda_\mu(\eta,\dot\eta\cdots,\overset{(\bar m)}{\eta})$

by the differential equation

(5.17) $\xi^k\, da^h_k(\xi,\dot\xi,\cdots,\overset{(m)}{\xi})=0$	$\eta^\kappa d\alpha^\mu_\kappa(\eta,\dot\eta,\cdots,\overset{(\bar m)}{\eta})=0$

for the non-locally line-elemented II-geodesic curves corresponding to

$a^h_k(\xi,\dot\xi,\cdots,\overset{(m)}{\xi}).$	$\alpha^\mu_\kappa(\eta,\dot\eta,\cdots,\overset{(\bar m)}{\eta}).$

Thus we have

$$
\begin{aligned}
(5.18)\qquad & a^h_o\, \bar{a}^l_h(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi}) \\
&= -\xi^k d\{a^h_k(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})\, \bar{b}^l_h(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})\} \\
&= -\xi^k d\{a^h_k(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})\, \bar{a}^l_h(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})\} \\
&= -\xi^k d b^l_k(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi}) = 0
\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned}
& \alpha^\mu_o\, d\bar{\alpha}^\lambda_\mu(\bar{\eta}, \dot{\bar{\eta}}, \cdots, \overset{(\bar{m})}{\bar{\eta}}) \\
&= -\eta^\kappa d\{\alpha^\mu_\kappa(\eta, \dot{\eta}, \cdots, \overset{(\bar{m})}{\eta})\, \bar{\beta}^\lambda_\mu(\eta, \dot{\eta}, \cdots, \overset{(\bar{m})}{\eta})\} \\
&= -\eta^\kappa d\{\alpha^\mu_\kappa(\eta, \dot{\eta}, \cdots, \overset{(\bar{m})}{\eta})\, \bar{\alpha}^\lambda_\mu(\eta, \dot{\eta}, \cdots, \overset{(\bar{m})}{\eta})\} \\
&= -\eta^\kappa d\beta^\lambda_\kappa(\eta, \dot{\eta}, \cdots, \overset{(\bar{m})}{\eta}) = 0
\end{aligned}
$$

by the differential equation

$$
(5.19)\qquad \xi^k d b^l_k(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi}) = 0 \qquad\Big|\qquad \eta^\kappa d\beta^\lambda_\kappa(\eta, \dot{\eta}, \cdots, \overset{(\bar{m})}{\eta}) = 0
$$

for the non-locally line-elemented II-geodesic curves corresponding to

$$
b^l_k(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi}). \qquad\Big|\qquad \beta^\lambda_\kappa(\eta, \dot{\eta}, \cdots, \overset{(\bar{m})}{\eta}).
$$

The (5.18) shows us that (5.14) follows from (5.15).

We will call the group

$$
\bar{\mathfrak{G}} \qquad\Big|\qquad \bar{\mathfrak{F}}
$$

the *doubly extended affine group.*

The most general doubly extended affine group

$$
\bar{\mathfrak{G}} \qquad\Big|\qquad \bar{\mathfrak{F}}
$$

contains the extended affine group

$$
\mathfrak{G} \qquad\Big|\qquad \mathfrak{F}
$$

as a snbgroup, which contains in turn the ordinary affine gronp as a subgroup.

The geometry under the doubly extended affine group will be called the *doubly extended affine geometry.*

6. Doubly Extended Equi-affine Group. The totality of the elements of the doubly extended affine group such that

$$
(6.1)\qquad |a^l_h(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})| = 1
$$

forms a subgroup of $\bar{\mathfrak{G}}$. We will call it the *doubly extended equi-affine group. It contains the extended equi-affine group* ([13]) *as a subgroup.*

The general n-volnme

$$
(6.2)\qquad |\xi \;\; \xi^1 + d_1\xi^1 \;\; \xi^2 + d_2\xi^2 \cdots \xi^n + d_n\xi^n| = |d_1\xi^1 \; d_2\xi^2 \cdots d_n\xi^n|
$$

is an invariant under the doubly extended equi-affine group.

7. Parameter of Curves under the Doubly Extended Equi-affine Group. Denoting the derivatives with respect to

t by dashes,	s by dots,

we introduce an invariant parameter s of curves under the doubly extended equi-affine group by the demand

$$(7.1)\qquad \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ \xi^1 & \dot{\xi}^1 & \ddot{\xi}^1 & \cdots & \overset{(n)}{\xi^1} \\ \xi^2 & \dot{\xi}^2 & \ddot{\xi}^2 & \cdots & \overset{(n)}{\xi^2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \xi^n & \dot{\xi}^n & \ddot{\xi}^n & \cdots & \overset{(n)}{\xi^n} \end{vmatrix} = |\dot{\xi}\,\ddot{\xi}\cdots\overset{(n)}{\xi}| = |\xi'\,\xi''\cdots\overset{(n)}{\xi}|_t\left(\frac{dt}{ds}\right)^{n(n+1)/2} = 1,$$

which tells us that $ds^{n(n+1)/2}$ represents $n!$ times the generalized n*-volume* of $(n+1)$ consecutive points, when (ξ^l) are ordinary parallel coordinates.

From (7.1), we obtain

$$(7.2)\qquad \begin{aligned} ds &= |d\xi\, d^2\xi \cdots d^n\xi|^{2/n(n+1)} \\ &= |(\omega'_\mu\, dx^\mu)\, d(\omega^2_\mu\, dx^\mu)\, d^2(\omega^3_\mu\, dx^\mu)\cdots d^{n-1}(\omega^n_\mu\, dx^\mu)|^{2/n(n+1)} \\ &= |\xi'\,\xi''\cdots\xi^{(n)}|_t^{2/n(n+1)}\, dt. \end{aligned}$$

8. Other Procedures.

I. A second	**II.** A third

procedure is to start with the fact that there exist in every differentiable manifolds $M=\bigcup\limits_\alpha U_\alpha$

II - geodesic	non-locally line-elemented II - geodesic

curves. For them, (4.1), ((4.12), (4.17)), (5.4) and (5.5) become respectively to

(8.1)	$\omega^l=\omega^l_\mu(x)\,dx^\mu,$	$\omega^l=\omega^l_\mu(x,\dot{x},\cdots,\overset{(m)}{x})\,dx^\mu,$
(8.2)	$\dfrac{dx^\lambda}{dt}=a^m\,\Omega^\lambda_m(x)=\alpha^\lambda,$	$\dfrac{dx^\lambda}{dt}=a^m\,\Omega^\lambda_m(x,\dot{x},\cdots,\overset{(m)}{x})=\alpha^\lambda,$
(8.3)	$\xi^l=a^l_\mu(x)\,x^\mu+a^l_o,$	$\xi^l=a^l_\mu(x,\dot{x},\cdots,\overset{(m)}{x})\,x^\mu+a^l_o,$
(8.4)	$da^l_\mu(x)\,x^\mu=0,$	$da^l_\mu(x,\dot{x},\cdots,\overset{(m)}{x})\,\dot{x}^\mu=0,$

provided that (x^λ) *themselves are*

II - *geodesic*	*doubly extended* II - *geodesic*

parallel coordinates corresponding to

$a^l_\mu(x)$, ([13]). | $a^l_\mu(x, \dot{x}, \cdots, \overset{(m)}{x})$.

If we utilize such special coordinates (x^λ), then (4.1), ((4.12), (4.17)), (5.4) and (5.5) become respectively to

(8.5) $$\omega^l = d\xi^l = a^l_\mu(x, \alpha, 0, \cdots, 0)\, dx^\mu,$$

(8.6) $$\frac{dx^\lambda}{dt} = a^m\, \Omega^\lambda_m(x, \alpha, 0, \cdots, 0) = \alpha^\lambda,$$

(8.7) $$d\bar{\xi}^l = a^l_h(\xi, \alpha, 0, \cdots, 0)\, d\xi^h,$$

(8.8) $$\xi^l = a^l_\mu(x, \alpha, 0, \cdots, 0)\, x^\mu + a^l_0,$$

(8.9) $$da^l_\mu(x, \alpha, 0, \cdots, 0)\, dx^\mu = 0,$$

(8.10) $$da^l_\mu(x, \alpha, 0, \cdots 0)\, x^\mu = 0.$$

The resulting theory is nothing other than the author's *extended affine geometry* ([13]) *but for that the n parameters α^λ arise in addition.*

9. Realization of the Doubly Extended Affine Geometry in the Differentiable Manifolds. Our results of Art. 4-8 show us that the author's doubly extended affine geometry is realized in the differentiable manifolds.

§ 4. Extension of the Domain of Validity of the Theory of Lie Groups to that of the Theory of the Doubly Extended Lie Groups.

10. The Fundamental Pfaffians for the Lie Group (Germ). The ordinary theory of the fundamental Extended Pfaffians for the Lie group germs applies still when the elements $a^l, (l=1, 2, \cdots, r; i=1, 2, \cdots, n)$ of the Lie group germs are doubly extended to the case $a^l(x, \dot{x}, \cdots, \overset{(m)}{x})$, which are appropriate functions of coordinates (x) of the base manifold and of their derivatives $(\dot{x}), (\ddot{x}), \cdots, (\overset{(m)}{x})$.[5] Such a theory will be exposed in the following lines, *writing a^l in place of $a^l(x, \dot{x}, \cdots, \overset{(m)}{x})$.*

We assume moreover the coordinates (x^i) to be doubly extended II-geodesic parallel coordinates (ξ^i), which are global. Then we may omit the term "germ" *without taking the Otto Schreier's Fundamental Theorems into account.*

We have assumed in Art. 3 that the composition functions

(10.1) $$c^i = \varphi^i(a^1, a^2, \cdots, a^r; b^1, b^2, \cdots, b^r), \quad (i=1, 2, \cdots, r)$$

are such that

(10.2) $$\varphi^i \in C^3.$$

We form the matrix

5) Cf. [15] for the case $m=0$.

$$\alpha_j^i(a)=\left(\frac{\partial\varphi^i(a;b)}{\partial b^j}\right)_{b=0} \qquad (i,j=1,2,\cdots,r). \tag{10.3}$$

Since

$$|\alpha_j^i(0)|=|\delta_j^i|=1, \tag{10.4}$$

we introduce the inverse $\beta_j^i(a)$ by the conditions

$$\alpha_k^i(a)\beta_j^k(a)=\delta_j^i \Longleftrightarrow \alpha_j^k(a)\beta_k^i(a)=\delta_j^i, \tag{10.5}$$

where δ_j^i are Kronecker deltas.

DEFINITION. We call

$$\omega^i(a,da)\overset{\text{def}}{=}\beta_j^i(a)\,da^j,\ a^l=a^l(x,\dot{x},\cdots,\overset{(m)}{x}),\ \omega^i\in A^{(1)}(C^2) \tag{10.6}$$

the *fundamental extended Pfaffians* (1-*forms*), where $A^{(1)}\subset(C^2)$ is a Lie algebra having $\omega^i(a,da)$ as base.

Multiplying (10.6) with $\alpha_i^j(a)$, we obtain

$$da^j=\alpha_i^j(a)\omega^j. \tag{10.7}$$

THEOREM. *The necessary and sufficieut condition for that the differential form*

$$\Phi=\sum_{i_1<\cdots<i_p} g_{i_1\cdots i_p}(a)\,da^{i_1}\wedge\cdots\wedge da^{i_p}\in A(C^0) \tag{10.8}$$

may be invariant:

$$\bar{\Phi}=\sum_{i_1<\cdots<i_p} g_{i_1\cdots i_p}(\bar{a})\,d\bar{a}^{i_1}\wedge\cdots\wedge d\bar{a}^{i_p}=\Phi \tag{10.9}$$

for all the transformations

$$\bar{a}^i=\varphi^i(k^1,k^2,\cdots,k^r;a^1,a^2,\cdots,a^r),\ (i=1,2,\cdots,r) \tag{10.10}$$

with parameters $(k^1(x,\dot{x},\cdots,\overset{(m)}{x}),\cdots,k^r(x,\dot{x},\cdots,\overset{(m)}{x}))$ belonging to *a vicinity of the origin* (O) *is that for*

$$\Phi=\sum_{i_1<\cdots<i_p} h_{i_1i_2\cdots i_p}\,\omega^{i_1}\wedge\cdots\wedge\omega^{i_p} \tag{10.11}$$

the coefficients $h_{i_1\cdots i_p}$ *are all constants.*

PROOF. We will begin with the proof for that (10.8) are invariant for (10.10). Apply the transformation (10.10) to (10.7); then we have

$$d\bar{a}^i=\alpha_j^i(\bar{a})\,\bar{\omega}^j,$$

i.e.

$$\frac{\partial\varphi^i(k;a)}{\partial a^l}\,da^l=\alpha_j^i(\varphi(k;a))\,\bar{\omega}^j \tag{10.12}$$

on one hand and

$$\alpha^i_j(\varphi(k;a))=\left(\frac{\partial\varphi^i(\varphi(k;a);c)}{\partial c^j}\right)_{c=0}=\left(\frac{\partial\varphi^i(k;\varphi(a;c))}{\partial c^j}\right)_{c=0} \tag{10.13}$$

$$=\left(\frac{\partial\varphi^i(k;b)}{\partial b^l}\frac{\partial\varphi^l(a;c)}{\partial c^j}\right)_{c=0}=\frac{\partial\varphi^i(k;a)}{\partial a^l}\alpha^l_j(a)$$

on the other hand, where $b^i=\varphi^i(a;c)$. Apply the inverse of

$$\left(\frac{\partial\varphi^i(k:a)}{\partial a^l}\right)$$

to (10.12). Then it results that

$$da^l=\alpha^l_j(a)\,\bar{\omega}^j.$$

Thus we have

$$\omega^j=\beta^j_l(a)\,da^l=\beta^j_l(a)\,\alpha^l_k(a)\,\bar{\omega}^k=\delta^j_k\,\bar{\omega}^k=\bar{\omega}^j.$$

Secondly, in order that Φ may be invariant, the relation

$$h_{i_1\cdots i_p}(a)=h_{i_1\cdots i_p}(\varphi(k;a))$$

must hold for all values of k. If we take $a\to 0$, since $\varphi^i(k;0)=k^i$, we must have

$$h_{i_1\cdots i_p}(0)=h_{i_1\cdots i_p}(k).$$

Hence $h_{i_1\cdots i_p}$ must all be constants. Q. E. D.

THEOREM. *For the fundamental extended Pfaffians of r-dimensional doubly extended Lie group (germ), it holds that*

$$d\omega^i=\frac{1}{2}C^i_{jk}\,\omega^j\wedge\omega^k, \tag{10.14}$$

where the r^3 constant coefficients C^i_{jk} obey the rules

$$C^i_{jk}=C^i_{kj},\quad C^i_{jj}=0, \tag{10.15}$$

$$C^h_{ij}C^l_{hk}+C^h_{jk}C^l_{hi}+C^h_{ki}C^l_{hj}=0. \tag{10.16}$$

PROOF. Since ω^i are invariant, $d\omega^i$ must also be invariant, since the operator d and coordinate transformation are commutative. Hence, by the last Theorem, we must have constants C^i_{jk} such that

$$d\omega^i=C^i_{jk}\,\omega^j\wedge\omega^k.$$

If we set (10.15):

$$C^i_{jk}=-C^i_{kj},\quad (j>k),\quad C^i_{jj}=0,$$

we have

$$d\omega^i=\frac{1}{2}C^i_{jk}\,\omega^j\wedge\omega^k,\quad \omega^i\epsilon A(C^2),\quad d\omega^i\epsilon A(C^1), \tag{10.17}$$

$$d(d\omega^i)=0. \tag{10.18}$$

Therefore

$$d(d\omega^i)=\frac{1}{2}C^i_{kl}\,d\omega^k\wedge\omega^l-\frac{1}{2}C^i_{kl}\,\omega^k\wedge d\omega^l$$
$$=C^i_{kl}\,d\omega^k\wedge\omega^l=\frac{1}{2}C^i_{kl}\,C^k_{pq}\,\omega^p\wedge\omega^q\wedge\omega^l=0.$$

Hence

$$C^h_{ij}\,C^l_{hk}+C^h_{jk}\,C^l_{hi}+C^h_{ki}\,C^l_{hj}=0,\quad (i,j,k=1,2,\cdots,r).$$

Definition. The r^3 constants C^i_{jk} are called the ***structure constants*** of the r-dimensional doubly extended Lie group (germ).

If we develop $\varphi^i(a(x,\dot{x},\cdots,\overset{(m)}{x});b(x,\dot{x},\cdots,\overset{(m)}{x}))$, by virtue of (3.1), then we obtain

(10.19) $$\varphi^i(a;b)=a^i+b^i+d^i_{jk}\,a^j\,a^k+\varepsilon^i,$$

where ε^i is an infinitesimal of an order higher than the second in the vicinity of the origin. From (10;19), it results that

$$\alpha^i_j(a)=\delta^i_j+d^i_{kj}\,a^k+\varepsilon^2,$$
$$\beta^i_j(b)=\delta^i_j-d^i_{kj}\,a^k+\varepsilon^3,$$

where ε^2 and ε^3 are infinitesimals. Hence

$$\omega^i(a,da)=da^i-d^i_{kj}\,a^k\,da^j+\varepsilon_{4j}\,da^j,$$

where ε_{4j} is an infinitesimal. Hence it results that

$$d\omega^i=-d^i_{kj}\,da^k\wedge da^j+d\varepsilon_{4j}\wedge da^j=C^i_{jk}\,\omega^j\wedge\omega^k.$$

Comparing the coefficients of $da^k\wedge da^j$, we obtain

(10.20) $$C^i_{jk}=d^i_{jk}-d^i_{kj}.$$

N.B. (i) In order to deduce (10.16) in terms of d^i_{jk} directly, we utilize (3.2) having written out the terms of the third degree in (10.19) ([20]).

(ii) As for the class C^v in the ordinary case, L. Pontrjagin ([20]) has taken $v=3$. L. van der Waerden ([21]) has assumed, that (1) $\varphi^i(a;b)$ is once differentiable, (2) $\varphi'_a(a;b)$ satisfies the Lipschitz's condition for b and (3) its converse. G. Birkhoff ([22]) has assumed the existence of the total differential of $\varphi^i(a;b)$ and its continuity in the origin. P. A. Smith ([23]) has proved that when for $\varphi^i(a;b)=a^i+b^i+\psi^i(a;b)$, the condition $\frac{\psi_i(a;b)}{|a|}\to 0,(a\to 0,\ b\to 0)$, where $(|a|=a^{12}+\cdots+a^{r2})$, is satisfied, the Lie group (germ) may be rendered into an analytic Lie group (germ).

In our case, we have assumed "$\varphi^i\epsilon\,C^3$." This condition is fully utilized in (10.18). But, it will be seen that the result of Art. 10 holds good also for $\varphi^i\epsilon\,C^2$, if we notice the following fact. Indeed, if $\varphi^i \epsilon C^2$, then we have $\omega^i \epsilon A(C^1), d\omega^i \epsilon A(C^0)$. Thus the first Theorem of Art. 8 is still applicable, so that (10.17) holds. Consequently we see that $d\omega^i \epsilon A(C^1)$, so that $d(d\omega^i)$ exists and the fact $d(d\omega^i)=0$ is a consequence of $\omega^i \epsilon A(C^2)$. Hence it suffices to deduce $d(d\omega^i)=0$ from $d\omega^i \epsilon A$ (C^1) in another way. For this purpose we utilize the generalized Stoke's theorem. When $\omega^r \epsilon A(C^v), (v \geqq 1)$ is an homogeneous expression of r-th degree and C^{r+1} be an algebraic complex composed of curved simplex of μ-th class ($\mu \geqq 2$), then the relation

$$\int_{\varDelta C^{r+1}} \omega^r = \int_{C^{r+1}} d\omega^r$$

holds. Thus for an arbitrary 3-dimensional curved simplex C^3, we have

$$(C^3, d(d\omega^i)) = (\varDelta C^3, d\omega^i) = (\varDelta(\varDelta C^3), \omega^i) = 0,$$

where

$$\int_{C^r} \omega^r = (C^r, \omega^r).$$

Hence we have

$$d(d\omega^i) = 0.$$

(iii) The name "fundamental extended Pfaffians" arises fom the following theorem.

THEOREM. *When r fundamental extended Pfaffians are invariant for*

$$a^i \to \bar{a}^i = \psi^i(a), \qquad (i=1, 2, \cdots, r),$$

which maps the points of a vicinity U of the origin into a vicinity of the origin:

(10.21) $$\omega^i(a, da) = \omega^i(\bar{a}, d\bar{a}), \qquad (i=1, 2, \cdots, r),$$

the $\psi^i(a)$ coincides with the composition function $\varphi^i(k; a)$:

(10.22) $$\psi^i(a) = \varphi^i(k; a), \qquad (i=1, 2, \cdots, r)$$

for

(10.23) $$\psi^i(0) = k^i, \qquad (i=1, 2, \cdots, r),$$

that is to say, the doubly extended Lie group (germ) is determined uniquely by given fundamental extended Pfaffians.

PROOF. Consider the simultaneous extended total differential equations

(10.24) $$\bar{\omega}^i - \omega^i = 0, \qquad (i=1, 2, \cdots, r),$$

putting

$$\bar{\omega}^i = \beta^i_j(\bar{a})\, d\bar{a}^j.$$

These are completely integrable. For,

$$d(\bar{\omega}^i-\omega^i)=\sum_{j<k} C^i_{jk}(\bar{\omega}^j\wedge\bar{\omega}^k-\omega^j\wedge\omega^k)$$
$$=\sum_{j<k} C^i_{jk}\{\bar{\omega}^j\wedge(\bar{\omega}^k-\omega^k)+(\bar{\omega}^j-\omega^j)\wedge\omega^k\}$$
$$\equiv 0, \qquad (\text{mod. } \bar{\omega}^1-\omega^1, \cdots, \bar{\omega}^r-\omega^r),$$

and since

$$|\beta^i_j(\bar{a})|\neq 0,$$

the solutions such that

$$\text{(10.25)} \qquad \begin{aligned} \bar{a}^i&=f^i(k^1,\cdots,k^r;a^1,\cdots,a^r),\\ k^i&=f^i(k^1,\cdots,k^r;0,\cdots\cdots,0), \end{aligned} \qquad (i=1,2,\cdots,r)$$

exist on one hand. $\bar{a}^i=\psi^i(a)$ are solutions of (10.14) for the initial conditions (10.23), so that, by the uniqueness of the solution, we have

$$\psi^i(a)=f^i(k;a), \qquad (i=1,2,\cdots,r).$$

On the other hand

$$\bar{a}^i=\psi^i(k^1,\cdots,k^r;a^1,\cdots,a^r)$$

are also the solutions of (10.24) for the same initial conditions by the First Theorem above. Therefore we must have

$$\text{(10.26)} \qquad \varphi^i(k;a)=f^i(k;a)=\psi^i(a), \qquad (i=1,2,\cdots,r).$$

11. Abstract Lie Ring. In order to make the structure of the *extended Lie groups* clear, we give the definition of the abstract Lie ring.

DEFINITION. A vector space R of rank r with

real | complex

coefficients is called an *abstract Lie ring*, when the following conditions (i) and (ii) are satisfied:

(i) For $A, B\in R$, a *commutator product* $(A, B)\in R$ is defined uniquely;

(ii) $$(\lambda_1A_1+\lambda_2A_2, B)=\lambda_1(A_1, B)+\lambda_2(A_2, B),$$

$$\text{(11.1)} \qquad (A, B)=-(B, A),$$

$$\text{(11.2)} \qquad ((A, B), C)+((B, C), A)+((C, A), B)=0.$$

THEOREM. *For given basis $E_1, E_2, \cdots, E_r$ of a vector space, there exists r-dimensional abstract Lie ring R having the structure constants of an r-dimensional (doubly extended) Lie group (germ) G as coefficients of*

$$\text{(11.3)} \qquad (E_i, E_j)=C^k_{ij}E_k.$$

PROOF. Since $E_1, E_2, \cdots, E_r$ form a basis of a vector space, we may set (11.3). Then from (11.1) and (11.2), we obtain

$$(11.4) \qquad \begin{gathered} C^k_{ij} = -C^k_{ji}, \\ C^h_{ij}\, C^l_{hk} + C^h_{jk}\, C^l_{hi} + C^h_{ki}\, C^l_{hj} = 0. \end{gathered}$$

Conversely, if (11.4) holds for certain r^3 constants C^i_{jk}, we can determine the basis $E_1, E_2, \cdots, E_r$ so that the commutator product of them is (11.3) and introduce the definition

$$(\alpha^i E_i, \beta^j E_j) = \alpha^i \beta^j (E_i, E_j),$$

then (11.1) and (11.2) hold. Hence the Theorem.

N. B. When a property of a doubly extended Lie group (germ) is given, we shall express it in terms of the corresponding abstract Lie ring.

12. Coordinate Transformation.

DEFINITION. When the relations

$$(12.1) \qquad \left\{ \begin{array}{ll} \bar{g}^i(\varphi(a;b)) = \bar{\varphi}^i(\bar{g}(a);\bar{g}(b)), & \\ g^i(\bar{\varphi}(\bar{a};\bar{b})) = \varphi^i(g(\bar{a});g(\bar{b})), & (i=1,2,\cdots,r) \end{array} \right.$$

hold for a certain one-to-one transformation

$$(12.2) \qquad \left\{ \begin{array}{lll} a^i = g^i(\bar{a}^1, \cdots, \bar{a}^r), & 0 = g^i(0, \cdots, 0), & \\ \bar{a}^i = \bar{g}^i(a^1, \cdots, a^r), & 0 = \bar{g}^i(0, \cdots, 0), & (i=1,2,\cdots,r), \end{array} \right.$$
$$g^i, \bar{g}^i \in C^1$$

between certain vicinities $U, \bar{U}$ of respective origin of two r-dimensional doubly extended Lie group (germ) G and $\bar{G}$ hold, G and $\bar{G}$ are said to be ***isomorphic*** to each other. Thereby $\varphi(a;b)$ and $\bar{\varphi}(\bar{a};\bar{b})$ are respective composition functions in G and $\bar{G}$.

The (12.2) may also be expressed as follows:

$$(12.3) \qquad \begin{array}{l} \text{If } S_a \cdot S_b = S_c, \text{ then } \bar{S}_{\bar{g}(a)} \cdot \bar{S}_{\bar{g}(b)} = \bar{S}_{\bar{g}(c)}, \\ \text{If } \bar{S}_{\bar{a}} \cdot \bar{S}_{\bar{b}} = \bar{S}_{\bar{c}}, \text{ then } S_{g(\bar{a})} \cdot S_{g(\bar{b})} = S_{g(\bar{c})}, \\ \qquad (S_a, S_b, \cdots \in G; \bar{S}_{\bar{a}}, \bar{S}_{\bar{b}}, \cdots \in \bar{G}). \end{array}$$

When g^i and $\bar{g}^i$ are, in particular, analytic functions, G and $\bar{G}$ are said to be ***analytically isomorphic.***

If we transform the doubly extended parameters $(a^1, \cdots, a^r)$ of an r-dimensional doubly extended Lie group (germ) G into $(\bar{a}^1, \cdots, \bar{a}^r)$ by $\bar{g}^1, \cdots, \bar{g}^r \in C^1$ such that

$$(12.4) \qquad \left\{ \begin{array}{lll} \bar{a}^i = g^i(a^1, \cdots, a^r), & 0 = g^i(0, \cdots, 0), & (i=1,2,\cdots,r), \\ \dfrac{\partial(\bar{g}^1, \cdots, \bar{g}^r)}{\partial(a^1, \cdots, a^r)} \neq 0, & & \end{array} \right.$$

then it results that

$$S_a=S_{\bar{a}},$$

which is a special case $G=\bar{G}$ of the above definition for isomorphism. Thus a treatment of the isomorphism consequences a transformation of the doubly extended parameters.

If G and $\bar{G}$ be isomorphic to each other, then introducing

$$d\bar{a}^i=d\bar{g}^i(a)=\frac{\partial \bar{g}^i}{\partial a^k}da^k$$

and

$$\left(\frac{\partial\bar{\varphi}^i(\bar{a};\bar{c})}{\partial\bar{c}^j}\right)_{\bar{c}=0}=\frac{\partial \bar{g}^i}{\partial a^k}\left(\frac{\partial\varphi^k(a:c)}{\partial c^l}\right)_{c=0}\left(\frac{\partial g^l(\bar{c})}{\partial\bar{c}^j}\right)_{\bar{c}=0}$$

obtained by differentiation of

$$\bar{\varphi}^i(\bar{a};\bar{c})=\bar{\varphi}^i(\bar{g}(a);\bar{g}(c)=g^i(\varphi(a;c)),$$

into

$$d\bar{a}^i=\left(\frac{\partial\bar{\varphi}^i(\bar{a};\bar{c})}{\partial\bar{c}^j}\right)_{\bar{c}=0}\bar{\omega}^j(\bar{a},d\bar{a})$$

and solving the resulting equations with respect to da^k, we obtain

$$da^k=\left(\frac{\partial\varphi^k(a;c)}{\partial c^l}\right)_{c=0}\left(\frac{\partial g^i(\bar{c})}{\partial\bar{c}^j}\right)_{\bar{c}=0}\bar{\omega}^j(\bar{a},d\bar{a}).$$

Comparing this with the fundamental extended Pfaffians $\omega^j(a,da)$, we obtain

(12.5) $$\omega^i(a,da)=h^i_j\,\bar{\omega}^j(\bar{a},d\bar{a}),$$

where

(12.6) $$h^i_j=\left(\frac{\partial g^i(\bar{c})}{\partial c^j}\right)_{\bar{c}=0}.$$

Thus the fundamental extended Pfaffians undergo a linear transformation with constant coefficients.

We introduce this into

$$d\omega^i=\frac{1}{2}C^i_{kl}\,\omega^k\wedge\omega^l.$$

Then it results that

$$d(h^i_j\,\bar{\omega}^j)=\frac{1}{2}C^i_{kl}\,h^k_p\,h^l_q\,\bar{\omega}^p\wedge\bar{\omega}^q.$$

Set $$|\bar{h}^i_j|=|h^i_j|^{-1},\qquad \left(\bar{h}^i_j=\left(\frac{\partial\bar{g}^i(c)}{\partial c^j}\right)_{c=0}\right).$$

Then we have

$$d\bar{\omega}^j = \frac{1}{2} C^i_{kl}\, \bar{h}^j_i\, h^k_p\, h^l_q\, \bar{\omega}^p \wedge \bar{\omega}^q.$$

Comparing this with

$$d\bar{\omega}^i = \frac{1}{2} \bar{C}^i_{pq}\, \bar{\omega}^p \wedge \bar{\omega}^q,$$

we see that

$$\bar{C}^i_{pq} = (\bar{h}^j_i\, h^k_p\, h^l_q)\, C^i_{kl}. \tag{12.7}$$

Taking this result with the converse, we shall prove the following theorem.

THEOREM. *The necessary and sufficient condition for that two r-dimensional (doubly extended) Lie group (germs) G and $\bar{G}$ may be isomorphic to each other, is that the structure constants of G and $\bar{G}$ are transformed by matrix* (12.7), *where (h^i_j) is a matrix of constants such that $|h^i_j| \neq 0$ and $(\bar{h}^i_j)$ its reciprocal matrix.*

PROOF. Setting

$$\bar{\theta}^i(\bar{a}, d\bar{a}) = h^i_j\, \bar{\omega}^j(\bar{a}, d\bar{a}), \tag{12.8}$$

we see

$$d\bar{\theta}^i = \frac{1}{2} C^i_{jk}\, \bar{\theta}^j \wedge \bar{\theta}_k$$

as in the case of $d\bar{\omega}^j$ above. Hence

$$\bar{\theta}^i(\bar{a}, d\bar{a}) - \omega^i(a, da) = 0, \qquad (i=1, 2, \cdots, r)$$

is completely integrable as in the case of (10.24) and the solution is given by

$$\bar{a}_i, = \bar{g}^i(a^1, \cdots, a^r), \quad 0 = \bar{g}^i(0, 0, \cdots, 0), \qquad (i=1, 2, \cdots, r).$$

Since these are one and the same integral, we must have

$$\begin{aligned} & g^i(\bar{g}^i(a)) = a^i, \quad \bar{g}^i(g(\bar{a}) = \bar{a}^i, \qquad (i=1, 2, \cdots, r), \\ & \omega^i(g(\bar{a}), dg(\bar{a})) = \bar{\theta}^i(\bar{a}, d\bar{a}), \\ & \bar{\theta}^i(\bar{g}(a), d\bar{g}(a)) = \omega^i(a, da), \qquad (i=1, 2, \cdots, r). \end{aligned} \tag{12.9}$$

Now the composition functions $\bar{\varphi}(\bar{a}\,;\bar{b})$ of $\bar{G}$ make $\bar{\omega}^1, \cdots, \bar{\omega}^r$ invariant for $\bar{a} \rightarrow \bar{\varphi}(\bar{k}\,;\bar{a})$ and consequently it makes also their linear combinations $\bar{\theta}^1, \cdots, \bar{\theta}^r$ invariant. Hence, for the transformation

$$a^i \rightarrow \bar{g}^i(a) \rightarrow \bar{\varphi}^i(\bar{g}(k)\,; \bar{g}(a)) \rightarrow g^i(\bar{\varphi}(\bar{g}(k)\,; g(a))), \quad (i=1, 2, \cdots, r),$$

we obtain

$$\omega(a, da) \rightarrow \bar{\theta}(\bar{a}, d\bar{a}) \rightarrow \theta(\bar{a}, d\bar{a}) \rightarrow \omega, (a, da)$$

together with

$$0 \rightarrow \bar{g}^i(0) = 0 \rightarrow \bar{\varphi}^i(\bar{g}(k)\,; 0) = \bar{g}^i(k) - g^i(\bar{g}(k)) = k^i, \quad (i=1, 2, \cdots, r)$$

in particular.

Now by the Theorem concerning (10.22) we must have

$$g^i(\bar{\varphi}(\bar{g}(k);\bar{g}(a)))=\varphi^i(k;a), \qquad (i=1,2,\cdots,r)$$

i. e.

$$\bar{\varphi}^i(\bar{g}(k);\bar{g}(a))=\bar{g}^i(\varphi(k;a)), \quad (i=1,2,\cdots,r)$$

by (12.9).

A similar result will be obtained when we interchange the situations of G and $\bar{G}$.

Taking these two results together, we arrive at (12.1).

If hereby $\varphi^i, \bar{\varphi}^i \epsilon C^3$, then $\omega^i, \bar{\omega}^i\, \bar{\theta}^j \epsilon A(C^2)$ and we see that

$$g^i, \bar{g}^i \epsilon C^2. \qquad\qquad \text{Q. E. D.}$$

Restating the last Theorem in terms of the abstract Lie ring, we obtain the following theorem.

THEOREM. *In order that two r-dimensional (doubly extended) Lie group (s) (germs) G and $\bar{G}$ may be isomorphic to each other, it is necessary and sufficient that the corresponding abstract Lie rings R and $\bar{R}$ become ring-isomorphic by an appropriate linear transformation between their bases, that is to say, that to $A \epsilon R$ there corresponds $f(A)=\bar{A} \epsilon \bar{R}$ uniquely and that the relations*

$$\begin{cases} f(\lambda A_1+\mu A_2)=\lambda f(A_1)+\mu f(A_2), \\ \quad f((A, B))=(f(A), f(B)) \end{cases}$$

hold, the linear transformations being

$$f(E_i)=h_i^j E_j, \qquad (i, j=1, 2, \cdots, r).$$

13. Inner Automorphic Transformations.

DEFINITION. The isomorphism $G \to G$ of the type

$$S_a \to S_{\bar{a}}=S_b S_a S_b^{-1}, \qquad (S_b \epsilon G) \tag{13.1}$$

is called an *inner automorphism* of G.

The transformation

$$\bar{a}^i=\bar{g}^i(a), \qquad (i=1,2,\cdots,r)$$

transforms a vicinity of the origin into a vicinity of the orgin is one-to-one manner and since $\bar{g}^i \epsilon C^3$, the first theorem of Art. 12 applies, so that we have

$$\bar{\omega}^i(\bar{a}, d\bar{a})=h_k^i\, \omega^k(a, da) \qquad (i=1,2,\cdots,r), \tag{13.2}$$

where the matrix $(h_k^i(b))$ is obtained as follows. Since from (13.1) follows;

$$S_{\bar{a}}\, S_b=S_b\, S_a,$$

the relation

$$\varphi^i(\bar{a}\,;b)=\varphi^i(b\,;a), \qquad (i=1,2,\cdots,r)$$

holds and consequently

$$\left(\frac{\partial\varphi^k(\bar{a}\,;b)}{\partial\bar{a}^l}\right)_{\bar{a}=0}\left(\frac{\partial\bar{g}^l(a)}{\partial a^j}\right)_{a=0}=\left(\frac{\partial\varphi^k(b\,;a)}{\partial a^j}\right)_{a=0}. \tag{13.3}$$

We set

$$\alpha_j^{*k}(a)=\left(\frac{\partial\varphi^k(c\,;a)}{\partial c^j}\right)_{c=0}, \qquad \alpha_j^{*k}(a)\,\beta_k^{*i}(a)=\delta_j^i \tag{13.4}$$

according to (10.3) and (10.5) and multiply (13.3) with β_k^{*i}, then it results that

$$h_j^i(b)=\left(\frac{\partial\bar{g}^i(a)}{\partial a^j}\right)_{a=0}=\alpha_j^k(b)\,\beta_k^{*i}(b).$$

Next, for

$$S_{\bar{\bar{a}}}=S_dS_{\bar{a}}S_d^{-1}=(S_dS_b)\,S_a\,(S_dS_b)^{-1},$$

we have

$$\bar{\bar{\omega}}^i(\bar{\bar{a}},d\bar{\bar{a}})=h_j^i(d)\,\bar{\omega}^j(\bar{a},d\bar{a})=h_j^i(d)\,h_k^j(b)\,\omega^k(a,da),$$

whence follows:

$$k_k^i(\varphi(b\,;d))=h_k^j(b)\,h_j^i(d). \tag{13.5}$$

Thus, if we set

$$H(S_b)\overset{\text{def}}{=\!=}(h_k^i(b)),$$

from (13.5), we obtain

$$H(S_b\cdot S_d)=H(S_b)\cdot H(S_d). \tag{13.6}$$

This tells us that the set

$$\{(h_k^i(b))\,;b\in U_0\} \tag{13.7}$$

forms a group (germ), which is homomorphic to the r-dimensional doubly extended Lie group (germ) G.

DEFINITION. We call (13.7) the *adjoint doubly extended group of G.*

N. B. *The adjoint doubly extended group is a doubly extended Lie group (germ).*

14. Existence Conditions and Canonical Parameter.

DEOINITION. An r-dimensional group (germ) is said *to have a canonical parameter,* when the following two conditions are satisfied: (i) it is a doubly extended analytic Lie group (germ) i. e. $\varphi^i(a\,;b)$ are analytic functions of a and b; (ii) for sufficiently small real values of s and t, the relation

$$a^i(s+t)=\varphi^i(a^1s, \cdots, a^rs; a^1t, \cdots, a^rt). \qquad (i=1, 2, \cdots, r) \tag{14.1}$$

holds in $a \in U_1$, i. e.

$$S_a : a^i = a_0^i t, |t| < \epsilon, \qquad (i=1, 2, \cdots, r) \tag{14.2}$$

forms a one-dimensional doubly extended subgroup (germ). The (14.2) is called a *one-parametric doubly extended subgroup* (*germ*).

THEOREM 1°. *It is possible to make any* (*doubly*) *extended Lie group* (*germ*) *G have a canonical parameter by an appropriate change of parameter, retaining the structure constants.*

This theorem implies also, that there exist an analytie (doubly extended) Lie group (germ) $\bar{G}$ having the structure constants in common with an arbitrary given doubly extended Lie group (germ) G, and the G and the $\bar{G}$ being isomorphic to each other.

This theorem is an immediate consequence of the following existence theorem having a stronger content.

THEOREM 2°. *If* r^3 *constants*

$$C^i_{jk}, \qquad (i, j, k=1, 2, \cdots, r) \tag{14.3}$$

have the properties (10.15) *and* (10.16), *there exists an r-dimensional* (*doubly extended*) *Lie group* (*germ*) *G having the canonical parameter and the* (14.3) *as structure constants.*

For, if we form an r-dimensional doubly extended Lie group (germ) of canonical parameter having the structure constants G^i_{jk} of the given r-dimensional Lie group (germ) as structure constants, the G and the $\bar{G}$ are isomorphic to each other by the first theorem of Art. 12.

N. B. The Theorem 2° shows us the complete correspondence between an r-dimensional Lie group (germ) and an abstract Lie ring of rank r. Thus taking the first theorem of Art. 12 together, we have the

THEOREM 3°. *There exists an r-dimensional doubly extended Lie group* (*germ*) *corresponding to an arbitrary given abstract Lie ring of rank r. Consequently a class of mutually isomorphic r-dimensional doubly extended Lie group* (*s*) (*germs*) *and a class of mutually ring-isomorphic doubly extended abstract Lie ring of rank r have one-to-one correspondence.*

Let us now prove Theorem 3° in three steps I, II and III.

I. If analytic functions $b^i_j(a)$ such that for constants C^i_{jk} the relations

$$d\omega^i = \frac{1}{2} C^i_{jk}\, \omega^j \wedge \omega^k, \tag{14.4}$$

$$(14.5)\qquad \begin{cases} \omega^i = b^i_j(a)\,da^j, & (i=1,2,\cdots,r),\\ \delta^i_j = b^i_j(0,\cdots,0), \end{cases}$$

hold, then there exists an r-dimensional analytic (doubly extended) Lie group (germ) G, for whose composition function φ the relation

$$(14.6)\qquad (b^i_j(a))^{-1} = \left(\left[\frac{\partial\varphi^i(a;c)}{\partial c^j}\right]_{c=0}\right)$$

holds, so that the C^i_{jk} become the structure constants for this G.

PROOF. (i) The simultaneous extended total differential equations

$$(14.7)\qquad \bar{\omega}^i - \omega^i = 0, \qquad (i=1,2,\cdots,r)$$

for $2r$ independent variables $a^1,\cdots,a^r;\bar{a}^1,\cdots,\bar{a}^r$ formed after (14.5) as in the case of (10.24) are completely integrable.

Taking their solutions such that

$$(14.8)\qquad \begin{cases} \bar{a}^i = \varphi^i(k^1,\cdots,k^r;a^1,\cdots,a^r), & (i=1,2,\cdots,r),\\ k^i = \varphi^i(k^1,\cdots,k^r;0,\cdots,0), & (i=1,2,\cdots,r), \end{cases}$$

we define the product

$$S_a\cdot S_b = S_c, \qquad (c^i = \varphi^i(a;b)), \qquad (i=1,2,\cdots,r)$$

for sufficiently small vicinity of the origin. Let us examine if a (doubly extended) Lie group (germ) $\bar{G}$ is fomed.

(ii) By (14.8), we have

$$\varphi^i(k;0) = k^i, \qquad (i=1,2,\cdots,r).$$

It is further seen that

$$\varphi_i(0;a) = a^i, \qquad (i=1,2,\cdots,r)$$

from the fact that both sides are solutions of (14.7) for the initial conditions $\varphi^i(0;0)=0$.

(iii) Since under the two transformations

$$a^i \to \bar{a}^i = \varphi^i(l;a) \to \bar{\bar{a}}^i = \varphi^i(k;\varphi(l;a)), \qquad (i=1,2,\cdots,r),$$

the Pfaffians $\omega^1,\cdots,\omega^r$ are invariants,

$$\bar{a}^i = \varphi^i(k;\varphi(l;a)), \qquad (i=1,2,\cdots,r)$$

are solutions of (14.7) and satisfy

$$\varphi^i(k;\varphi(l;0)) = \varphi^i(k;l), \qquad (i=1,2,\cdots,r).$$

Hence by the uniqueness of the solution, they coincide with $\varphi^i(\varphi(k;l);a)$ taking the same values in $a=0$:

$$\varphi^i(k;\varphi(l;a)) = \varphi^i(\varphi(k;l);a), \qquad (i=1,2,\cdots,r).$$

Finally, comparing

$$\begin{pmatrix} d\bar{a}^1 \\ \vdots \\ d\bar{a}^r \end{pmatrix} = (b^i_k(\bar{a}))^{-1} (b^k_j(a)) \begin{pmatrix} da^1 \\ \vdots \\ da^r \end{pmatrix}$$

deduced from (14.7) with

$$d\bar{a}^i = \frac{\partial \varphi^i(k\,; a)}{\partial a^j} da^j$$

deduced from (14.8), we see that $\bar{a}^i = k^i$ on putting $a=0$, so that we obtain (14.6). Q. E. D.

II. Since the solutions $b^i_j(a)$ such that (14.4), (14.5) hold, are determinable not uniquely, we shall solve the problem under an additional demand (14.11) below.

If we introduce (14.5) into (14.4), then it results that

$$\left(\frac{\partial b^i_k}{\partial a^l} da^l\right) \wedge da^k = \frac{1}{2} C^i_{pq}\, b^p_l\, b^q_k\, da^l \wedge da^k.$$

Comparing the coefficients of $da^k \wedge da^l$, $(k<l)$, we are led to solve

(14.9) $$\frac{\partial b^i_k}{\partial a^l} - \frac{\partial b^i_l}{\partial a^k} = C^i_{pq}\, b^p_l\, b^q_k, \qquad (i, l, k = 1, 2, \cdots, r).$$

(These equations were *Maurer-Cartan differential equations* in the classical case).

Let us prove:

There exist analytic functions $b^i_j(a^1, \cdots, a^r)$ satisfying the *doubly extended* Maurer-Cartan differential equations such that

(14.10) $$b^i_j(0, \cdots, 0) = \delta^i_j, \qquad (i, j = 1, 2, \cdots, r),$$

(14.11) $$b^i_j(a)\, a^j = a^i.$$

PROOF.[6] Before all we shall solve the simultaneous ordinary differential equations of the first order (in the doubly extended sense)

(14.12) $$\frac{df^i_l}{dt} = \delta^i_l + C^i_{pq}\, a^p f^q_l, \qquad (i, l = 1, 2, \cdots, r)$$

having $a^1(x, \dot{x}, \cdots, \overset{(m)}{x}), \cdots, a^r(x, \dot{x}, \cdots, \overset{(m)}{x})$ as parameters, under the initial condition

(14.13) $$f^i_l = 0, \quad \text{in} \quad t = 0.$$

Their solutions

6) Substantially due to F. Schur. Another substantial solution will be found in : J. H. Whitehead, Note on Maurer's equations. Jour. London Math. Soc., 7 (1932) in the classical case.

$$f^i(a^l, \cdots, a^r; t) \tag{14.14}$$

are analytic functions of $a^1, \cdots, a^r$ and t. Setting

$$b^i_j(a^1, \cdots, a^r) = f^i_j(a^1, \cdots, a^r; 1),$$

we see that (14.9) holds. For it, we set

$$F^i_{kl} = \frac{\partial f^i_k}{\partial a^l} - \frac{\partial f^i_l}{\partial a^k} - C^i_{pq} f^p_l f^q_k, \qquad (i, k, l = 1, 2, \cdots, r). \tag{14.15}$$

Since

$$f^p_l = f^p_k = 0, \ \frac{\partial f^i_l}{\partial a^k} = \frac{\partial f^i_k}{\partial a^l} = 0 \qquad \text{for } t = 0,$$

we have $F^i_{kl} = 0$ for $t = 0$.

If we could show

$$\frac{dF^i_{lk}}{dt} = C^i_{pz} a^p F^z_{lk}, \qquad (i, k, l = 1, 2, \cdots, r) \tag{14.16}$$

by virtue of $F^i_{lk}(0) = 0$, it would follow that

$$F^i_{lk} \equiv 0,$$

so that (14.9) holds. Hence we shall examine (14.16).

$$\frac{dF^i_{lk}}{dt} = -\frac{\partial}{\partial a^k}(\delta^i_l - C^i_{pz} a^z f^p_l) + \frac{\partial}{\partial a^l}(\delta^i_k - C^i_{pz} a^z f^p_k) - C^i_{pq} f^p_l (\delta^q_k - C^q_{xz} a^z f^x_k)$$
$$- C^i_{pq} f^p_l (\delta^q_k - C^q_{xz} a^z f^x_k)$$
$$= C^i_{pk} f^p_l - C^i_{pl} f^p_k + C^i_{pz} a^z \left(\frac{\partial f^p_l}{\partial a^k} - \frac{\partial f^p_k}{\partial a^l} \right)$$
$$= C^i_{pk} f^p_l - C^i_{lq} f^q_k + C^i_{pq} C^q_{xz} a^z f^p_l f^x_k + C^i_{pq} C^p_{xz} a^z f^q_k f^x_l.$$

If we introduce

$$\frac{\partial f^p_l}{\partial a^k} - \frac{\partial f^p_k}{\partial a^l} = -F^p_{lk} - C^p_{xy} f^x_l f^y_k,$$

obtained from (14.15) into the last equation, then it follows that

$$\frac{dF^i_{lk}}{dt} = -C^i_{pz} a^z F^p_{lk} - C^i_{pz} C^p_{xy} f^x_l f^y_k a^z + C^i_{pq} C^q_{xz} f^p_l f^x_k a^z + C^i_{pq} C^p_{xz} f^q_k f^x_l a^z.$$

Replacing the indices (x, p, q) by (y, x, p), (x, p, y) respectively and utilizing (10.15) and (10.16), we obtain

$$\frac{dF^i_{lk}}{dt} = -C^i_{pz} a^z F^p_{lk} - (C^p_{xy} C^i_{pz} + C^p_{yz} C^i_{px} + C^p_{zx} C^i_{py}) f^x_l f^y_k a^z = -C^i_{pz} a^z F^p_{lk}.$$

In a similar way, for

$$G^i(t)=f^i_j(a\,;t)\,a^j-ta^i, \tag{14.17}$$

we have $G^i(0)=0$. For (14.17), we examine

$$\frac{dG^i}{dt}=C^i_{pj}\,a^p\,G^j. \tag{14.18}$$

We see that $G^i(t)=0$ and in particular, $G^i(1)=0$. Now

$$\frac{dG^i}{dt}=(\delta^i_j+C^i_{pq}\,a^p f^q_j)\,a^j-a^i$$

$$=C^i_{pq}\,a^p\,a^j f^q_j=C^i_{pq}\,a^p\,(f^q_j\,a^j-ta^q),$$

since $C^i_{pp}=0, C^i_{pq}=-C^i_{qp}, (p>q)$, so that (14.18) is regitimate.[7] That (14.10) holds, follows from the fact that the solution of (14.12) for $a^1=\cdots=a^r=0$ becomes $f^i_l=\delta^i_l t$.

III. Lastly, we shall prove that when (14.11) holds, the (doubly extended) Lie group (germ) obtained under I is of the canonical parameter.

By (14.6), for the $\bar{G}$ obtained under I the relation $b^i_j(a)=\beta^i_j(a)$ holds for the $\beta^i_j(a)$ in (10.5). Hence by (14.11), we have

$$\alpha^i_j(a)\,a^j=a^i \tag{14.19}$$

also.

Next we shall prove that

$$a^i=a^i_0(s+t), \qquad (i=1,2,\cdots,r) \tag{14.20}$$

for

$$a^i=a^i_0 s, \qquad b^i=b^i_0 t, \qquad (i=1,2,\cdots,r). \tag{14.21}$$

Consider

$$c^i=\varphi^i(as,at)=c^i(t)$$

fixing s for a while. Then for (14.21), we have

$$\frac{dc^i}{dt}=\frac{\partial\varphi^i(a\,;b)}{\partial b^j}\frac{db^j}{dt}=\frac{\partial\varphi^i}{\partial b^j}\,a^j_0. \tag{14.22}$$

Now we introduce

$$\alpha^i_j(a_0 t)\,a^j_0=\frac{1}{t}\,\alpha^i_j(a_0 t)\,a^j_0 t=a^i_0, \tag{14.23}$$

---

7) The reason why we considered (14.12) consists in that when conversely (14.8) and (14.11) hold, it is seen that $f^i_j(t)=tb^i_j(t)$ satisfies (14.12). Cf. Pontrjagin, ([20]), p. 253.

obtained from (14.19), into (14.22), it results that

$$\frac{dc^i}{dt}=\frac{\partial\varphi^i}{\partial b^j}\,\alpha^j_k(b)\,a^k_0.$$

Utilizing (10.13) herein, we obtain

$$\frac{dc^i}{dt}=\alpha^i_k(c)\,a^k_0. \tag{14.24}$$

The solution of (14.24) such that $c^i(0)=c^i_0 s$ for $t=0$ is, by (14.23) and (3.1):

$$c^i(t)=a^i_0(s+t).$$

Thus (14.20) is proved.

N. B. It is easily seen that conversely the (14.19) holds for the canonical parameter.

15. Reciprocal Isomorphism.

THEOREM. *If two r-dimensional (doubly extended) Lie group(s) (germs) G and $\bar{G}$ be reciprocally isomorphic, then their structure constants C^i_{jk} and C^{*i}_{jk} are related to each other by*

$$C^i_{jk}=-C^{*i}_{jk},\qquad (i,j,k=1,2,\cdots,r). \tag{15.1}$$

PROOF. Consider the Pfaffians

$$\omega^{*i}(a,da)=\beta^i_j(a)\,da^j, \tag{15.2}$$

where

$$\alpha^{*i}_j(a)=\left(\frac{\partial\varphi^i(c\,;a)}{\partial a^j}\right)_{c=0}, \tag{15.3}$$

$$\alpha^{*i}_k(a)\,\beta^{*k}_j(a)=\delta^i_j,\qquad \alpha^{*k}_j(a)\,\beta^{*i}_k(a)=\delta^i_j. \tag{15.4}$$

Then as in the case of the Third Theorem of Art. 10, the transformation under which $\omega^{*1},\cdots,\omega^{*r}$ are invariant, is

$$a^i-\bar{a}^i=\varphi^i(a\,;k),\qquad (i=1,2,\cdots,r). \tag{15.5}$$

The $d\omega^{*i}$ is expressible in the form

$$d\omega^{*i}=C^{*i}_{jk}\,\omega^{*j}\wedge\omega^{*k}. \tag{15.6}$$

We consider the expansion analogous to those in Art. 10:

$$\alpha^{*i}_j(b)=\delta^i_j+d^i_{jk}\,b^k+\varepsilon^{*2},$$
$$\beta^{*i}_j(b)=\delta^i_j-d^i_{jk}\,b^k+\varepsilon^{*3},$$
$$\omega^{*i}(b,db)=db^i-d^i_{jk}\,b^k\,db^j+\varepsilon^{*i}_j\,db^j,$$

whence we have

$$C^{*i}_{jk}=d^i_{kj}-d^i_{jk}=-C^i_{jk} \tag{15.7}$$

quite as in the case of (10.20).

Consider the totality G^* of

$$T_a \stackrel{\text{def}}{=} S_a^{-1}, \qquad (S_a \epsilon\, G).$$

Then we have

$$T_a\, T_b = (S_b\, S_a)^{-1},$$

so that G and G^* become reciprocally isomorphic.

If we set

(15.8) $$T_a T_b = T_c, \quad c^i = \varphi^{*i}(a\,;b). \qquad (i=1,2,\cdots,r),$$

then it follows that

(15.9) $$\varphi^{*i}(a\,;b) = \varphi^i(b\,;a).$$

Hence the Theorem.

§ 5. Doubly Extended Lie Transformation Group.

16. The Lie Transformation Group (**Germ**). Let G be an r-dimensional Lie group (germ); let D_0 be a vicinity of a point (x_0) of an n-dimensional Euclidean space E^n taken merely auxiliarily.

(i) Let

(16.1) $$x'^i = f^i(x^1, \cdots, x^n\,; a^1, \cdots, a^r), \qquad (i=1,2,\cdots,n),$$

be a one-to-one transformation T_a mapping a vicinity $D_1 \subset D_0$ of (x_0) into D_0;

$$x' \epsilon\, D_0, f^i(x\,;a) \epsilon\, C^3, \qquad (i=1,2,\cdots,n),$$

(ii) $$x'^i = f^i(x\,;0) = x^i, \qquad (i=1,2,\cdots,n)$$

is the *unit transformation.*

(iii) If $S_a \cdot S_b = S_c$ in G, i.e. if fot φ we have

(16.2) $$f^i(f(x\,;a)\,;b) \equiv f^i(x\,;c),$$

where

(16.3) $$c^k = \varphi^k(a\,;b), \qquad (k=1,2,\cdots r),$$

and the G is called the *parameter group* (*germ*) of $T=\{T_a\}$.

When the functions $f^i(x;a)$ of (16.1) are regular analytic functions of x and a for the analytic Lie group (germ) G, the T is called the ***analytic Lie transformation group (germ)***. $T=\{T_a\}$ is called the ***Lie's transformation group (germ)***.

When, in particular, $n=r$ and in D_0,

$$f^i(x;a)=\varphi^i(a;x), \qquad (i=1,2,\cdots r,),$$

$T=\{T_a\}$ is called a ***regular representation of the transformation group (germ)***. $f^i(x;0)=x^i$ is $\varphi^i(0;x)=x^i$.

(16.2) holds also as follows:

$$(16.2)' \qquad f^i(f(x;b);a)=\varphi^i(a;\varphi(b;x))=\varphi^i(\varphi(a;b);x)=f^i(x;\varphi(a;b)).$$

17. Doubly Extended Lie Transformation Group in the Large. The element

(17.1) $x=(x^1, x^2, \cdots, x^n)$	$a=(a^1, a^2, \cdots, a^r)$

of the

base manifold M	Lie group space G

admits of being made ***global*** by the principle stated in Art. 5, so that we have

(17.2) $x^i=\xi^i$, $(i=1,2,\cdots,n)$,	$a^l=\eta^l$, $(l=1,2,\cdots,r)$,

where the

ξ^i	η^l

are non-locally line-elemented II-geodesic parallel coordinates in the global

base manifold M.	Lie group space G.

Hereafter, we assume the

x^i	a^l

themselves to be the global ones:

ξ^i,	η^l,

and ***doubly extend*** the Lie transformation group to the case that a^l *are functions* of $x, \dot{x}, \cdots, \overset{(m)}{x}$:

$$a^l = a^l(x, \dot{x}, \cdots, \overset{(m)}{x}). \tag{17.3}$$

Thus we obtain a *doubly extended Lie transformation group G.*

A concrete *example* will be found in the case, where

$$a = (a^l_m(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})) \qquad (r=n)$$

in the sense of the right-hand side of Art. 5.

If we interpret

$$\frac{\partial f^i(x\,;a(x, \dot{x}, \cdots, \overset{(m)}{x}))}{\partial x^j} = \alpha^i_j(x, \dot{x}, \cdots, \overset{(m)}{x}) \quad \text{as} \quad a^i_j(x, \dot{x}, \cdots, \overset{(m)}{x}), \qquad (r=n^2),$$

then, *for the general* $a^l(x, \dot{x}, \cdots, \overset{(m)}{x})$, *we obtain* $a^i_j(x, \dot{x}, \cdots, \overset{(m)}{x})$ *correspondingly and the results for the right-hand side of Art.* 5 *applies to the case of general* $a^l(x, \dot{x}, \cdots, \overset{(m)}{x})$.

In the following articles, the following Fundamental Theorem will be established.

FUNDAMENTAL THEOREM. *For the doubly extended Lie Transformation groups, the theory* (Art. 18-21) *of the ordinary Lie transformation groups applies.*

18. Some Theorems on Simultaneous Extended Pfaffian Differential Equations. Before all I will give three known existence theorems.

EXISTENCE THEOREM OF IMPLICIT FUNCTIONS. If for n functions $f^i(x^1, \cdots, x^m, y^1, \cdots, y^n)$, $(i=1, 2, \cdots, n)$ of class C^v in a vicinity $(a^1, \cdots, a^m, b^1, \cdots, b^n)$ of $(m+n)$ variables such that

$$f^i(a^1, \cdots, a^m, b^1, \cdots, b^n)=0 \qquad (i=1, 2, \cdots, n),$$

the condition

$$\frac{\partial(f^1, \cdots, f^n)}{\partial(y^1, \cdots, y^n)} \neq 0$$

be satisfied, then there exists one and only one system of n functions

$$g^i(x^1, \cdots, x^m), \qquad (i=1, 2, \cdots, n)$$

of the class C^v defined in the vicinity of $(a^1, a^2, \cdots, a^m)$ such that

$$f^i(x^1, \cdots, x^m, g^1, \cdots, g^n)=0, \qquad (i=1, 2, \cdots, n),$$
$$b^i = g^i(a^1, \cdots, a^m), \qquad (i=1, 2, \cdots, n).$$

EXISTENCE THEOREM OF THE INVERSE FUNCTIONS. If for n functions

$$y^i = f^i(x^1, \cdots, x^n), (i=1, 2, \cdots, n)$$

of the class C^v defined in the vicinity of a point $(a^1, \cdots, a^n)$, the conditions

$$b^i=f^i(a^1,\cdots,a^n),\qquad (i=1,2,\cdots,n),$$

$$\frac{\partial(f^1,\cdots,f^n)}{\partial(x^1,\cdots,x^n)}\neq 0$$

are satisfied, then there exists one and only one system of n functions

$$x^i=g^i(y^1,\cdots,y^n),\qquad (i=1,2,\cdots,n)$$

of class C^v defined in the vicinity of the point $(b^1,\cdots,b^n)$ such that

$$y^i=f^i(g^1,\cdots,g^n),\qquad (i=1,2,\cdots,n),$$

$$a^i=g^i(b^1,\cdots,b^n),\qquad (i=1,2,\cdots,n).$$

EXISTENCE THEOREM OF THE SOLUTIONS OF n SIMULTANEOUS ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER. For the $(n-1)$ simultaneous ordinary differential equations

$$\frac{dy^i}{dx^n}=f^i(y^1,\cdots,y^{n-1},x^n),\qquad (i=1,2,\cdots,n-1),$$

where $f^i(x^1,x^2,\cdots,x^n)$, $(i=1,2,\cdots,n-1)$ are functions of the class C^v defined in the vicinity of a point $(a^1,\cdots,a^n)$, there exists one and only one system of $(n-1)$ functions $g^i(x^1,\cdots,x^n)$, $(i=1,2,\cdots,n-1)$ such that

$$\frac{\partial g^i}{\partial x^n}=f^i(g^1,\cdots,g^{n-1},x^n),$$

$$x^i=g^i(x^1,\cdots,x^{n-1},a^n).$$

If f^i, $(i=1,2,\cdots,n-1)$ are functions of class C^v of r parameters $(\lambda^1,\cdots,\lambda^r)$ defined in the vicinity of $(\lambda^1_0,\cdots,\lambda^r_0)$, then the $(n-1)$ solutions g^i are also functions of the class C^v of $\lambda^1,\cdots\lambda^r$.

DEFINITION. The n functions

(18.1) $$x^i=f^i(u^1,\cdots,u^s),\qquad (i=1,2,\cdots,n)$$

of the class C^v defined in the vicinity of a point $(u^1_0,\cdots,u^s_0)$ of an s-dimensional space are called the solution of the r equations

(18.2) $$\theta^{p_k}_k=0,\qquad (k=1,2,\cdots,r),$$

where

(18.3) $$\theta^{p_k}_k \sum_{i_1<\cdots<i_p} a^{(k)}_{i_1\cdots i_p}(x^1,\cdots,x^n;\dot{x},\ddot{x},\cdots,\overset{(m)}{x})\,dx^{i_1}\cdots dx^{i_p},\ (p=p_k;k=1,2,\cdots,r),$$

when (18.3) are transformed into 0 by the transformation (18.1), that is to say, when for each

$$1\leqq j_1<j_2<\cdots\cdots<j_p\leqq s;\ k=1,2,\cdots,r,$$

the relation

$$(18.4)\qquad \sum_{i_1<\cdots<i_p} a^{(k)}_{i_1\cdots i_p}(f^1(n),\cdots,f^n(n);\dot{f},\cdots,\overset{(m)}{f})\frac{\partial(f^{i_1},\cdots,f^{i_p})}{\partial(n^{i_1},\cdots,u^{i_p})}=0$$

holds.

When, in particular, (18.2) is

$$(18.5)\qquad \omega^i=0,$$

where

$$(18.6)\qquad \omega^i=a^i_j(x,\dot{x},,\cdots,\overset{(m)}{x})\,dx^j,\qquad (i=1,2,\cdots,r;j=1,2,\cdots,n),$$

the (18.5) is called a ***system of simultaneous extended total differential equations*** and its solution $f^i(u^1,\cdots,u^s)$ is defined by the condition that it satisfies the differential equation:

$$(18.7)\qquad a^i_j(f(u);\dot{f}(u),\cdots,\overset{(m)}{f}(u))\frac{\partial f^j}{\partial u^k}=0,\qquad (k=1,2,\cdots,s).$$

When, in particular, (18.6) are of the forms

$$(18.8)\qquad \omega^i=dx^i-C^i_k(x,\dot{x},\cdots,\overset{(m)}{x})\,dx^{r+k},\qquad (r+s=n;i=1,2,\cdots,r;k=1,2,\cdots,s),$$

the condition for that

$$(18.9)\qquad \begin{cases} x^i=f^i(u^1,\cdots,u^s), & (i=1,2,\cdots,r),\\ x^j=u^{j-r}, & (j=r+1,\cdots,n)\end{cases}$$

may be the ***solution*** of (18.5) is that (18.9) satisfies the simultaneous extended linear partial differential equations

$$(18.10)\qquad \frac{\partial f^i(x)}{\partial u^k}=C^i_k(x^1,x^2,\cdots,x^r;u^1,\cdots,u^s;\dot{u},\cdots,\overset{(m)}{u};\dot{x},\cdots,\overset{(m)}{x}),$$
$$(i=1,2,\cdots,r;k=1,2,\cdots,s).$$

When $\omega^1,\cdots,\omega^r\in C^v$ are not linearly independent ones among them and form simultaneous extended total differential equations, their solutions coincide with those of $\omega^1=0,\cdots,\omega^r=0$. Thus it suffices to treat the case of r linearly independent ones only.

The condition for that $\omega^1=0,\cdots,\omega^r=0$ are linearly independent, is that rank of $a^i_j(x,\dot{x},\cdots,\overset{(m)}{x}),(i=1,2,\cdots,r;j=1,2,\cdots,n)$ is r in every point of D_0.

In the case $r=n$, (18.7) demands that

$$\frac{\partial f^j}{\partial u^k}=0,\qquad (j=1,2,\cdots n;k=1,2,\cdots,s),$$

i.e. that

$$f^i=\text{const.}$$

In the general case, where

$$|\,a^i_j(x, \dot{x}, \cdots, \overset{(m)}{x})\,|_{i,j=1,2,\cdots,r} \neq 0, \tag{18.11}$$

we form

$$\omega^i_0 = b^i_j(x, x, \cdots, \overset{(m)}{x})\,\omega^j, \qquad (i, j=1, 2, \cdots, r), \tag{18.12}$$

where

$$b^i_h\, b^h_j = \delta^i_j, \qquad (i, j, k=1, 2, \cdots, r),$$

we have

$$\omega^i_0 = dx^i - C^i_j(x, \dot{x}, \cdots, \overset{(m)}{x})\,dx^j, \qquad (i=1, 2, \cdots, r; j=r+1, \cdots, n), \tag{18.13}$$

which is of the same form as (18.8).

Taking (18.12) together with its inverse transformation

$$\omega^j = a^j_i(x, \dot{x}, \cdots, \overset{(m)}{x})\,\omega^i_0, \qquad (i, j=1, 2, \cdots, r),$$

we see that the equations $\omega^i=0, i=1, 2, \cdots, r)$ and $\omega^i_0=0, (i=1, 2, \cdots, r)$ have solutions in common.

From this consideration, we introduce the following definition.

Definition. When, for a subset I of a Grassmann algebra A composed of extended differential forms, the two conditions

(I) I forms an ideal, i. e.

(i) $a, b \in I \rightarrow \pm b \in I$, (ii) $a \in I, R\,(\text{ring}), b \in R \rightarrow a \cdot b \subset I, b \cdot a \in I$,

(II) $\mathrm{I} \ni \theta \rightarrow I \ni d\theta$

are satisfied, we say that I *is a differential ideal.*

The differential ideal composed of r arbitrary homogeneous extended differential forms $\theta^1, \theta^2, \cdots, \theta^r \in A$ is

$$(\theta^1, \cdots, \theta^r) = (\varphi_1\,\theta^1 + \cdots + \varphi_r\,\theta^r\,;\, \varphi \in A, i=1, 2, \cdots, r)$$

and the *least minimal extended differential ideal* is

$$(\theta^1, \theta^2, \cdots, \theta^r\,; d\theta^1, d\theta^2, \cdots, d\theta^r). \tag{18.14}$$

For, if we differentiate the element

$$a = \varphi_1\,\theta^1 + \cdots + \varphi_r\,\theta^r + \psi_1\,d\theta^1 + \cdots + \psi_r\,d\theta^r$$

of (18.14), we obtain

$$da = d\varphi_1 \wedge \theta^1 + \cdots + d\varphi_r \wedge \theta^r + (\pm\varphi_1 + d\psi_1) \wedge d\theta^1 + \cdots + (\pm\varphi_r + d\psi_r) \wedge d\theta^r,$$

when φ_i are all homogeneous extended differential forms. And similarly for general φ_i.

DEFINITION. When two systems ot homogeneous extended differential equations

$$(18.15)\qquad \theta^1=0,\quad \theta^2=0,\cdots,\quad \theta^p=0$$

and

$$(18.16)\qquad \bar{\theta}^1=0,\quad \bar{\theta}^2=0,\cdots,\quad \bar{\theta}^p=0$$

composed of homogeneous extended differential rorms $\theta^1,\theta^2,\cdots,\cdots,\theta^p$ and $\bar{\theta}^1,\bar{\theta}^2,\cdots,\bar{\theta}^p$ respectively, have solutions in common, we say that (18.15) and (18.16) are *equivalent.*

THEOREM. *A sufficient condition for that* (18.15) *and* (18.16) *are equivalent is that*

$$(18.17)\qquad (\theta^1,\theta^2,\cdots,\theta^p\,;d\theta^1,d\theta^2,\cdots,d\theta^p)=(\bar{\theta}^1,\bar{\theta}^2,\cdots,\bar{\theta}^p\,;d\bar{\theta}^1,d\bar{\theta}^2,\cdots,d\bar{\theta}^p)$$

and in particular that

$$(18.18)\qquad (\theta^1,\theta^2,\cdots,\theta^p)=(\bar{\theta}^1,\bar{\theta}^2,\cdots,\bar{\theta}^p).$$

PROOF. For variable transformation

$$(18.19)\qquad x^i=f^i(u^1,u^2,\cdots,u^n),\qquad (i=1,2,\cdots,n),$$

we have

(i) $dx^i \longleftrightarrow \overline{dx^i}=\dfrac{\partial x^i}{\partial u^j}du^j,$

(ii) $a(x,\dot{x},\cdots,\overset{(m)}{x})\,dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_n}\longleftrightarrow\overline{(a(x,\dot{x},\cdots\overset{(m)}{x})\,dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_n})}$

$$=\overline{a(x,\dot{x},\cdots,\overset{(m)}{x})}\,\overline{dx^{i_1}}\wedge\overline{dx^{i_2}}\wedge\cdots\wedge\overline{dx^{i_n}}$$

$$=a\left(x(u),\frac{\partial x(u)}{\partial u^j}\dot{u}^j,\cdots\right)\sum_{j_1<\cdots<j_r}\frac{\partial(x^{i_1},\cdots,x^{i_n})}{\partial(u^{j_1},\cdots,u^{j_n})}du^{j_1}\wedge\cdots\wedge du^{j_n},$$

(iii) Generally, $(\omega^1+\omega^2)\longleftrightarrow\overline{(\omega^1+\omega^2)}=\bar{\omega}^1+\bar{\omega}^2.$

Hence we obtain

$$(18.20)\qquad \overline{(\omega^1\wedge\omega^2)}=\bar{\omega}^1\wedge\bar{\omega}^2$$

and generally

$$(18.21)\qquad \overline{(d_x\omega)}=d_u\bar{\omega}.$$

In order to prove (18.20), it suffices to treat the case

$$\omega^1=a(x,\dot{x},\cdots,\overset{(m)}{x})\,\omega'^1,\qquad \omega'^1=dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_r},$$
$$\omega^2=b(x,\dot{x},\cdots,\overset{(m)}{x})\,\omega'^2,\qquad \omega'^2=dx^{j_1}\wedge dx^{j_2}\wedge\cdots\wedge dx^{j_s}$$

only.

$$\overline{(\omega^1\wedge\omega^2)}=\overline{(ab\omega'^1\wedge\omega'^2)}=\overline{(ab)}\,\overline{(\omega'^1\wedge\omega'^2)}$$
$$=\bar a\bar b\bar\omega'^1\wedge\bar\omega'^2=(\bar a\bar\omega'^1)\wedge(\bar b\bar\omega'^2)=\bar\omega^1\wedge\bar\omega^2.$$

Thus (18.20) is proved.

In order to prove (18.21), it suffices to treat the case

$$\omega^r=a\,(x,\dot x,\cdots,\overset{(m)}{x})\,dx^1\wedge dx^2\wedge\cdots\wedge dx^r$$

only.

In the case

$$\omega^0=a\,(x,\dot x,\cdots,\overset{(m)}{x}),$$

we have

$$\overline{(d_x a)}=\overline{\left(\frac{\partial a}{\partial x^l}dx^l+\frac{\partial a}{\partial \dot x^l}d\dot x^l+\cdots+\frac{\partial a}{\partial \overset{(m)}{x}}d\overset{(m)}{x^l}\right)}=d\bar a,$$

$$d_u\,(\bar a)=\frac{\partial a}{\partial u^l}du^l+\frac{\partial a}{\partial \dot u^l}d\dot u^l+\cdots+\frac{\partial a}{\partial \overset{(m)}{u^l}}d\overset{(m)}{u^l}=d\bar a,$$

so that

$$\overline{(d_x a)}=d_u\bar a.$$

$$\overline{(d_x\omega)}=\overline{\{(d_x a)\,dx^1\wedge\cdots\wedge dx^r\}}=\overline{(d_x a)}\cdot\overline{(dx^1\wedge\cdots\wedge dx\ )}=d_u\,(\bar a)\,\overline{dx^1\wedge\cdots\wedge dx^r},$$
$$d_u\,(\bar\omega)=d_u\{(\bar a)\,\overline{dx^1\wedge\cdots\wedge dx^r}\}=d_u\,(\bar a)\wedge\overline{(dx^1\wedge\cdots\wedge dx^r)}+\bar a d_u\,\overline{(dx^1\wedge\cdots\wedge dx^r)}.$$

Now

$$d_u\,\overline{(dx^1\wedge\cdots\wedge dx^r)}=d_u\,(\overline{dx^1}\wedge\cdots\wedge\overline{dx^r})=d_u\,(\overline{d_x\,(x^1)}\wedge\cdots\wedge\overline{d_x\,(x^r)})$$
$$=d_u\,(d_u\,(\bar x^1)\wedge\cdots\wedge d_u\,(\bar x^r))=0.$$

Hence

$$\overline{(d_x\omega)}=d_u\,(\bar\omega).$$

Thus (18.21) is proved.

If the transformation (18.19) transforms $\theta^1,\theta^2,\cdots,\theta^p$ into 0, then, by (18.20), and the (18.21), the elements of the differential ideal produced by $\theta^1,\theta^2,\cdots,\theta^p$ are transformed into 0. Hence, if (18.17) or (18.18) holds, the two systems (18.15) and (18.16) become equivalent. Q. E. D.

When extended Pfaffians $\omega^1,\omega^2,\cdots,\omega^r$ are linearly independent, so that (18.11) is satisfied, we could take the extended Pfaffian differential equations

$$(18.22)\qquad \begin{cases}\omega_0^1=0,\ \omega_0^2=0,\ \cdots,\omega_0^r=0,\\ \omega_0^i=dx^i-C_j^i\,(x,\dot x,\cdots,\overset{(m)}{x})\,dx^j,\quad (i=1,2,\cdots,r\,;j=r+1,\cdots,n),\end{cases}$$

which are equivalent to

$$\omega^1=0,\ \omega^2=0,\ \cdots,\omega^r=0. \tag{18.20}$$

DEFINITION. That the extended total differential equations (19.20) satisfying (18.11) is ***completely integrable*** means that r functions $f^i(x^1, x^2, \cdots, x^p), (i=1, 2, \cdots, r)$ of the class C^v defined in the vicinity of a point $(x_0^1, \cdots, x_0^n)$ such that

$$\begin{cases} x^i=f^i(c^1, \cdots, c^r; u^1, \cdots, u^s), & (i=1, 2, \cdots, r; r+s=n), \\ x^{r+j}=u^j, \end{cases} \tag{18.24}$$

are solutions of (18.23), where $c^1, c^2, \cdots, c^r$ are parameters, and

$$x^i=f^i(x^1, x^2, \cdots, x^r, x_0^{r+1}, \cdots, x_0^{r+s}), \qquad (i=1, 2, \cdots, r) \tag{18.25}$$

is satisfied:

N. B. (i) In the case $r=n$, we can consider (18.20) to be always completely integrable. (ii) In the case $r=n-1$, (18.22) is of the form

$$\frac{dx^i}{du}=c_n^i(x^1, \cdots, x^{n-1}, u), \qquad (i=1, 2, \cdots, n-1),$$

so that by the existence theorem of solutions of the ordinary differential equations, (18.20) becomes always completely integrable.

Next, let us express the above conditions of complete integrability in another form (18.32).

By (18.25), for

$$y^i=f^i(x^1, x^2, \cdots, x^n), \qquad (i=1, 2, \cdots, r), \tag{18.26}$$

we obtain

$$\left(\frac{\partial(f^1, \cdots, f^r)}{\partial(x^1, \cdots, x^r)}\right)_{(x_0)}=(\delta_j^i). \tag{18.27}$$

Hence we can solve (18.26) in the form

$$x^i=F^i(y^1, \cdots, y^r, x^{r+1}, \cdots, x^{r+s}), \qquad (i=1, 2, \cdots, r), \tag{18.28}$$

where

$$F^1, F^2, \cdots, F^r \subset C^v.$$

Furthermore, by (18.25), the conditions

$$x^i=F^i(x^1, \cdots, x^r, x_0^{r+1}, \cdots, x_0^{r+s}), \qquad (i=1, 2, \cdots, r) \tag{18.29}$$

are satisfied.

If we replace (18.23) by the equivalent ones (18.22), since (18.26) are their solutions, by (18.10), they satisfy

$$\frac{\partial y^i}{\partial x^{r+1}}=C_j^i(y^1, \cdots, y^r; x^{r+1}, \cdots, x^{r+s}; \dot{x}, \cdots \overset{(m)}{x}; \dot{y}, \cdots, \overset{(m)}{y}).$$

Hence, by (18.26) and (18.28), we see that

$$\frac{\partial x^i}{\partial y^{r+j}} 0, \qquad (i=1, 2, \cdots, r; j=1, 2, \cdots, s),$$

which may be rewritten as follows

$$\frac{\partial F^i}{\partial x^{r+1}} + \frac{\partial F^i}{\partial y^k} C^k_j = 0, \qquad (i=1, 2, \cdots, r; j=1, 2, \cdots, s).$$

This contains independent variables

$$y^1, \cdots, y^r; x^{r+1}, \cdots, x^{r+s}, \tag{18.30}$$

which have values in the vicinity of $(x^1_0, x^2_0, \cdots, x^n_0)$, we may rewrite it as follows

$$x^1, x^2, \cdots, x^r, \cdots, x^n,$$

so that, by (18.22), we see

$$dF^i = \frac{\partial F^i}{\partial x^1} \omega^1_0 + \cdots + \frac{\partial F^r}{\partial x^r} \omega^r_0 \tag{18.31}$$

i. e.

$$dF^i \in (\omega^1_0, \cdots, \omega^r_0) = (\omega^1, \cdots, \omega^r).$$

Now, since

$$\left(\frac{\partial (F^1, \cdots, F^r)}{\partial (x^1, \cdots, x^r)} \right)_{(x_0)} = \left(\frac{\partial (f^1, \cdots, f^r)}{\partial (x^1, \cdots, x^r)} \right)^{-1}_{(x_0)} = (\delta_{ij}),$$

$dF^1, dF^2, \cdots, dF^r$ are linearly independent, so that

$$(dF^1, \cdots, dF^r) = (\omega^1, \cdots, \omega^r), \tag{18.32}$$

which is the desired condition of complete integrability.

DEFINITION. That the function $F(x^1, x^2, \cdots, x^n) \in C^v$ is a *first integral* of the simultaneous extended total differential equations (18.23) means that

$$dF \in (\omega^1, \cdots, \omega^r). \tag{18.33}$$

In the case (18.33), ***the relation***

$$dF(f^1(u), \cdots, f^n(u)) = 0$$

i. e.

$$F(f^1(u), \cdots, f^n(u)) = \text{const.} \tag{18.34}$$

holds for arbitrary solutions

$$x^i = f^i(u), \qquad (i=1, 2, \cdots, n)$$

of (18.23).

The following theorem follows at once.

THEOREM. *If* $F^1(x), \cdots, F^k(x)$ *are first integrals of* (18.23), *then*

$$\Phi(x^1, \cdots, x^r) \stackrel{\text{def}}{=} \varphi(F^1(x), \cdots, F^k(x)) \tag{18.35}$$

is also a first integral of (18.23), *where*

$$\varphi(y^1, \cdots, y^k) \in C^v$$

is an arbitrary function. Conversely, if (18.23) *be completely integrable, then any first integral of* (18.23), *is expressible as a function of first integrals* $F^1, \cdots, F^r$ *of* (18.23).

PROOF. The first half holds good, because for the Φ in (18.35), we have

$$d\Phi = \frac{\partial\varphi}{\partial y^j}\frac{\partial F^j}{\partial x^i}dx^i = \frac{\partial\varphi}{\partial y^i}dF^j \in (\omega^1, \omega^2 \cdots, \omega^r), (i=1, 2, \cdots, n; j=1, 2, \cdots, k).$$

The remaining half may be proved as follows.

Taking $F^1, \cdots, F^r, x^{r+1}, \cdots, x^n$ as independent variables and setting

$$\Phi(x) = \varphi(F^1, \cdots, F^r, x^{r+1}, \cdots, x^n)$$

for the first integral $\Phi(x)$, we have

$$d\Phi = \frac{\partial\Phi}{\partial F^i}dF^i + \frac{\partial\varphi}{\partial x^{r+j}}dx^{r+j} \in (\omega^1, \cdots, \omega^r) = (dF^1, \cdots, dF^r), (i=1, 2, \cdots, r; j=1, 2, \cdots, s),$$

so that we must have

$$\frac{\partial\varphi}{\partial x^{r+j}} = 0, \qquad (j=1, 2, \cdots, s),$$

that is, $\Phi(x)$ is expressible in the form

$$\Phi(x) = \varphi(F^1, \cdots, F^r). \quad \text{Q. E. D}$$

N. B. (18.33) expresses that, when (18.3) is completely integrable, there exist r independent first integrals $F^1, F^2, \cdots, F^r$ (i. e. r first integrals $F^1, F^2, \cdots, F^r$, such that $dF^1, dF^2, \cdots, dF^r$ are linearly independent). Considering the converse, we have

THFOREM. *In order that the linearly independent simultaneous extended total differential equations* (18.23) *may be completely integrable, it is necessary and sufficient that, there* r *independent first integrals* $F^1, F^2, \cdots, F^r$ *exist, that is, that the following expressibility holds good*:

$$(\omega^1, \cdots, \omega^r) = (dF^1, \cdots, dF^r). \tag{18.36}$$

PROOF FOR THF SUFFICIENCY. Since $dF^1, \cdots, dF^r$ are linearly independent, we have

$$\frac{\partial (F^1, \cdots, F^r}{\partial (x^1, \cdots, x^r)}=0.$$

Solving

(18.37) $$y^i=F^i(x^1, \cdots, x^n), \qquad (i=1, 2, \cdots, r)$$

conversely, let us have

(18.38) $$x^i=\Psi^i(y^1, \cdots, y^r, x^{r+1}, \cdots, x^n), \qquad (i=1, 2, \cdots, r).$$

Herewith we form r first integrals

(18.39) $$\Phi^i(x^1, \cdots, x^n)=\Psi^i(F^1(x), \cdots, F^r(x), x_0^{r+1}, \cdots, x_0^n),$$

for which there exist the relations

(18.40) $$\frac{\partial (\Phi^1, \cdots, \Phi^r)}{\partial (x^1, \cdots, x^r)}=\frac{\partial (\Psi^1, \cdots, \Psi^r)}{\partial (y^1, \cdots, y^r)}\frac{\partial (F^1, \cdots, F^r)}{\partial (x^1, \cdots, x^r)}\neq 0,$$

and the conditions

(18.41) $$\begin{aligned}&\Phi^i(x^1, x, \cdots, x^r, x_0^{r+1}, \cdots, x_0^n)\\ &=\Psi^i(F^1(x^1, \cdots, x^r, x_0^{r+1}, \cdots, x_0^n), \cdots, F^n(x^1, \cdots, x^r, x_0^{r+1}, \cdots, x_0^n), \quad (i=1, 2, \cdots, r)\\ &=x^i\end{aligned}$$

are satisfied. Hence, if we solve

(18.42) $$c^i=\Phi^i(x^1, \cdots, x^n), \qquad (i=1, 2, \cdots, r),$$

we have

(18.43) $$\begin{cases} x^i=f^i(c^1, \cdots, c^r, x^{r+1}, \cdots, x^n), \\ c^i=f^i(c^1, \cdots, c^r, x_0^{r+1}, \cdots, x_0^n), \qquad (i=1, 2, \cdots, r)\end{cases}$$

and thus the condition

$$\frac{\partial (f^1, \cdots, f^r)}{\partial (c^1, \cdots, c^r)}=\left(\frac{\partial (\Phi^1, \cdots, \Phi^r)}{\partial (x^1, \cdots, x^r)}\right)^{-1}\neq 0$$

is satisfied,

We consider $(c^1, \cdots, c^r)$ as parameters taking the values in the vicinity of $(x_0^1, \cdots, x_0^r)$. Utilizing (18.43) and taking $n=r+s$, we set

(18.44) $$\begin{cases} x^i=f^i(c^1, \cdots, c^r; u^1, \cdots, u^s), & (i=1, 2, \cdots, r), \\ x^{r+j}=u^j, & (j=1, 2, \cdots, s),\end{cases}$$

and from (18.42), we have

(18.45) $$\Phi^i(f^1(c; u), \cdots, f^r(c; u), u^1, \cdots, u^s)=c^i, \qquad (i=1, 2, \cdots, r).$$

Hence, considering $c^1, \cdots, c^r, u^1, \cdots, u^s$ as independent variables in place of

$x^1, \cdots, x^n$,

$$\frac{\partial \Phi^i}{\partial u^k}=\frac{\partial \Phi^i}{\partial x^j}\frac{\partial f^j}{\partial u^k}+\frac{\partial \Phi^i}{\partial u^{r+k}}=0, \qquad (j=1, 2, \cdots, r; k=1, 2, \cdots, s).$$

Consequently, (18.44) satisfies

$$d\Phi^i=\frac{\partial \Phi^i}{\partial u^k}du^k=0, \qquad (i=1, 2, \cdots, r; k=1, 2, \cdots, s),$$

if we consider $c^1, \cdots, c^r$ as parameters. Thus (18.44) are solutions of

$$d\Phi^1=0, \cdots, d\Phi^r=0.$$

These are equivalent to

$$dF^1=0, \cdots, dF^r=0$$

by virtue of (18.40) and (18.41), so that they are also equivalent to

$$\omega^1=0, \cdots, \omega^r=0$$

by (18.36). Thus (18.23) has become completely integrable owing to (18.43) and (18.44). Q. E. D.

N. B. Let us prove that *there exists only one solution of the form* (18.25).

If for $(c^1, \cdots, c^r)$ arbitrarily taken in the vicinity of $(x_0^1, \cdots, x_0^r)$, solutions such that

$$\begin{cases} x^i=g^i(c^1, \cdots, c^r; u^1, \cdots, u^s), & (i=1, 2, \cdots, r), \\ x^{j+r}=u^j & (j=1, 2, \cdots, s) \end{cases}$$

satisfy

$$c^i=g^i(c^1, c^2, \cdots, c^r; x_0^{r+1}, \cdots, x_0^n), \qquad (i=1, 2, \cdots, r),$$

they are also solutions of

$$d\Phi^1=0, \cdots, d\Phi^r=0$$

and consequently

$$\Phi^i(g^1(c; u), \cdots, g^r(c; u); u^1, \cdots, u^s) \tag{18.46}$$

do not depend upon $u^1, \cdots, u^s$. If, in particular,

$$u^i=x_0^{r+i} \qquad (i=1, 2, \cdots, s),$$

then the value of (18.46) becomes c^i. Now, since the f in (18.44) have been obtained by solving (18.42), we must have

$$f^i(c; u)=g^i(c; u), \qquad (i=1, 2, \cdots, r),$$

what shows us the uniqueness of the said solution.

THEOREM. *The necessary and sufficient condition for that the simultaneous extended total differential equations* (18.23) *may be completely integrable, is that*

(18.47) $$d\omega^i \in (\omega^1, \cdots, \omega^r), \qquad (i=1, 2, \cdots, r),$$

that is that $(\omega^1, \cdots, \omega^r)$ *forms a differential ideal.*

PROOF. *Necessity.* The above Theorem tells us the necessity of the existence of $F^1(x), \cdots, F^r(x)$ such that

(18.48) $$(\omega^1, \cdots, \omega^r) = (dF^1, \cdots, dF^r),$$

in which case we have

(18.49) $$\omega^i = \alpha^i_j(x, \dot{x}, \cdots, \overset{(m)}{x})\, dF^j(x), \qquad (j=1, 2, \cdots, r)$$

by (18.6) and (18.29), so that

$$d\omega^i = d\alpha^i_j(x, \dot{x}, \cdots, \overset{(m)}{x}) \wedge dF^j(x) \in (dF^1, \cdots, dF^r) = (\omega^1, \cdots, \omega^r),$$

whence the necessity follows.

Sufficiency. As in G. Frobenius, [42].

19. Some Theorems on Extended Linear Partial Differential Equations of the First Order. Consider s linearly independent ***extended***[8] linear partial differential equations of the first order

(19.1) $$C^j_i(x, \dot{x}, \cdots, \overset{(m)}{x}) \frac{\partial f}{\partial x^j} = 0, \qquad (i=1, 2, \cdots, s; j=1, 2, \cdots, n),$$

where $C^j_i(x, \dot{x}, \cdots, \overset{(m)}{x}) \in C^v$ in the vicinity D_0 of a point (x_0) of M.

(19.1) is said to be ***completely integrable***, when it has $n-s$ independent solutions in the vicinity of (x_0).

Solving n linearly independent 1-forms

(19.2) $$\omega^i = a^i_j(x, \dot{x}, \cdots, \overset{(m)}{x})\, dx^j, \qquad (i, j=1, 2, \cdots, n)$$

inversedly, we have

(19.3) $$dx^i = b^i_j(x, \dot{x}, \cdots, \overset{(m)}{x})\, \omega^j, \qquad (i, j=1, 2, \cdots, n),$$

where

(19.4) $$a^l_j(x, \dot{x}, \cdots, \overset{(m)}{x})\, b^i_l(x, \dot{x}, \cdots, \overset{(m)}{x}) = \delta^i_j \iff a^i_l(x, \dot{x}, \cdots, \overset{(m)}{x})\, b^l_j(x, \dot{x}, \cdots, \overset{(m)}{x}) = \delta^i_j.$$

Then, for an arbitrary $f(x) \in C^v$, we have

8) "Extended" in the sense that $\dot{x}, \ddot{x}, \cdots, \overset{(m)}{x}$ enter.

$$df=df=\frac{\partial f}{\partial x^i}dx^i=\frac{\partial f}{\partial x^i}\,b^i_j(x,\dot{x},\cdots,\overset{(m)}{x})\,\omega^j=(X_jf)\,\omega^j, \tag{19.5}$$

where

$$X_jf=b^i_j(x,\dot{x},\cdots,\overset{(m)}{x})\frac{\partial f}{\partial x^i},\qquad (i,j=1,2,\cdots,n). \tag{19.6}$$

THEOREM. *Simultaneous extended total differential eqnations*

$$\omega^1=0,\ \omega^2=0,\cdots,\omega^r=0 \tag{19.7}$$

are completely integrable when and only when the simultaneous extended linear partial differential equations

$$X_{r+1}f=0,\ X_{r+2}f=0,\cdots,X_{r+s}f=0,\qquad (n=r+s) \tag{19.8}$$

are completely integrable. Thereby (19.7) *and* (19.8) *have the first integrals in common.*

DEFINITION.

(19.8)	(19.7)
is called the *adjoint extended partial*	*total*
differential equations of	
(19.7).	(19.8).

PROOF OF THF THEOREM. By (19.5), that

$$df\in(\omega^1,\cdots,\omega^r)$$

and that

$$X_{r+1}f=0,\cdots,X_nf=0$$

are equivalent. Hence that (19.7) is completely integrable means that

$$(\omega^1\cdots,\omega^r)=(df^1,\cdots,df^r),$$

i.e. that r independent solutions $f^1,\cdots,f^r$ such that

$$df^i\in(\omega^1,\cdots,\omega^r),\qquad (i=1,2,\cdots,r)$$

exist, what is nothing other than that there exist r independent solutions of (19.8). Q. E, D.

Next, we seek for the concrete condition expressing the last condition.

Set

$$d\omega^i=\frac{1}{2}C^i_{jk}(x, \dot{x}, \cdots, \overset{(m)}{x})\,\omega^j\wedge\omega^k, \qquad (i, j, k=1, 2, \cdots, n), \tag{19.9}$$

$$C^i_{jk}(x, \dot{x}, \cdots, \overset{(m)}{x})=-C^i_{kj}(x, \dot{x}, \cdots, \overset{(m)}{x}). \tag{19.10}$$

Differentiating

$$df=df=(X_i f)\,\omega^i, \qquad (i=1, 2, \cdots, n),$$

we have

$$\begin{aligned}
0=d(df)=d(df)&=(X_i f)\,d\omega^i+d(X_i f)\wedge\omega^i\\
&=\frac{1}{2}C^i_{jk}(x, \dot{x}, \cdots, \overset{(m)}{x})(X_i f)\,\omega^j\wedge\omega^k+X_i df\wedge\omega^i\\
&=\frac{1}{2}C^i_{jk}(x, \dot{x}, \cdots, \overset{(m)}{x})(X_i f)\,\omega^j\wedge\omega^k+X_i(X_j f)\,\omega^j\wedge\omega^i\\
&=\{C^i_{jk}(x, \dot{x}, \cdots, \overset{(m)}{x})(X_i f)+(X_j X_k-X_k X_j)f\}\,\omega^j\wedge\omega^k,
\end{aligned}$$

that is, for all pairs (j, k), we have

$$\{C^i_{jk}(x, \dot{x}, \cdots, \overset{(m)}{x})X_i+(X_j X_k-X_k X_j)\}f=0 \tag{19.11}$$

for all f.

After Jacobi, we set

$$(X_j, X_k)=X_j X_k-X_k X_j. \tag{19.12}$$

This means that for the differential operator

$$X_i=b^j_i(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial}{\partial x^j}, \qquad (i, j=1, 2, \cdots, n), \tag{19.13}$$

we have

$$\begin{aligned}
(X_j, X_k)&=b^l_j(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial}{\partial x^l}\left(b^i_k(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial}{\partial x^i}\right)\\
&\quad-b^l_k(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial}{\partial x^l}\left(b^i_j(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial}{\partial x^i}\right)\\
&=\left\{b^l_j(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial b^i_k}{\partial x^l}-b^l_k(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial b^i_j}{\partial x^l}\right\}\frac{\partial}{\partial x^i}\\
&=\{(X_j b^i_k(x, \dot{x}, \cdots, \overset{(m)}{x}))-(X_k b^i_j(x, \dot{x}, \cdots, \overset{(m)}{x}))\}\frac{\partial}{\partial x^i}\,.
\end{aligned} \tag{19.14}$$

Thus (19.11) is nothing other than the relation

$$(X_j, X_k)=-C^i_{jk}(x, \dot{x}, \cdots, \overset{(m)}{x})X_i, \qquad (i, j, k=1, 2, \cdots, n). \tag{19.15}$$

By virtue of this relation, we obtain the

THEOREM 2°. *The necessary and sufficient condition for that the simulta-*

neous extended

total differential equations (19.7)	***linear partial differential equations*** (19.8)

may become completely integrable is that

$$C^i_{jk}(x, \dot{x}, \cdots, \overset{(m)}{x})=0, \qquad (i=1, 2, \cdots, r; j, k=r+1, \cdots, n) \tag{19.16}$$

holds in

(19.9).	(19.15).

Definition. It is said that the *extended* differential operators

$$X_i=b^j_i(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial}{\partial x^j}, \qquad (i=1, 2, \cdots, s; j=1, 2, \cdots, n) \tag{19.17}$$

form a *complete system*, when for X_i the relation

$$(X_j, X_k)=-C^i_{jk}(x, \dot{x}, \cdots, \overset{(m)}{x})\, X_i \tag{19.18}$$

holds. Thus we obtain the

Theorem 3[9]. *The necessary and sufficient condition for that the linearly independent simultaneous extended linear partial differential equations*

$$X_i f=0, \qquad (i=1, 2, \cdots, s)$$

is completely integrable, is that $X_1, \cdots, X_s$ form a complete system. When

$$f^i(x, \dot{x}, \cdots, \overset{(m)}{x}), \qquad (i=1, 2, \cdots, r)$$

are $r=(n-s)$ independent solutions, the general solution is expressible in the form

$$F(x, \dot{x}, \cdots, \overset{(m)}{x})=\Phi(f^1(x, \dot{x}, \cdots \overset{(m)}{x}), \cdots, f^r(x, \dot{x}, \cdots, \overset{(m)}{x})). \tag{19.19}$$

If, in particular,

$$|b^j_i(x, \dot{x}, \cdots, \overset{(m)}{x})|_{i,j=1,2,\ldots,s} \neq 0$$

is a point (x_0) of M, for an arbitrarily given $\varphi(x^{s+1}, \cdots, x^n; \dot{x}^{s+1}, \cdots, x^n; \cdots, \overset{(m)}{x}{}^{s+1}, \cdots, \overset{(m)}{x}{}^n)$, there exists one and only one solution F such that

$$\begin{aligned} F(x^1_0, \cdots, x^s_0, x^{s+1}, \cdots, x^n; \dot{x}^1_0, \cdots, \dot{x}^s_0, \dot{x}^{s+1}, \cdots, \dot{x}^n; \ddot{x}^1_0, \cdots, \ddot{x}^s_0, \ddot{x}^{s+1}, \cdots, \ddot{x}^n, \cdots, x^1_0, \\ \cdots, \overset{(m)}{x}{}^s_0, \overset{(m)}{x}{}^{s+1}, \cdots, \overset{(m)}{x}{}^n) \\ =\varphi(x^{s+1}, \cdots, x^n; \dot{x}^{s+1}, \cdots, \dot{x}^n, x^{s+1}, \cdots, \ddot{x}^n, \cdots, \overset{(m)}{x}{}^{s+1}, \cdots, \overset{(m)}{x}{}^n). \end{aligned} \tag{19.20}$$

Proof. It suffices to treat the last part only. We have

$$b_i^j(x, \dot{x}, \cdots, \overset{(m)}{x}) = \delta_i^j, \qquad (i = s+1, \cdots, n; j = 1, 2, \cdots, n).$$

Completing the matrix B to a square matrix, let the reciprocal matrix be A. Taking (19.2) and (19.3) into account, by virtue of Theorem 1^0, we see that

$$\omega^{s+1} = 0, \cdots, \omega^n = 0 \tag{19.21}$$

is completely integrable. Since

$$|a_i^j(x, \dot{x}, \cdots, \overset{(m)}{x})|_{i,j=s+1,\dots,n} \neq 0,$$

there exists one and only one first integral of (19.21) satisfying (19.20) as we have seen in the N. B. after the proof of the Theorem about (18.36).

20. Fundamental Theorems. We set

$$\xi_j^i(x, \dot{x}, \cdots, \overset{(m)}{x}) = \left(\frac{\partial f^i(x; a(x, \dot{x}, \cdots, \overset{(m)}{x}))}{\partial a^j}\right)_{a=0}, \quad \begin{pmatrix} i = 1, 2, \cdots, n; \\ j = 1, 2, \cdots, r \end{pmatrix} \tag{20.1}$$

and

$$\omega^l(a(x, \dot{x}, \cdots, \overset{(m)}{x}), da(x, \dot{x}, \cdots, \overset{(m)}{x})) = \beta_j^l(a(x, \dot{x}, \cdots, \overset{(m)}{x}), da^j(x, \dot{x}, \cdots, \overset{(m)}{x})), (l = 1, 2, \cdots, r) \tag{20.2}$$

as before.

Further we set

$$\theta^i = dx^i + \omega^j(a(x, \dot{x}, \cdots, \overset{(m)}{x}), da(x, \dot{x}, \cdots, \overset{(m)}{x}))\,\xi_j^i(x, \dot{x}, \cdots, \overset{(m)}{x}), (i = 1, 2, \cdots, n). \tag{20.3}$$

THEOREM 1^0. *The simultaneous extended total differential equations*

$$\theta^1 = 0, \quad \theta^2 = 0, \quad \cdots, \quad \theta^n = 0 \tag{20.4}$$

are completely integrable and

$$f^1(x; a(x, \dot{x}, \cdots, \overset{(m)}{x})), \cdots, f^n(x; a(x, \dot{x}, \cdots, \overset{(m)}{x})) \tag{20.5}$$

are n independent first integrals of (20.4) *such that*

$$f^i(x; 0) = x^i, \qquad (i = 1, 2, \cdots, n),$$

so that

$$(\theta^1, \theta^2, \cdots, \theta^n) = (df^1(x; a(x, \dot{x}, \cdots, \overset{(m)}{x})), \cdots, df^n(x; a(x, \dot{x}, \cdots, \overset{(m)}{x}))) \tag{20.6}$$

for the ideals.

PROOF. We differentiate (16.2):

$$\begin{aligned} f^i(f^1(x; b(x, \dot{x}, \cdots, \overset{(m)}{x})), \cdots, f^n(x; b(x, \dot{x}, \cdots, \overset{(m)}{x})); a^1(x, \dot{x}, \cdots, \overset{(m)}{x}), \cdots, a^r(x, \dot{x}, \cdots, \overset{(m)}{x})) \\ = f^i(x^1, x^2, \cdots, x^n; \varphi^1(a(x, \dot{x}, \cdots, \overset{(m)}{x}); b(x, \dot{x}, \cdots, \overset{(m)}{x})), \cdots, \varphi^r(a(x, \dot{x}, \cdots, \overset{(m)}{x}); \\ b(x, \dot{x}, \cdots, \overset{(m)}{x}))), (i = 1, 2, \cdots, n) \end{aligned} \tag{20.7}$$

with respect to b and set $b=0$. Then it follows that

$$\left(\frac{\partial f^i(f(x;b);a)}{\partial b^l}\right)_{b=0}=\left(\frac{\partial f^i(x;\varphi(a;b))}{\partial b^l}\right)_{b=0},$$

$$\left(\frac{\partial f^i(f(x;b);a)}{\partial x^k}\right)_{b=0}\left(\frac{\partial x^k}{\partial b^l}\right)_{b=0}=\left(\frac{\partial f^i(x;\varphi(a;b))}{\partial a^j}\right)_{b=0}\left(\frac{\partial a^j}{\partial b^l}\right)_{b=0},$$

$$\left(\frac{\partial f^i(f(x;0);a)}{\partial x^k}\right)\left(\frac{\partial f^k(x;0)}{\partial b^l}\right)=\left(\frac{\partial f^i(f(x;b);a)}{\partial a^j}\right)_{b=0}\left(\frac{\partial \varphi^j(a;b)}{\partial b^l}\right)_{b=0},$$

$$\frac{\partial f^i(x;a)}{\partial x^k}\left(\frac{\partial f^k(x;b)}{\partial b^l}\right)_{b=0}=\frac{\partial f^i(x;a)}{\partial a^j}\alpha^j_l(a),$$

$$\frac{\partial f^i(x;a)}{\partial x^k}\xi^k_l(x,\dot{x},\cdots,\overset{(m)}{x})=\frac{\partial f^i(x;a)}{\partial a^j}\alpha^j_l(a). \tag{20.8}$$

From (10.7) and (20.3), we obtain

$$\begin{aligned}df^i(x;a)&=\frac{\partial f^i}{\partial x^k}dx^k+\frac{\partial f^i}{\partial a^j}da^j\\&=\frac{\partial f^i}{\partial x^k}(\theta^k-\omega^l\xi^k_l)+\frac{\partial f^i}{\partial a^j}(\alpha^j_l\omega^l)\\&\equiv\frac{\partial f^i}{\partial x^k}\theta^k-\left\{\frac{\partial f^i}{\partial x^k}\xi^k_l-\frac{\partial f^i}{\partial a^j}\alpha^j_l\right\}\omega^l\\&=\frac{\partial f^i}{\partial x^k}\theta^k\epsilon(\theta^1,\cdots,\theta^k).\end{aligned}$$

Since $df^1(x;a),\cdots,df^n(x;a)$ are linearly independent, the (20.6) holds. Q. E. D.

The converse of the Theorem 1^0 holds as will be seen as follows.

THEOREM 2^0. *When we introduce*

$$\xi^i_j(x,\dot{x},\cdots,\overset{(m)}{x})\in C^2,\qquad (i=1,2,\cdots n;j=1,2,\cdots,r)$$

appropriately for the fundamental extended Pfaffians $\omega^1,\cdots,\omega^r$, of an r-dimensional doubly extended Lie group (germ) G and the simultaneous equations:

$$\theta^1=0,\ \theta^2=0,\cdots,\ \theta^n=0 \tag{20.9}$$

are completely integrable, the n independent first integrals $f^1,\cdots,f^n$ such that

$$f^i(k;0)=x^i,\qquad (i=1,2,\cdots,r) \tag{20.10}$$

determine an n-dimensional doubly extended Lie transformation group (germ) and the given $\xi^j_l(x,\dot{x},\cdots,\overset{(m)}{x})$ satisfy (20.10).

PROOF. If (20.9) be completely integrable, then there exist n first integrals $f^1,f^2,\cdots,f^n$ satisfying (20.10). It suffices to show that these satisfy (20.7). Since

$$\bar{a}^i(x, \dot{x}, \cdots, \overset{(m)}{x}) = \varphi^i(k(x, \dot{x}, \cdots, \overset{(m)}{x}); a(x, \dot{x}, \cdots, \overset{(m)}{x})), \quad (i=1, 2, \cdots, n)$$

satisfies

$$\omega^i(\bar{a}(x, \dot{x}, \cdots, \overset{(m)}{x}), d\bar{a}(x, \dot{x}, \cdots, \overset{(m)}{x})) = \omega^i(a(x, \dot{x}, \cdots, \overset{(m)}{x}), da(x, \dot{x}, \cdots, \overset{(m)}{x})), (i=1, 2, \cdots, r),$$

the functions

$$f^i(x; \bar{a}(x, \dot{x}, \cdots, \overset{(m)}{x})) = f^i(x; \varphi(k(x, \dot{x}, \cdots, \overset{(m)}{x})); a(x, \dot{x}, \cdots, \overset{(m)}{x})), (i=1, 2, \cdots, n), \tag{20.11}$$

satisfy

$$\bar{\theta}^i = dx^i + \bar{\omega}^j \xi^i_j = dx^i + \omega^j \xi^i_j = \theta^i, \qquad (i=1, 2, \cdots, n), \tag{20.12}$$

i. e. (20.11) becomes the first integrals of (20.9).

Since (20.11) implies

$$f^i(x; \varphi(k(x, \dot{x}, \cdots, \overset{(m)}{x}); 0)) = f^i(x; k(x, \dot{x}, \cdots, \overset{(m)}{x})),$$

they take values for $a=0$ with the integrals $f^i(f(x; a(x, \dot{x}, \cdots, \overset{(m)}{x})))$ of (20.9) in common. Hence we must have

$$f^i(x; \varphi(k; x); a(x, \dot{x}, \cdots, \overset{(m)}{x})) = f^i(f(x; k(x, \dot{x}, \cdots, \overset{(m)}{x})); a(x, \dot{x}, \cdots, \overset{(m)}{x})), \quad (i=1, 2, \cdots, n).$$

Since thereby $df^i \epsilon (\theta^1, \cdots, \theta^n)$, pursuing the process of proof for Theorem 1^0 reversedly, we see that (20.8) must hold. If we set $a=0$ in (20.8), then we obtain (20.1), since

$$\alpha^j_i = \delta^j_i, \ \frac{\partial f^i}{\partial x^j} = \delta^i_j, \text{ Q. E. D.}$$

The First Fundamental Theorem of the doubly extended Lie transformation group (germ) below makes a liaison between the property of the doubly extended Lie transformation group (germ) and the fundamental differential operators. In order to prove it, we shall try to replace the above properties with those of the simultaneous extended linear partial differential equations of the first order by virtue of the following Lemma (Theorem 1^0, Art. 19).

LEMMA. *That the simultaneous extended total differential equations*

$$\omega^i = a^i_l(x, \dot{x}, \cdots, \overset{(m)}{x})\, dx^l = 0, \qquad (i=1, 2, \cdots, n) \tag{20.13}$$

are completely integrable is equivalent to that the simultaneous doubly extended linear partial differential equations of the first order

$$X_{r+1}f = 0, \cdots, X_{r+s}f = 0, \qquad (n = r+s) \tag{20.14}$$

are completely integrable. The first integrals are thereby common to (20.13) ***and*** (20.14). ***Thereby we have put***

$$X_i f = b^l_i(x, \dot{x}, \cdots, \overset{(m)}{x}) \frac{\partial f}{\partial x^l}, \qquad (i, l = 1, 2, \cdots, n), \tag{20.15}$$

where $(b^i_l(x, \dot{x}, \cdots, \overset{(m)}{x}))$ *is the inverse of* $(a^l_i(x, \dot{x}, \cdots, \overset{(m)}{x}))$.

THE FIRST FUNDAMENTAL THEOREM. *In a doubly extended Lie transformation group* (*germ*) G *having* G *as doubly extended parameter group* (*germ*), *the functions*

$$f^k(x\,;a(x, \dot{x}, \cdots, \overset{(m)}{x})), \qquad (k=1, 2, \cdots, n)$$

are n *independent solutions of the completely integrable simultaneous doubly extended linear partial differential equations* (cf. (20.1))

$$\frac{\partial f}{\partial a^l} = \xi^k_j(x, \dot{x}, \cdots, \overset{(m)}{x})\,\beta^j_l(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\,\frac{\partial f}{\partial x^k}, \quad (k=1, 2, \cdots, r), \tag{20.16}$$

such that

$$x^k = f^k(x\,;0). \tag{20.17}$$

Conversely, when an r*-dimensional doubly extended Lie group* (*germ*) *is given, the* (20.16) *are completely integrable for certain*

$$\xi^i_j(x, \dot{x}, \cdots, \overset{(m)}{x}) \in C^2, \qquad (i=1, 2, \cdots, n\,; j=1, 2, \cdots, r),$$

the solutions

$$f^1(x\,;a(x, \dot{x}, \cdots, \overset{(m)}{x})), f^2(x\,;a(x, \dot{x}, \cdots, \overset{(m)}{x})), \cdots, f^n(x\,;a(x, \dot{x}, \cdots, \overset{(m)}{x}))$$

satisfying (20.17) *determine a doubly extended Lie transformation group* (*germ*) *having* G *as doubly extended parameter group* (*germ*).

PROOF. We consider two r-dimensional square matrices A and B defined by

$$A=(\alpha^i_l(a(x, \dot{x}, \cdots, \overset{(m)}{x}))), B=(\beta^i_k(a(x, \dot{x}, \cdots, \overset{(m)}{x}))), AB=BA=(\delta^i_k)$$

having defined $\alpha^i_k(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ and $\beta^i_k(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ by (10.3) and (10.5). Then

$$\left\{\begin{array}{l} \theta^1 = dx^1 + \{\beta^j_k(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\,\xi^1_j(x, \dot{x}, \cdots, \overset{(m)}{x})\}\,da^k(x, \dot{x}, \cdots, \overset{(m)}{x}), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \theta^n = dx^n + \{\beta^j_k(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\,\xi^n_j(x, \dot{x}, \cdots, \overset{(m)}{x})\}\,da^k(x, \dot{x}, \cdots, \overset{(m)}{x}), \end{array}\right.$$

and

$$\left\{\begin{array}{l} \theta^{n+1} = \beta^1_k(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\,da^k(x, \dot{x}, \cdots, \overset{(m)}{x}), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \theta^{n+r} = \beta^r_k(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\,da^k(x, \dot{x}, \cdots, \overset{(m)}{x}), \end{array}\right.$$

are linearly independent and the determinant D of their coefficients may be expressed as follows:

$$D = \begin{pmatrix} E_n & \Xi' B \\ 0 & B \end{pmatrix}.$$

$$E_n = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

is the unit determinant of the n-th order and Ξ' the determinant obtained from $|\xi^l_j(x, \dot{x}, \cdots, \overset{(m)}{x})|$ by interchanging the rows with columns. The reciprocal determinant of D is
$$D^{-1} = \begin{pmatrix} E_n & \Xi' \\ 0 & A \end{pmatrix}.$$

We set

$$A_l f = \left(\alpha^j_l(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\frac{\partial}{\partial a^j}\right)f, \qquad (l=1, 2, \cdots, r), \tag{20.18}$$

$$X_j f = \left(\xi^k_j(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial}{\partial x^k}\right)f, \qquad (j=1, 2, \cdots, r). \tag{20.19}$$

By the above Lemma, when the simultaneous extended total differential equations

$$\theta^1 = 0, \theta^2 = 0, \cdots, \theta^n = 0 \tag{20.20}$$

are completely integrable, the simultaneous doubly extended linear partial differential equations

$$\overline{X}_1 f = 0, \cdots, \overline{X}_r f = 0, \tag{20.21}$$

where

$$\begin{aligned} \overline{X}_j &= -\xi^k_j(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial}{\partial x^k} + \alpha^l_j(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\frac{\partial}{\partial a^l}, \qquad (k=1, 2, \cdots, n) \\ &= -X_j + A_j, \qquad (j=1, 2, \cdots, r), \end{aligned} \tag{20.22}$$

are also completely integrable, the first integrals of (20.9)=(20.20) coincide with the solution of (20.21).

Now (20.21) and the simultaneous doubly extended linear partial differential equations

$$Y_1 f = 0, \cdots, Y_r f = 0, \tag{20.23}$$

where

$$Y_j = \beta^l_j(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\overline{X}_l = \frac{\partial}{\partial a^j} - \xi^t_l(x, \dot{x}, \cdots, \overset{(m)}{x})\beta^l_j(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\frac{\partial}{\partial x^t}, (j, l=1, 2, \cdots, r), \tag{20.24}$$

are equivalent.

Hence the Theorems 1^0 and 2^0 may be restated in the form of our First Fundamental Theorem.

tarnsformations with appropriate functions of coordinates as coefficients, the concerning invariants being retained. We will refer to such transformations as ***extended orthogonal transformations.*** They constitute a group, an ***extended orthogonal transformation group.***

We utilize the ω^l appearing in (14.4) as the ω^l in (2.1).

It is readily seen that

$$\omega^l\omega^l=\omega^l_\mu\omega^l_\nu dx^\mu dx^\nu, \tag{14.6}$$

so that

$$g_{\mu\nu}=\omega^l_\mu\omega^l_\nu, \tag{14.7}$$

$$|g_{\mu\nu}|=|\omega^l_\mu|\cdot|\omega^l_\nu|=|\omega^l_\mu|^2\neq 0. \tag{14.8}$$

$$g^{\mu\nu}=[\text{cofactor of } g_{\mu\nu} \text{ in } |g_{\mu\nu}|]\,/\,|g_{\mu\nu}|$$

$$=\frac{\text{cofactor of } \omega^l_\mu \text{ in } |\omega^l_\mu|}{|\omega^l_\mu|}\cdot\frac{\text{cofactor of } \omega^l_\nu \text{ in } |\omega^l_\nu|}{|\omega^l_\nu|},$$

$$g^{\mu\nu}=\Omega^\mu_l\Omega^\nu_l. \tag{14.9}$$

Hence

$$\left\{{\lambda \atop \mu\nu}\right\}=\tfrac{1}{2}g^{\lambda\sigma}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu}+\frac{\partial g_{\sigma\nu}}{dx^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right)$$

$$=\tfrac{1}{2}\Omega^\lambda_h\Omega^\sigma_h\left(\frac{\partial\omega^l_\mu}{\partial x^\nu}\omega^l_\sigma+\omega^l_\mu\frac{\partial\omega^l_\sigma}{\partial x^\nu}+\frac{\partial\omega^l_\sigma}{\partial x^\mu}\omega^l_\nu+\omega^l_\sigma\frac{\partial\omega^l_\nu}{\partial x^\mu}-\frac{\partial\omega^l_\mu}{\partial x^\sigma}\omega^l_\nu-\omega^l_\mu\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right),$$

$$\left\{{\lambda \atop \mu\nu}\right\}=\tfrac{1}{2}(\Lambda^\lambda_{\mu\nu}+\Lambda^\lambda_{\nu\mu})+\tfrac{1}{2}\Omega^\sigma_h\left[\omega^l_\mu\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x_\sigma}\right)+\omega^l_\nu\left(\frac{\partial\omega^l_\sigma}{\partial x^\mu}-\frac{\partial\omega^l_\mu}{\partial x^\sigma}\right)\right]. \tag{14.10}$$

According to (8.14), we set

$$\left\{{\lambda \atop \mu\nu}\right\}=\tfrac{1}{2}(\Lambda^\lambda_{\mu\nu}+\Lambda^\lambda_{\nu\mu})+\delta^\lambda_\mu\psi_\nu+\delta^\lambda_\nu\psi_\mu, \tag{14.11}$$

so that

$$\delta^\lambda_\mu\psi_\nu+\delta^\lambda_\nu\psi_\mu=\Omega^\lambda_h\Omega^\sigma_h\left[\omega^l_\mu\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right)+\omega^l_\nu\left(\frac{\partial\omega^l_\sigma}{\partial x^\mu}-\frac{\partial\omega^l_\mu}{\partial x^\sigma}\right)\right]. \tag{14.12}$$

Contracting $\mu\rightarrow\lambda$:

$$(n+1)\,\psi_\nu=\Omega^\sigma_h\Omega^\lambda_h\left[\omega^l_\lambda\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right)+\omega^l_\nu\left(\frac{\partial\omega^l_\sigma}{\partial x^\lambda}-\frac{\partial\omega^l_\lambda}{\partial x^\sigma}\right)\right]$$

$$=\Omega^\sigma_l\left(\frac{\partial\omega^l_\sigma}{\partial x^\nu}-\frac{\partial\omega^l_\nu}{\partial x^\sigma}\right)=\Lambda^\sigma_{\sigma\nu}-\Lambda^\sigma_{\nu\sigma},$$

$$(n+1)\,\psi_\nu=\Lambda^\sigma_{\sigma\nu}-\Lambda^\sigma_{\nu\sigma}, \tag{14.13}$$

what proves the unexpected result:

$$\frac{d^2x^\lambda}{ds^2}+\left\{{\lambda \atop \mu\nu}\right\}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}=\frac{d^2x^\lambda}{ds^2}+\Lambda^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}. \tag{14.14}$$

Second proof for (14.14). It suffices to prove that *linear partial differential equations*

$$\alpha_l^j \frac{\partial f}{\partial a^j} = \xi_l^i \frac{\partial f}{\partial x^i}, \qquad (j, l=1, 2, \cdots, r; i, k=1, 2, \cdots, n), \tag{20.29}$$

such that

$$x^k = f^k(x; 0). \tag{20.30}$$

Conversely, when an r-dimensionl doubly extended Lie group (germ) G is given, (20.29) *are completely integrable for certain*

$$\xi_j^i(x, \dot{x}, \cdots, \overset{(m)}{x}) \in C^2, \qquad (i=1, 2, \cdots, n; j=1, 2, \cdots, r),$$

their solutions

$$f^1(x; a(x, \dot{x}, \cdots, \overset{(m)}{x})), \cdots, f^n(x; a(x, \dot{x}, \cdots, \overset{(m)}{x}))$$

satisfying (20.26)–(20.30), *determine a doubly extended Lie transformation group (germ).*

PROOF. Now it suffices to show that (20.1)=(20.29). For it, multiplying (20.16):

$$\frac{\partial f}{\partial a^l} = \xi_j^i(x, \dot{x}, \cdots, \overset{(m)}{x})\, \beta_l^j(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\frac{\partial f}{\partial x^i} \tag{20.31}$$

with $\alpha_h^l(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$, we see that

$$\alpha_h^l(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\frac{\partial f}{\partial a^l} = \xi_j^i(x, \dot{x}, \cdots, \overset{(m)}{x})\, \alpha_h^l(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\, \beta_l^j(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\frac{\partial f}{\partial x^i},$$

$$= \xi_j^i(x, \dot{x}, \cdots, \overset{(m)}{x})\, \delta_h^j \frac{\partial f}{\partial x^i} = \xi_h^i \frac{\partial f}{\partial x^i}, \qquad (i=1, 2, \cdots, n; j, h, k=1, 2, \cdots, r)$$

by (10.5) and consequently, multiplying the last relation with $\beta_l^h(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$, we return to (20.31).

THE SECOND FUNDAMENTAL THEOREM. (A Double Extension of the Lie's Second Fundamental Theorem). *When a given r-dimensionl doubly extended Lie group (germ) G as a douby extended parameter group (germ) has the structure constants* C_{ij}^k, $(i, j, k=1, 2, \cdots, r)$, *the necessary and sufficient condition for that* (20.16) *may be completely integrable, is that the relations*

$$(X_j, X_l) = C_{jl}^h X_h, \qquad (h, j, l=1, 2, \cdots, r) \tag{20.32}$$

hotd for the fundamental operators

$$X_j = \xi_j^i(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial}{\partial x^i}, \qquad (i=1, 2, \cdots, n; j=1, 2, \cdots, r). \tag{20.33}$$

Hereby (X_i, X_j) *is the Jacobi's parentheses.*

PROOF. We have seen that the (20.16)=(20.23) are completely integrable is equivalent to that (20.21) is completely integrable. Now it is known that the necessary and sufficient condition for that (20.21) is completely integrable is that $\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_r$ form a *complete system* i.e. that $\bar{X}_1, \cdots, \bar{X}_r$ satisfy

$$(20.34)\qquad (\bar{X}_j, \bar{X}_l) = -C^h_{jl}(x; a(x, \dot{x}, \cdots, \overset{(m)}{x}))\,\bar{X}_h, \qquad (j, l, h = 1, 2, \cdots, r).$$

Now (20.22):

$$(20.35)\qquad \bar{X}_h = -X_h + A_h$$

gives

$$(20.36)\qquad (\bar{X}_j, \bar{X}_l) \equiv (X_j, X_l) + (A_j, A_l)$$

and after setting

$$(20.37)\qquad d\omega^l = \frac{1}{2} C^h_{jl}\,\omega^j \wedge \omega^l, \qquad \omega^l = \beta^l_j(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\,da^j(x, \dot{x}, \cdots \overset{(m)}{x}),$$

$$(20.38)\qquad C^h_{jl} = -C^h_{lj},$$

apply the operator d to

$$df = \omega^l(A_l f):$$

$$\begin{aligned} 0 = d(df) &= (A_l f)\,d\omega^l + d(A_l f) \wedge \omega^l \\ &= \frac{1}{2} C^h_{jl}(A_h f)\,\omega^j \wedge \omega^l + A_j(A_h f)\,\omega^j \wedge \omega^h \\ &= \sum_{j<l} \{C^h_{jl}(A_h f) + (A_j, A_l) f\}\,\omega^j \wedge \omega^l. \end{aligned}$$

Thus we obtain

$$(20.39)\qquad (A_j, A_l) = -C^h_{jl} A_h. \qquad (j, l = 1, 2, \cdots, r).$$

Owing to (20.35), (20.36) and (20.39), the (20.34) becomes

$$(X_j, X_l) - C^h_{jl} A_h = -C^h_{jl}(x; a)(-X_h + A_h),$$

so that

$$(20.40)\qquad C^h_{jl}(x; a) = C^h_{jl}$$

and thus finally we have

$$(20.41)\qquad (X_j, X_l) = C^h_{jl} X_h, \qquad (l, j, h = 1, 2, \cdots, r).$$

THE THIRD FUNDAMENTAL THEOREM. *When r linearly independent differential operators*

$$(20.42)\qquad X_j = \xi^i_j(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial}{\partial x^i}, \qquad (i = 1, 2, \cdots, n; j = 1, 2, \cdots, r), (\xi^i_j(x, \dot{x}, \cdots \overset{(m)}{x}) \in C^2)$$

are given, the necessary and sufficient condition for that they are the fundamental differential operators for a doubly extended Lie transformation group (germ), is *that the relations*

$$(X_j, X_l) = C^h_{jl} X_h, \qquad (h, j, l = 1, 2, \cdots, r) \tag{20.43}$$

hold for certain constants

$$C^k_{jl}, \qquad (j, k, l = 1, 2, \cdots, r). \tag{20.44}$$

PROOF. The necessity is implied in the last theorem. It is known that when $\xi^i_j(x, \dot{x}, \cdots, \overset{(m)}{x}) \in C^v, (v \geqq 2)$, the Jacobi's parentheses satisfy the identities

$$(X_i, X_j) = -(X_j, X_i), \tag{20.45}$$

$$((X_i, X_j), X_k) + ((X_j, X_k), X_i) + ((X_k, X_i), X_j) = 0. \tag{20.46}$$

For the complete system accompanied by (20.43), the relations (11.4):

$$C^k_{ij} = -C^k_{ji}, \tag{20.47}$$

$$C^h_{ij} C^l_{hk} + C^h_{jk} C^l_{hi} + C^h_{ki} C^l_{hj} = 0, \qquad (i, j, k, l = 1, 2, \cdots, n) \tag{20.48}$$

hold. Hence, by Theorem 2⁰ of Art. 14, there exists an r-dimensional doubly Extended Lie group (germ) G having C^k_{ij} as structure constants. If we adopt this G, we are led to the last Theorem for sufficiency.

THE FOURTH FUNDAMENTAL THEOREM. (A Double Extension of S. Lie's Third Fundamental Theorem). *The necessary and sufficient condition for that the r^3 given constants C^h_{jl}, $(h, j, l = 1, 2, \cdots, r)$ may establish the relations*

$$(X_i, X_j) = C^k_{ij} X_k$$

for the fundamental differential operators $X_1, \cdots, X_r$ of a doubly extended Lie transformation group (germ), is that they satisfy the following two conditions (20.47), (20.48):

$$C^k_{ij} = -C^k_{ji}, \tag{20.49}$$

$$C^h_{ij} C^l_{hk} + C^h_{jk} C^l_{hi} + C^h_{ki} C^l_{hj} = 0, \qquad (i, j, k, l = 1, 2, \cdots, r). \tag{20.50}$$

21. The Lie Ring composed of the Fundamental Differential Operators. We have represented the (doubly extended) parameter group (germ) G by the doubly extended transformation group (germ) T, so that the abstract (doubly extended) Lie ring $\boldsymbol{R}$ has become homeomorphic to the doubly extended Lie ring $\boldsymbol{S}$ consisting of the totality

$$X = \lambda_i X_i, \qquad (\lambda_i = \text{constants}).$$

Thus we obtain the following homeomorphic correspondence:

Abstract (doubly extended) Lie ring $\boldsymbol{R}$	Doubly Extended Lie ring $\boldsymbol{S}$
(Doubly extended) parameter group (germ)	Doubly extended transformation group (germ)
G	T
Basis	Fundamental differential operators
$E_1, E_2, \cdots, E_r$	$X_1, X_2, \cdots, X_r$
$A=\lambda_i E_i \in \boldsymbol{R}$	$X=\lambda_i X_i \in \boldsymbol{S}$
$B=\mu_i E_i \in \boldsymbol{R}$	$Y=\mu_i X_i \in \boldsymbol{S}$
$\alpha A+\beta B$	$\alpha X+\beta Y$
(A, B)	(X, Y)

Concerning this correspondence, we get the following theorem.

THEOREM 1^0. *In order that a doubly extended transformation group (germ) may be a faithful representation of the doubly extended parameter group (germ) G, is that the*

doubly extended Lie ring composed of the fundamental differential operators and the abstract (doubly extended) Lie ring $\boldsymbol{R}$ may be isomorphic to each other.	*correspondence of the two sides of the above table is one-to-one.*

PROOF. We utilize the canonical parameter t of the doubly extended Lie group (germ) G. Taking a point $(a^1, \cdots, a^r)$ in the vicinity of the origin (unit element), we set

$$f^i(x^1, \cdots, x^n; a^1 t, \cdots, a^r t)=f^i(x^1, \cdots, x^n; t), \qquad (i=1, 2, \cdots, n).$$

Then by (20.16) and (14.11), we have

$$\text{(21.1)} \qquad \frac{\partial f^i}{\partial t}=\frac{\partial f^i}{\partial a^l} a^l=(\beta_l^k(a)\, a^l)\, \xi_j^k(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial f^i}{\partial x^j}$$

$$=a^k\left(\xi_k^j(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial}{\partial x^j}\right) f^j=(a^k X_k) f^i.$$

Hence, in the case that the correspondence between the two sides of the above table is not one-to-one, we have

$$\text{(21.2)} \qquad X=\lambda^1 X_1+\cdots+\lambda^r X_r=0,$$

where $\lambda^1, \cdots, \lambda^r$ are sufficiently small values, which are not zero at the same time.

If we take them for $(a^1, \cdots, a^r)=(\lambda^1, \cdots, \lambda^r)$, from (21.1), we obtain

$$\frac{\partial f^i}{\partial t}=0, \qquad (i=1, 2, \cdots, n)$$

i. e.

$$f^i(x^1, \cdots, x^r; a^1t, \cdots, a^rt)=x^i, \qquad (i=1, 2, \cdots, n).$$

Thus G and T do not correspond one-to-one.

In the case, where (21.2) holds when and only when $\lambda^1=\lambda^2=\cdots=\lambda^r$, take a hypersphere with sufficiently small radius ε and with the origin as center. Then, since $a^k X_k \neq 0$ in (21.1) for each point $(a^1, \cdots, a^r)$ on it, we get

(21.3) $$f^i(x^1, \cdots, x^n; a^1t, \cdots, a^nt) \neq x^i, (i=1, 2, \cdots, r; |t| < \delta(a^1, \cdots, a^r)).$$

Since $\delta(a^1, \cdots, a^r)$ is evidently a continuous function of $(a^1, \cdots, a^r)$, for the least value δ_0 of it, we must hnve

(21.4) $$T_a \neq T_0, \qquad (a^ia^i<\delta_0).$$

Since T makes a doubly extended group (germ) from (21.4), we can conclude that G and T correspond one-to-one in a suffieiently small vicinity of the origin. Q. E. D.

Let us consider now the case, where $\boldsymbol{R}$ and $\boldsymbol{S}$ are not isomorphic to each other generally. Let $s(\leqq r)$ out of the r fundamental differential operators $X_1, \cdots, X_r$ be linearly independent with constant coefficients. Let

(21.5) $$Y_i=h_i^j X_j, \qquad (i=1, 2, \cdots, s)$$

be linearly independent and suppose that in terms of them we have

(21.6) $$X_j=g_j^i X_i, \qquad (j=1, 2, \cdots, r).$$

Since $Y_1, \cdots, Y_s$ are linearly independent, we have

(21.7) $$h_i^j g_j^k=\delta_i^k, \qquad (i, k=1, 2, \cdots, s).$$

Utilizing

$$(X_k, X_l)=C_{kl}^m X_m,$$

we obtain

$$(Y_i, Y_j)=h_i^k h_j^l (X_k, X_l)=h_i^k h_j^l C_{kl}^m X_m=h_i^k h_j^l C_{kl}^m g_m^p Y_p,$$

i. e.

(21.8) $$(Y_i, Y_j)=\gamma_{ij}^p Y_p, \qquad (i, j=1, 2, \cdots, s),$$

where

(21.9) $$r_{ij}^p=h_i^k h_j^l g_m^p C_{kl}^m.$$

Further we set

(21.10) $$\tau^i(a, da)=g_j^i \omega^j(a, da), \qquad (i=1, 2, \cdots, s).$$

From (14.4) and (20.41), it results that

$$(21.11) \qquad d\omega^m (X_m f) - \frac{1}{2} \omega^p \wedge \omega^q (X_p, X_q) f = 0,$$

which becomes

$$(21.12) \qquad d\tau^i (Y_i f) - \frac{1}{2} \tau^j \wedge \tau^k (Y_j, Y_k) f = 0$$

by virtue of (21.6) and (21.10).

Utilizing (21.8), thence we obtain

$$(21.13) \qquad (d\tau^i - \frac{1}{2} \gamma^i_{jk} \tau^j \wedge \tau^k)(Y_i f) = 0.$$

Now, since $Y_1, \cdots, Y_r$ are linearly independent, their coefficients must vanish severally, i. e.

$$(21.14) \qquad d\tau^i (a, da) = \frac{1}{2} \gamma^i_{jk} \tau^j \wedge \tau^k, \qquad (i = 1, 2, \cdots, s).$$

Consequently the simultaneous extended total differential equations

$$(21.15) \qquad \tau^1 (a, da) = 0, \cdots, \tau^j (a, da) = 0$$

are completely integrable. Since furthcr $Y_1, \cdots, Y_r$ are linearly independent, the rank of (g^i_j) is s. Hence $\tau^1, \cdots, \tau^s$ are also linearly independent by virtue of (21.10). Thus there exist s independent first integrals of (21.15), which are zero at the origin. Let them be

$$b^1 (a^1, \cdots, a^r), \cdots, b^s (a^1, \cdots, a^r) \in C^2,$$

where

$$b^i (0, \cdots, 0) = 0, \qquad (i = 1, 2, \cdots, s).$$

Taking $(r-s)$ adequate functions

$$b^{s+1} (a^1, \cdots, a^r), \cdots, b^r (a^1, \cdots, a^r) \in C^2,$$

where

$$b^j (0, \cdots, 0) = 0, \qquad (j = s+1, s+2, \cdots, r)$$

in addition, we have one-to-one correspondence

$$(a^1, \cdots, a^r) \to (b^1, \cdots, b^r)$$

in the vicinity of the origin. Noticing this transformation of the variables, we write

$$\tau^i (a, da) = \pi^i (b, db), \qquad (i = 1, 2, \cdots, s).$$

Since $b^1, \cdots, b^r$ are s independent first integrals of (21.15), the relation

$$(\pi^1, \cdots, \pi^s) = (db^1, \cdots, db^s)$$

holds, so that we may write

$$\pi^i (b, db) = \psi^i_j (b^1, \cdots, b^r) \, db^j, \qquad (i = 1, 2, \cdots, s),$$

Now, by (21.14), we must have

$$d\pi^i(b, db) = \frac{\partial\psi^i_l(b^1, \cdots, b^r)}{\partial b^h} db^h \wedge db^l, \qquad (h, l = 1, 2, \cdots, r)$$

$$= \frac{1}{2}\gamma^i_{jk}\pi^j \wedge \pi^k \qquad (j, k = 1, 2, \cdots, s)$$

$$= \frac{1}{2}\gamma^i_{jk}\psi^j_h(b^1, \cdots, b^r)\, db^h \wedge \psi^k_l(b^1, \cdots, b^r)\, db^l$$

$$= \frac{1}{2}\gamma^i_{jk}\psi^j_h(b^1, \cdots, b^r)\,\psi^k_l(b^1, \cdots, b^r)\, db^h \wedge db^l,$$

so that

$$\frac{\partial\psi^i_j}{\partial b^k} = 0, \qquad (i, j = 1, 2, \cdots, s\,; k = s+1, \cdots, r).$$

Hence we have

$$\psi^i_j = \psi^i_j(b^1, \cdots, b^s),$$

and consequently π^i must be expressible in terms of $b^1, \cdots, b^s, db^1, \cdots, db^s$ only.

We denote the s-dimensional (doubly extended) Lie group (germ) defined uniquely by

(21.16) $$d\pi^i = \frac{1}{2}\pi^j \wedge \pi^k, \qquad (i = 1, 2, \cdots, s)$$

in the s-dimensional neighborhood of the origin of $(b^1, \cdots, b^r)$ by $\bar{G}$. Now by Theorem 1° of Art. 20, the $f^i(x; a)$ are the first integrals of

$$dx^i + \omega^j(a, da)\,\xi^i_j(x, \dot{x}, \cdots, \overset{(m)}{x}) = 0, \qquad (i = 1, 2, \cdots, n)$$

such that $f^i(x; 0) = x^i$. Taking the last differential equation together with (21.5), (21.6) and (21.10), we can deduce

(21.17) $$dx^i + \pi^j(b, db)\,\eta^i_j(x, \dot{x}, \cdots, \overset{(m)}{x}) = 0, \qquad (i = 1, 2, \cdots, n),$$

(21.18) $$Y_i = h^j_i X_j = \eta^j_i(x, \dot{x}, \cdots, \overset{(m)}{x})\frac{\partial}{\partial x^j}$$

for

$$\eta^j_i(x, \dot{x}, \cdots, \overset{(m)}{x}) = k^j_k\,\xi^k_i(x, \dot{x}, \cdots, \overset{(m)}{x}).$$

Hence (21.17) are also completely integrable and its first integral is expressible as

(21.19) $$f^i(x^1, \cdots, x^n; a^1, \cdots, a^r) = g^i(x^1, \cdots, x^n; b^1(a), \cdots, b^s(a)), \; (i = 1, 2, \cdots, n).$$

Thus we obtain the following theorem.

THEOREM 2°. *When the rank of the fundamental (doubly extended) Lie ring composed of the fundamental differetial operators* $X_1, \cdots, X_r$ *is* $s\,(\leqq r)$, *there exists an s-dimensional (doubly extended) Lie group (germ)* G *as doubly extended parameter group (germ) having linearly independent* (21.18)*as fundamental differential operators, for which we have* (21.19). *In this case, the given doubly extended transformation group (germ) becomes a faithful representation of* $\bar{G}$.

§ 6. The Relation between the Non-Locally Line-Elemented II-Geodesic Curves in the Base Manifold M and Those in the Doubly Extended Lie Transformation Group Manifold G.

22. The Relation between the Non-Locally Line-Elemented II-Geodesic Curves in the Base Manifold M and Those in the Doubly Extended Lie Transformation Group Manifold G. We must not overlook that *we are considering both the non-locally line-elemented II-geodesic curves in the*

base manifold M.	*doubly extended Lie transformation group manifold G.*

Now we will seek for how the non-locally line-elemented II-geodesic curves in the

base manifold M	doubly extended Lie transformation group manifold G

behave in the

doubly extended Lie transformation group manifold G.	base manifold M.

If we multiply (20.3) with, $\bar{\xi}^l_i(x, \dot{x} \cdots, \overset{(m)}{x})$ defined by $\bar{\xi}^l_i \xi^i_k = \delta^l_k$ [(25.1)], then

$$\bar{\xi}^l_i(x, \dot{x}, \cdots, \overset{(m)}{x})\, \theta^i = \bar{\xi}^l_i(x, \dot{x}, \cdots, \overset{(m)}{x})\, dx^i + \beta^l_j(a(x, \dot{x}, \cdots \overset{(m)}{x}))\, da^j(x, \dot{x}, \cdots, \overset{(m)}{x}),$$

so that the differential equations (20.20) $\theta^1 = 0, \cdots, \theta^n = 0$ give

$$\bar{\xi}^l_i(x, \dot{x}, \cdots, \overset{(m)}{x})\, dx^i = -\beta^l_j(a(x, \dot{x}, \cdots \overset{(m)}{x}))\, da^j(x, \dot{x}, \cdots, \overset{(m)}{x}) \tag{22.1}$$

i.e. (25.13)

$$d\xi^j = -da^j = e^j\, dt \tag{22.2}$$

by (4.12). Thus to *the non-locally line-elemented II-geodesic curves $d\xi^j = e^j dt$ in the base manifold M, there correspond the non-locally line-elemented* II-*geodesic curves $da^j = -e^j dt$ in the doubly extended Lie transformation group manifold G.*

§ 7. Two Systems of Equipollences of Vectors in the Doubly Extended Lie Transformation Group Space.

23. Two Systems of Equipollences of Vectors in the Doubly Extended Lie Transformation Group Space.

(i) Consider a doubly extended Lie transformation group G with r doubly extended parameters $a^1(x, \dot{x}, \cdots, \overset{(m)}{x})$, $a^2(x, \dot{x}, \cdots, \overset{(m)}{x})$, $\cdots$, $a^r(x, \dot{x}, \cdots \overset{(m)}{x})$. The coordinates $x=(x^1, \cdots, x^n)$, which undergo the doubly extended Lie transformations $a(x, \dot{x}, \cdots, \overset{(m)}{x})$, will play quite an accessory rôle in the following lines. We will extend the E. Cartan's theory ([19]) of two kinds of parallelisms of the vectors in the group space to the case of our doubly extended Lie transformation group space $\mathfrak{E}$.

Let us denote the elements of G corresponding to $a(x, \dot{x}, \cdots, \overset{(m)}{x})$ as an operator by T_a and the product of T_a and T_b by T_bT_a, and the inverse of T_a by T_a^{-1} so that $(T_b T_a)^{-1}=T_a^{-1} T_b^{-1}$.

We will call a pair of points $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ and $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$ taken in this order a *vector* $\overrightarrow{ab}$ of $\mathfrak{E}$ and when $a(x, \dot{x}, \cdots, \overset{(m)}{x})=b(x, \dot{x}, \cdots \overset{(m)}{x})$, we will call the vector a ***null vector.***

(ii) DEFINITION. We will say that two vectors $\overrightarrow{ab}$ and $\overrightarrow{a'b'}$ are *equipollent of the*

first	*second*

kind, when

(23.1)	$T_bT_a^{-1}=T_{b'}T_{a'}^{-1}.$	$T_a^{-1}T_b=T_{a'}^{-1}T_{b'}.$

Considering the inverses, we may replace (23.1) by

$T_aT_b^{-1}=T_{a'}T_{b'}^{-1}.$	$T_b^{-1}T_a=T_{b'}^{-1}T_{a'}.$

The equipollences have the following properties.

1°. *Every vector, which is equipollent to a null vector is null.*

2°. *Every vector is equipollent to itself.*

3°. *If a vector is equipollent to a second vector, then the second vector is equipollent to the first.*

4°. *If two vectors are equipollent, then their inverses are also equipollent.*

5°. *Every point of the group space $\mathfrak{E}$ may be considered as the origin of one and only one vector, which is equipollent to a given vector.*

6°. *Two vectors, which are equipollent to a third vector, are equipollent to each other.*

7°. *If $\overrightarrow{ab}$ is equipollent to $\overrightarrow{a'b'}$ and $\overrightarrow{bc}$ equipollent to $\overrightarrow{b'c'}$, then the vector $\overrightarrow{ac}$ is equipollent to $\overrightarrow{a'c'}$.*

The 7° may be proved as follows. From

$$T_bT_a^{-1}=T_{b'}T_{a'}^{-1}, \quad T_cT_b^{-1}=T_{c'}T_{b'}^{-1},$$

we obtain

$$(T_c T_b^{-1})(T_b T_a^{-1}) = (T_{c'} T_{b'}^{-1})(T_{b'} T_{a'}^{-1})$$

i. e.

$$T_c T_a^{-1} = T_{c'} T_{a'}^{-1}.$$

(iii) THEOREM. *When $\overrightarrow{ab}$ is equipollent to the first kind to $a'b'$, the vector $\overrightarrow{aa'}$ is equipollent of the second kind to $\overrightarrow{bb'}$ and vice versa.*

PROOF. From (23.1), we have

$$T_b^{-1} T_b T_a^{-1} T_{a'} = T_b^{-1} T_{b'} T_{a'}^{-1} T_{a'},$$

i. e.

$$T_a^{-1} T_{a'} = T_b^{-1} T_{b'}, \tag{23.2}$$

which is of the form (23.1) for $\overrightarrow{aa'}$ and $\overrightarrow{bb'}$.

THEOREM. *When the first equipollence plays property* 7^0, *the second equipollence plays the property* 6^0 *and vice versa.*

PROOF. Suppose that an equipollence satisfying the properties 1^0-6^0 is defined in an r-dimensional space in a certain way. Thence we deduce an equipollence of the second kind saying that $\overrightarrow{aa'}$ is equipollent of the second kind to $\overrightarrow{bb'}$ and $\overrightarrow{ab}$ is equipollent of the first kind to $\overrightarrow{a'b'}$. It is easy to see that the properties 1^0-5^0 are verified for this equipollence of the second kind. But as for the property 6^0, it is not necessarily the case. Suppose $\overrightarrow{aa'}$ and $\overrightarrow{bb'}$ are equipollent of the second kind to $\overrightarrow{cc'}$. This means that $\overrightarrow{ac}$ is equipollent of the first kind to $\overrightarrow{a'c'}$ and that $\overrightarrow{bc}$ is equipollent of the first kind to $\overrightarrow{b'c'}$. In order that $\overrightarrow{aa'}$ and $\overrightarrow{bb'}$ may be equipollent of the second kind of each other, it is necessary and sufficient that $\overrightarrow{ab}$ is equipollent of the first kind to $\overrightarrow{a'b'}$; in other words, the equipollence of the second kind will verify 6^0, when the equipollence of the first kind verifies 7^0 and vice versa.

(iv) The two kinds of equipollences are in close relation to ***two groups*** of doubly extended parameters of G. Indeed, let us consider the geometrical operation consisting of laying through a variable point $(\xi(x, \dot{x}, \cdots, \overset{(m)}{x}))$ a vector $\overrightarrow{\xi\xi'}$, which is equipollent of the first kind to a fixed vector. Let $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ be the extremity of the vector, which is equipollent to the fixed vector and is drawn through the origin of $\mathfrak{E}$. The operation considered is expressed analytically by

$$T_{\xi'} T_a^{-1} = T_\xi$$

or by

$$T_{\xi'} = T_a T_\xi. \tag{23.3}$$

This is thus analytically identical to one of the transformations of the first group of doubly extended parameters [9].

Similarly the operation consisting in drawing a vector $\overrightarrow{\xi\xi'}$ through a variable point $(\xi(x, \dot{x}, \cdots, \overset{(m)}{x}))$, which is equipollent of the second kind to a fixed vector $\overrightarrow{Oa}$, may be expressed analytically by

$$T_{\xi'} = T_{\xi} T_a. \tag{23.4}$$

This is thus analytically identical to one of the transformations of the second group of doubly extended parameters [10].

(v) The property explained by the Theorem under (ii) is a geometrical interpretation of the fact that the doubly extended transformations of the two groups of extended parameters are interchanged among themselves.

The properties 1^0-7^0 are the characteristic properties of the equipollence attached to the groups. We shall prove that *when we have defined an equipollence of vectors in doubly extended group space $\mathfrak{E}$ playing the seven properties* 1^0-7^0, *the space $\mathfrak{E}$ can be considered as a space of group, the equipollence defined in $\mathfrak{E}$ being the first equipollence attached to doubly extended group.*

For this purpose, let us take an origin (O) in the space $\mathfrak{E}$ quite arbitrarily. Let $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ be any point of $\mathfrak{E}$. Consider an operator S_a, by which we pass from a variable point $(\xi(x, \dot{x}, \cdots, \overset{(m)}{x}))$ to the extremity $(\xi'(x, \dot{x}, \cdots, \overset{(m)}{x}))$ of the vector $\overrightarrow{\xi\xi'}$, which is equipollent to $\overrightarrow{Oa}$ (a vector, which exists by 5^0). We will prove first that *these operations constitute a group.*

To prove this, we proceed as follows. Those operations contain evidently the identical operation (by 1^0). Let S_a and S_b be two such operators. Let $(c(x, \dot{x}, \cdots, \overset{(m)}{x}))$ be the transform of $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ by S_b. Executing the operation S_a and S_b successively, we pass from the point $(\xi(x, \dot{x}, \cdots, \overset{(m)}{x}))$ to the point $(\xi'(x, \dot{x}, \cdots, \overset{(m)}{x}))$ and then to $(\xi''(x, \dot{x}, \cdots, \overset{(m)}{x}))$ by virtue of

$$(S_a)\quad \overrightarrow{\xi\xi'} = \overrightarrow{Oa}, \qquad\qquad (S_b)\quad \overrightarrow{\xi'\xi''} = \overrightarrow{Ob}.$$

Now, by the hypothesis, $\overrightarrow{ac}$ is equipollent to $\overrightarrow{Ob}$. Hence $\overrightarrow{\xi'\xi''}$ is equipollent to $\overrightarrow{ac}$ (by 6^0). From the equipollences

$$\overrightarrow{\xi\xi'} = \overrightarrow{Oa}, \qquad\qquad \overrightarrow{\xi'\xi''} = \overrightarrow{ac},$$

follows thus (7^0) that

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(9) A double extension of the analogous result in [16]. p. 449.

(10) [16], p. 633.

$$\overrightarrow{\xi\xi'}=\overrightarrow{Oc},$$

whence we obtain $S_b S_a = S_c$. Q. E. D.

Next, let G be the group composed of the operations S_a. *This group is simply transitive.* This means that it contains one and only one transformation, which maps a given point $(\xi(x, \dot{x}, \cdots \overset{(m)}{x}))$ to another given point $(\xi'(x, \dot{x}, \cdots, \overset{(m)}{x}))$, obtaining the transformation S_a corresponding to the extremity of the vector $\overrightarrow{Oa}$, which is equipollent to $\overrightarrow{\xi\xi'}$. Consider next two arbitrary equipollent vectors $\overrightarrow{ab}$ and $\overrightarrow{a'b'}$. The vector $\overrightarrow{Oc}$, which is equipollent to $\overrightarrow{ab}$, is also equipollent to $\overrightarrow{a'b'}$ (property 6^0). Hence the transformation S_c maps $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ to $(b(x, \dot{x}, \cdots \overset{(m)}{x}))$ and $(a'(x, \dot{x}, \cdots, \overset{(m)}{x}))$ to $(b'(x, \dot{x}, \cdots, \overset{(m)}{x}))$ simultaneously. Now the transformation $S_b S_a^{-1}$ also maps $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ to $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$ (by the mediation of the origin (O)), and transformation $S_{b'} S_{a'}^{-1}$ maps likewise $(a'(x, \dot{x}, \cdots, \overset{(m)}{x}))$ to $(b'(x, \dot{x}, \cdots, \overset{(m)}{x}))$. Hence we have

$$S_c = S_b S_a^{-1} = S_{b'} S_{a'}^{-1},$$

what shows us that the equipollence defined in $\mathfrak{E}$ is identical with the equipollence of the first kind attached to the group G.

(**vi**) The results of the last Theorem that the equipollence of the second kind of the space of group may be considered as equipollence of the first kind attached to another group admitting the same representative space. It is easy to see that *the second group of doubly extended parameters* will admit the second equipollence of the group G for the first equipollence.

Now we encounter another important remark. Consider a set of transformation T_a depending on r doubly extended parameters, *not forming a group*, but playing the property that *the transformations* $T_b T_a^{-1}$ *do not depend on not more than r doubly oriented parameters* (when $a(x, \dot{x}, \cdots, \overset{(m)}{x})$ and $b(x, \dot{x}, \cdots, \overset{(m)}{x})$ take all possible values). We can define an equipollence of vectors in the space of this set of transformations by the equality

$$T_b T_a^{-1} = T_{b'} T_{a'}^{-1}, \tag{23.5}$$

and this equipollence plays the seven properties 1^0-7^0 as we can easily verify. Choose an *arbitrary origin transformation* T_0. The transformation S_a defined above may be expressed as follows:

$$T_{\xi'} T_{\xi}^{-1} = T_a T_0^{-1}$$

i. e.

$$(S_a) \quad T_{\xi'} = T_a T_0^{-1} T_{\xi}.$$

Execute the transformations S_a and S_b successively and set

$$S_b\, S_a = S_c.$$

We shall obtain

$$T_{\xi'} = T_a\, T_0^{-1}\, T_\xi,$$
$$T_{\xi''} = T_b\, T_0^{-1}\, T_{\xi'} = T_b\, T_0^{-1}\, T_a\, T_0^{-1}\, T_\xi = T_c\, T_0^{-1}\, T_\xi$$

by S_a and S_b successively. Hence the equality

$$(T_b\, T_a^{-1})(T_a\, T_0^{-1}) = T_c\, T_0^{-1}$$

results, so that the *transformations* $T_a\, T_0^{-1}$ *form a group.*

This theorem, which is of purely analytical nature, may be proved else directly. Consider a set of transformations $T_b\, T_a^{-1}$ of r doubly extended parameters. From the product

$$(T_{b'}\, T_{a'}^{-1})(T_b\, T_a^{-1})$$

of such transformations, we see that there exists a transformation T_c such that

$$T_{a'}^{-1}\, T_b = T_{b'}^{-1}\, T_c. \tag{23.6}$$

For, the transformations $T_b\, T_\xi^{-1}$, where we let the doubly extended parameters ξ vary, must have all the transformations of set $T_\zeta\, T_\eta^{-1}$, so, in particular, the transformation $T_{a'}\, T_{b'}^{-1}$. Therefore there exists a point $(c(x, \dot{x}, \cdots, \overset{(m)}{x}))$ such that

$$T_b T_c^{-1} = T_{a'}\, T_{b'}^{-1}.$$

This equality is equivalent to the equality (23.6). Thus from (23.6), we deduce

$$(T_{b'}\, T_{a'}^{-1})(T_b\, T_c^{-1}) = T_{b'}\, T_{b'}^{-1}\, T_c\, T_a^{-1} = T_c\, T_a^{-1},$$

which shows us that the transformations $T_b\, T_a^{-1}$ form a group. Moreover all the transformations of this group are obtainable by letting $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ fix and letting $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$ vary.

(vii) We know that two groups G and G' of the same order are said to be ***isomorphic*** (holoédrique),(11) when we can establish among their transformations a correspondence such that to the product of two arbitrary transformations of the first group there corresponds the product of two corresponding transformations of the second group. In the correspondence, which realize the isomorphism, the identity transformations correspond to each other. Moreover to the inverse of transformation of the first group there corresponds the inverse of the corresponding transformation of the second group.

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(11) "Hémiédrique" in the case of one-to-one correspondence. ([44], (1930), p. 11)

Let G and G' be two isomorphic groups and $\mathfrak{E}$ and $\mathfrak{E}'$ their spaces.

All correspondences by isomorphism of two groups may be interpreted by the point-correspondence of two spaces $\mathfrak{E}$ *and* $\mathfrak{E}'$, *such that to two vectors of* $\mathfrak{E}$, *which are equipollent of the first* (*second*) *kind to each other, there correspond two vectors of* $\mathfrak{E}'$, *which are equipollent of the first* (*second*) *kind to each other.*

Indeed, we can choose the doubly extended parameters of two groups in such a way that in the correspondence by isomorphism under consideration, the doubly extended parameters of two corresponding transformations are the same. We denote the transformations of the two groups by T and Θ. Then the equality

$$T_b T_a^{-1} = T_{b'} T_{a'}^{-1}$$

signifies that there exists a transformation T_c such that we have

$$T_b = T_c\, T_a, \qquad T_{b'} = T_c\, T_{a'},$$

whence follows;

$$\Theta_b = \Theta_c \Theta_a, \qquad \Theta_{b'} = \Theta_c\, \Theta_{a'},$$

so that

$$\Theta_b\, \Theta_a^{-1} = \Theta_{b'}\, \Theta_{a'}^{-1}. \qquad \text{Q. E. D.}$$

The demonstration will be the same for the parallelism of the second kind.

(viii) Conversely, *suppose that we can establish a point correspondence between the spaces* $\mathfrak{E}$ *and* $\mathfrak{E}'$ *of two groups* G *and* G' *of the same order* r *such that to two vectors of* $\mathfrak{E}$, *which are equipollent of the first kind, there correspond two vectors of* $\mathfrak{E}'$, *which are equipollent of the first kind, then the two groups* G *and* G' *are isomorphic.*

To prove this, let (ω) be the point of $\mathfrak{E}'$ corresponding to the origin (O) of $\mathfrak{E}$, and let $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$, $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$ and $(c(x, \dot{x}, \cdots, \overset{(m)}{x}))$ be three arbitrary points of $\mathfrak{E}$ and $(\alpha(x, \dot{x}, \cdots, \overset{(m)}{x}))$, $(\beta(x, \dot{x}, \cdots, \overset{(m)}{x}))$, $(\gamma(x, \dot{x}, \cdots, \overset{(m)}{x}))$ the corresponding points of $\mathfrak{E}'$. From the equality

$$T_b T_a^{-1} = T_c = T_c\, T_0^{-1}$$

follows:

$$\Theta_\beta\, \Theta_\alpha^{-1} = \Theta_\gamma\, \Theta_\omega^{-1}$$

by hypothesis. In other words, from the equality

$$T_b = T_c\, T_a$$

follows:

$$\Theta_\beta\, \Theta_\omega^{-1} = (\Theta_\gamma\, \Theta_\omega^{-1})\,(\Theta_\alpha\, \Theta_\omega^{-1}).$$

Then we let the transformation $\Theta_\alpha\, \Theta_\omega^{-1}$ of G' correspond to the transformation T_a of G. This correspondence shows the isomorphism of the two groups by the last equality.

We can make the remark that it is very easy to establish a correspondence with a given group by interchanging the two kinds of equipollence attached to the group : it suffices to make the transformation $T_\alpha = T_a^{-1}$ correspond to the transformation T_a. Then the equality

$$T_b T_a^{-1} = T_{b'} T_{a'}^{-1},$$

which defines the equipollence of the first kind, becomes changed into the equality

$$T_\beta^{-1} T_\alpha = T_{\beta'}^{-1} T_{\alpha'},$$

which defines the equipollence of the second kind.

It results from this remark, that *in order that two groups of the same order may be isomorphic, it is necessary and sufficient that we can establish a point correspondence between the spaces of two groups transforming one of the spaces into a certain of the spaces by an equipollence of the second kind.*

(**ix**) The preceding consideration proposes the question of determination of all the point transformations of a space of group into itself, which play the property to conserve the two kinds of equipollence of the space.

It is firstly evident that a point transformation, which conserves the equipollence of the first kind, conserves the equipollence of the second kind and vice versa. Let $(\alpha(x, \dot{x}, \cdots, \overset{(m)}{x}))$, $(\beta(x, \dot{x}, \cdots, \overset{(m)}{x}))$, etc. be the points transformed from $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$, $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$, etc. From the equipollence of the first kind of $\overrightarrow{ab}$ and $\overrightarrow{a'b'}$ follows that of $\overrightarrow{\alpha\beta}$ and $\overrightarrow{\alpha'\beta'}$ by hypothesis, whence follows that from the equipollence of the second kind of $\overrightarrow{aa'}$ and $\overrightarrow{bb'}$ follows that of $\overrightarrow{\alpha\alpha'}$ and $\overrightarrow{\beta\beta'}$.

Let us commence with determination of the point transformations, which conserve the equipollence of the first kind and *let the point origin be invariant.*

The equality

$$T_c = T_b T_a^{-1}$$

expresses simply the equipollence of the first kind of vectors $\overrightarrow{Ob}$ and $\overrightarrow{ac}$, whence follows the equipollence of $\overrightarrow{O\beta}$ and $\overrightarrow{\alpha\gamma}$, and consequently the equality holds:

$$T_\gamma = T_\beta T_\gamma^{-1}.$$

Hence the transformation sought for is *autoisomorphism* of group G. Among the autoisomorphisms, there exist in particular the transformations of the *adjoint group*:

$$T_{\xi'} = T_a^{-1} T_\xi T_a,$$

where $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ is a fixed point.

If the group G is semi-simple, the adjoint group is the largest maximum continuous group of automorphisms of G [41].

In order to obtain all the point transformations conserving the equipollence of the first kind, it will suffice to combine the preceding point transformations with the transformations

$$T_{\xi'} = T_\xi T_a \qquad \text{or} \qquad T_{\xi'} = T_a T_\xi$$

or further with the trannsformations

$$T_{\xi'} = T_a T_\xi T_b, \tag{23.7}$$

$(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ and $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$ denoting two fixed points. The point transformations (23.7) transform the equality

$$T_\eta T_\xi^{-1} = T_{\bar{\eta}} T_{\bar{\xi}}^{-1}$$

into the equality

$$T_{\eta'} T_{\xi'}^{-1} = T_{\bar{\eta}'} T_{\bar{\xi}'}^{-1}.$$

Evidently the transformations (23.9) form a group Γ_0, which is a subgroup of the total group Γ of transformations, which conserve the equipollence of the first kind. It is likewise easy to see that *Γ_0 is an invariant subgroup of Γ*. It suffices to prove that all the transformations of Γ_0 are changed into other transformations of Γ_0 by an automorphism of the group G. If the points $(a'(x, \dot{x}, \cdots, \overset{(m)}{x}))$, $(b'(x, \dot{x}, \cdots, \overset{(m)}{x}))$, $(\xi'(x, \dot{x}, \cdots, \overset{(m)}{x}))$, $(\eta'(x, \dot{x}, \cdots \overset{(m)}{x}))$ correspond to $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$, $(b(x, \dot{x}, \cdots \overset{(m)}{x}))$, $(\xi(x, \dot{x}, \cdots, \overset{(m)}{x}))$, $(\eta(x, \dot{x}, \cdots, \overset{(m)}{x}))$ by this isomorphism, th relation

$$T_\eta = T_a T_\xi T_b$$

is changed into

$$T_{\eta'} = T_{a'} T_{\xi'} T_{b'},$$

the transformation of Γ_0 corresponding to points $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ and $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$ is changed into another transformation of Γ_0, that which correspond to points $(a'(x, \dot{x}, \cdots, \overset{(m)}{x}))$ and $(b'(x, \dot{x}, \cdots, \overset{(m)}{x}))$.

We will give the name ***group of isomorphism*** of $\mathfrak{G}$ to Γ.

The group of point transformations of $\mathfrak{G}$, which conserve the set of two ***equipollences*** will easily be deduced from Γ by combining it with the transformation

$$T_\xi = T_{\xi'}.$$

It may be remarked that the group Γ_0 defined by the equations (23.7) is at most of $2r$ doubly extended parameters. Precisely, it is of $2r-\rho$ doubly extended parameters, where ρ denotes the order of the subgroup formed of those transformations of G, which are interchangeable with all the other transformations of G. The group Γ_0 contains evidently the adjoint group (23.6), which is itself of $r-\rho$ doubly extended parameters.

§ 8. Extension of E. Cartan's Geodesics, His Two Kinds of Parallelisms and His Transformations.

24. Extension of E. Cartan's Geodesics, His Two Kinds of Parallelisms and His Transformations.

(**i**) In case of the ordinary equipollence of two vectors, the straight line play the following characteristic property:

If we take three arbitrary points $(a), (b), (c)$ on a straight line, the vector $\overrightarrow{cd}$, which is equipollent to $\overrightarrow{ab}$ has its extremity (d) on the straight line.

E. Cartan ([19]) has generalized this notion in his space of group. Now we will generalize his notion further to the case *of the groups of doubly extended parameters* as follows.

Definition. A curve (C) traced in a space of group of doubly extended parameters will be called a II′-*geodesic* (read: the first geodesics of the second grade), when three arbitrary points $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$, $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$ and $(c(x, \dot{x}, \cdots, \overset{(m)}{x}))$ are taken on this curve, the extremity $(d(x, \dot{x}, \cdots, \overset{(m)}{x}))$ of the vector $\overrightarrow{cd}$, which is epuipollent of the first kind to $\overrightarrow{ab}$, lie also on this curve. The II_2'-geodesics may be defined similarly with respect to the equipollence of the second kind.

But we have to make the following important remark.

All the

II_1'-*geodesics*	II_2'-*geodesics*
are	
II_2'-*geodesics.*	II_1'-*geodesics.*

For, if $\overrightarrow{cd}$ be equiollent of the first kind to $\overrightarrow{ab}$, then this implies that $\overrightarrow{bd}$ is equipollent of the second kind to $\overrightarrow{ac}$ and vice versa.

Thus *there exist really only* II-*geodesics.*

(**ii**) The Primary question arising is that of the existence of the II-geodesics.

Now it is easy to find a priori an infinity of II′-geodesics in the spaces of groups with doubly extended parameters. For this purpose, take a fixed point of $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$. Let us consider a 1-parametric subgroup g of G. Denote its general transformation by Θ_u. The point $(\xi(x, \dot{x}, \cdots, \overset{(m)}{x}))$ defined by

$$T_\xi = \Theta_u T_a \tag{24.1}$$

describes a II′-geodesic. For, if u_1, u_2 and u_3 be three particular values of the parameter u, and $(\xi_1(x, \dot{x}, \cdots \overset{(m)}{x}))$, $(\xi_2(x, \dot{x}, \cdots, \overset{(m)}{x}))$, $(\xi_3(x, \dot{x}, \cdots, \overset{(m)}{x}))$ the three corresponding points and if $(\xi_4(x, \dot{x}, \cdots, \overset{(m)}{x}))$ be the extremity of the vector $\overrightarrow{\xi_3 \xi_4}$, which is equipollent of the first kind to $\xi_1\xi_2$, then we have

$$T_{\xi_4} T_{\xi_3}^{-1} = T_{\xi_2} T_{\xi_1}^{-1},$$

i. e.

$$T_{\xi_4} = T_{\xi_2} T_{\xi_1}^{-1} T_{\xi_3} = \Theta_{u_2} \Theta_{u_1} \Theta_{u_3} T_a = \Theta_{u_4} T_a. \qquad \text{Q. E. D.}$$

Conversely, we obtain all the II′-geodesics in this manner.

For, if $(\xi(x, \dot{x}, \cdots, \overset{(m)}{x}))$ and $(\eta(x, \dot{x}, \cdots, \overset{(m)}{x}))$ be two variable points and $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ a fixed point on a II′-geodesic, then there exists on this II′-geodesic a point $(\zeta(x, \dot{x}, \cdots, \overset{(m)}{x}))$ such that

$$T_\eta T_\xi^{-1} = T_\zeta T_a^{-1}$$

and consequently the transformations $T_\eta T_\xi^{-1}$ depend on a single parameter, whence follows that these transformations and especially the transformations $T_\xi T_a^{-1}$ form a one-parametric subgroup g of G. Denoting its general transformation by Θ_u, we obtain

$$T_\xi = \Theta_u T_a. \qquad \text{Q. E. D.}$$

It should bc remarked that any II-geodesic may be defined also by

$$T_\xi = T_a \Theta_u, \tag{24.2}$$

the Θ_u forming a one-parametric group, or more generally by

$$T_\xi = T_a \Theta_u T_b. \tag{24.3}$$

Moreover the (24.1) may be rewritten as follows;

$$T_\xi = (T_a \Theta_u T_a^{-1})(T_a T_b),$$

and the transformation $T_a \Theta_u T_a^{-1}$ constitute a group being led to the transformation group of g by T_a. Thus we fall on the expression (24.1) again.

(iii) Hitherto we have considered a vector $\overrightarrow{ab}$ to be defined uniquely by its origin $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ and its extremity $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$. When the parameters of $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$ do not differ much from those of $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$, the transformation $T_b T_a^{-1}$ belongs to one and only one-parametric subgroup g of G as in the case of the theory of continuous groups of S. Lie; consequently the two points $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ and $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$ belong to one and only one II′-geodesic, which is the locus of the point $(\xi(x, \dot{x}, \cdots, \overset{(m)}{x}))$ defined by

$$T_\xi = \Theta_u T_a,$$

where Θ_u is the general transformation of g. Thus the vector assimilates to the II′-geodesic segment limited by $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ and $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$.

We can then state as follows:

All vectors lying on a II′*-geodesic is equipollent of the first and the second kind to a determined vector lying on the* II*-geodesic and having for the origin a given point of this* II′*-geodesic.*

If we define the equality of two segments by the equipollence of corresponding vectors, we can measure the segment of one and the same II′-geodesic as soon as we choose a unit segment on this II′-geodesic segment.

If, in particular, we have taken our parameter t (*the affine length* : a generalization of the canonic parameter of S. Lie) introduced in (4.8) for the parameter u of the general transformation g such that

$$\Theta_u \Theta_{u'} = \Theta_{u+u'}, \qquad (u=t, u'=t')$$

the measure of the segment $\overrightarrow{\xi_1 \xi_2}$ with

$$T_{\xi_1} = \Theta_{u_1} T_a, \qquad T_{\xi_2} = \Theta_{u_2} T_a,$$

will be $|u_2 - u_1| = |t_2 - t_1|$. The change of u into ku means a change of the unit of length. The algebraic ratio of two vectors $\overrightarrow{\xi_1 \xi_2}$ and $\overrightarrow{\xi_3 \xi_4}$ taken on one and the same II′-geodesic has the determinate value

$$\frac{u_4 - u_3}{u_2 - u_1} = \frac{t_4 - t_3}{t_2 - t_1}.$$

Thus *we may now drop the dashes* (primes) *from* II′*-geodesics and* write down merely II*-geodesics* in place of II′-geodesics.

THEOREM. *The* II′*-geodesics in this section are the non-locally line-elemented* II*-geodesics in the sense of our* Art. 4.

(iv) PARALLELISMS. If we draw through a point $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$ outside of a non-locally line-elemented II-geodesic (C) passin through $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ vectors, which are equipollent of the first kind to several vectors lying on (C), wc obtain the vector $\overrightarrow{b\eta}$, which is equipollent of the first kind to the vector $\overrightarrow{a\xi}$, whose extremity $(\xi(x, \dot{x}, \cdots, \overset{(m)}{x}))$ describes (C). Hence the point (η) describes a curve (C') and this curve is a non-locally line-elemented II-geodesic. If we have

$$T_\xi = T_u\, T_a,$$

then we deduce thence:

$$T_\eta = T_u\, T_b.$$

We say that (C') is ***parallel of the first kind to*** (C) and any vector lying on (C') is equipollent of the first kind to a vector lying on (C).

Two non-locally line-elemented II-***geodesics, which are parallel of the first kind to a third, are parallel of the first kind to each other.***

We can define similarly ***non-locally line-elemented*** II-***geodesics, which are parallel of the second kind to each other.*** When this is defined by

$$T_\xi = T_a\, \Theta_u,$$

we obtain non-locally line-elemented II-geodesics defined by

$$T_\eta = T_b\, \Theta_u,$$

where $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$ is an arbitrary fixed point.

Thus we have defined two kinds of parallelisms for the non-locally line-elemented II-geodesics and for *each of these kinds*, we have the following properties.

1°. ***Each non-locally line-elemented*** II-***geodesic is parallel to itself.***

2°. ***Two non-locally line-elemented*** II-***geodesics, which are parallel to a third, are parallel to each other.***

3°. ***Through any point taken out-side of a non-locally line-elemented*** II-***geodesics, there exists one and only one non-locally line-elemented*** II-***geodesic, which is parallel to the former.***

It should be remarked that the two parallelisms permit us easily to construct the vector $\overrightarrow{\xi\eta}$ equipollent of the

first	second

kind to a given vector $\overrightarrow{ab}$ and having a given origin $(\xi(x, \dot{x}, \cdots, \overset{(m)}{x}))$; for this it suffices to draw through $(\xi(x, \dot{x}, \cdots, \overset{(m)}{x}))$ the non-locally line-elemented II-geodesic, which is parallel of the

first	second

kind to $\overrightarrow{ab}$ and then through $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$ the II-geodesic, which is parallel of the

second	first

kind to $\overrightarrow{a\xi}$; these two non-locally line-elemented II-geodesics meet in the point $(\eta(x, \dot{x}, \cdots, \overset{(m)}{x}))$ sought for.

(v) It is convenient to say that two non-locally line-elemented II-geodesics, which are parallel of the

first | second

kind, have the *same direction of the*

first kind. | *second kind.*

If we draw through the origin the parallel of the

first | second

kind to a given non-locally line-elemented II-geodesic, then several points of this parallel represent the transformations of a one-parametric group g. Hence we can say that *any direction of the*

first | *second*

kind is defined by a one-parametric subgroup of G.

If a one-parametric subgroup g of G together with a point $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ of the space is given, starting from the point $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ we can make a displacement in the direction of the

first | second

kind definend by g, and thus we obtain *two distinct non-locally line-elemented* II-*geodesics starting from* $(a(x, \dot{x}, \cdots \overset{(m)}{x}))$.

(**vi**) The equipollences of the first and the second kind permits us, as we have done in (iii), to define the equality and then the *ratio* of two segments lying on two non-locally line-elemented II-geodesics, which are parallel of the first or second kind. If on a given non-locally line-elemented II-geodesic, we can thus measure the segment on all the non-locally line-elemented II-geodesics, which are parallel of the first kind to given non-locally line-elemented II-geodesic and then on any non-locally line-elemented II-geodesic, which is parallel of the second kind to one of those latter and so on. Suppose that the given non-locally line-elemented II-geodesic starting from the point of origin and defined by a subgroup g of transformations Θ_u, the u being the affine length (canonical parameter). The non-locally line-elemented II-geodesics, which thus arise by the indicated process are the loci of the point $(\xi(x, \dot{x}, \cdots, \overset{(m)}{x}))$ given by

$$T_\xi = T_a \, \Theta_u \, T_b,$$

the $(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$ and the $(b(x, \dot{x}, \cdots, \overset{(m)}{x}))$ denoting two arbitrary fixed points, in particular, those among such non-locally line-elemented II-geodesics, which pass through the point of origin, are given by

$$T_\xi = T_a \Theta_u T_a^{-1};$$

their directions are defined by the various homogeneous (gleichberechtigte, [16], p. 474) subgroup of g in the total group G. It is only in the set of these directions, that the space admits of an intrinsic metric.

(**vii**) Any point transformation of the group of isomorphism of the space $\mathfrak{E}$ transforms evidently a non-locally line-elemented II-geodesic, into a non-locally line-elemented II-geodesic, the ratio of segments being conserved. It transforms further two parallel non-locally line-elemented II-geodesics into two parallel non-locally line-elemented II-geodesics.

Consider, in particular, the transformation

$$T_{\xi'} = T_a T_\xi.$$

By this transformation, the points of the space describe the vectors, which are equipollent of the first kind to one another. Moreover any vector is transformed into another vector, which is equipollent of the second kind to the former, and any non-locally line-elemented II-geodesie into another non-locally line-elemented II-geodesic, which is parallel of the second kind. We may give to such a transformation the name *"the translation of the first kind"*. These transformations are the transformations of the first group of doubly extended parameters ((ii) of Art. 24).

The equation

$$T_{\xi'} = T_\xi T_a$$

defines similarly *a translation of the second kind.*

The continuous translation of the first kind

$$T_{\xi'} = \Theta_u T_\xi$$

where Θ_u denotes an arbitrary transformation of the one-parametric group g (u playing the rôle of the time), plays the property that respective points of the space describe the non-locally line-elemented II-geodesics, which are parallel of the first kind to another, while respective non-locally line-elemented II-geodesics displace remaining parallel of the second kind to another. We will call this continuous translation the *non-locally line-elemented* II-*geodesic translation of the first kind.* We define similarly the non-locally line-elemented II-*geodesic translation of the second kind.*

§ 9. Simplification of the Fundamental Theorems on the Doubly Extended Lie Transformation Groups by Means of the Doubly Non-Local II-Geodesic Parallel Coordinates.

25. Non-Local II-Geodesic Parallel Coordinates in the Base Manifold and the Group Space. In (5.4), we have already introduced *doubly non-local* II-*geodesic. parallel coordinates* $\overline{\eta}^\lambda$ *in the doubly extended Lie group manifolds.* Now we shall introduce *doubly non-local* II-*geodesic parallel coordinates* ξ *in the base manifold.* For this purpose we introduce a matrix:

$$\overline{\xi}^j_i(x, \dot{x}, \cdots, \overset{(m)}{x}) \in C^2, \qquad (i=1, 2, \cdots, n\,; j= 1, 2, \cdots, r)$$

corresponding to the matrix $\xi^i_j(x, \dot{x}, \cdots, \overset{(m)}{x})$ introduced by (20.1) by the conditions:

$$\overline{\xi}^j_i \xi^i_l = \delta^j_l, \qquad (i, k, p, q, \cdots=1, 2, \cdots, n\,; h, j, l, \cdots=1, 2, \cdots, r). \tag{25.1}$$

Multiplying

$$\overline{\Xi}^l_k \xi^i_l = \delta^i_k,$$

where $\overline{\Xi}^l_k$ are unkowns, by $\overline{\xi}^j_i$, we obtain $\overline{\Xi}^l_k = \overline{\xi}^l_k$ by virtue of (25.1), so that it results that

$$\overline{\xi}^l_k \xi^i_l = \delta^i_k \tag{25.2}$$

and multiplying $\overline{\Xi}^l_k \xi^i_l = \delta^i_k$ by $\overline{\xi}^j_i$, we obtain $\overline{\Xi}^j_k = \overline{\xi}^j_k$ arriving at (25.1). Thus we see that

$$(25.1) \qquad \Longleftrightarrow \qquad (25.2).$$

For

$$|\overline{\xi}^j_i \xi^i_l| = \|\overline{\xi}^j_i\| \cdot \|\xi^i_l\| = \begin{vmatrix} \eta^1_1\, \eta^1_2 \cdots \eta^1_n\, 0 \cdots 0 \\ \eta^2_1\, \eta^2_2 \cdots \eta^2_n\, 0 \cdots 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ \eta^r_1\, \eta^r_2 \cdots \eta^r_n\, 0 \cdots 0 \end{vmatrix} \cdot \begin{vmatrix} \xi^1_1\, \xi^1_2 \cdots \xi^1_r \\ \xi^2_1\, \xi^2_2 \cdots \xi^2_r \\ \cdots\cdots\cdots \\ \xi^n_1\, \xi^n_2 \cdots \xi^n_r \\ \left.\begin{matrix} 0\ 0\ \cdots\ 0 \\ \cdots\cdots\cdots \\ 0\ 0\ \cdots\ 0 \end{matrix}\right\} r-m \end{vmatrix},$$

$$|\overline{\xi}^l_k \xi^i_l| = \|\overline{\xi}^l_k\| \cdot \|\xi^i_l\| = \begin{vmatrix} \xi^1_1\, \xi^1_2 \cdots \xi^1_r \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ \xi^n_1\, \xi^n_2 \cdots \xi^n_r \\ r-m\left\{\begin{matrix} 0\ 0\ \cdots\ 0 \\ \cdots\cdots\cdots \\ 0\ 0\ \cdots\ 0 \end{matrix}\right. \end{vmatrix} \cdot \begin{vmatrix} \eta^1_1\, \eta^1_2 \cdots \eta^1_n\, 0\ 0 \cdots 0 \\ \eta^2_1\, \eta^2_2 \cdots \eta^2_n\, 0\ 0 \cdots 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ \eta^r_1\, \eta^r_2 \cdots \eta^r_n\, 0\ 0 \cdots 0 \end{vmatrix},$$

we have

$$|\overline{\xi}^j_i \xi^i_l| = |\delta^j_l| = 1. \qquad\qquad |\overline{\xi}^j_i \xi^i_l| = |\delta^j_l| = 1. \tag{25.3}$$

Replace $\omega^l_\mu(x, \dot{x}, \cdots, \overset{(m)}{x})$ of Art. 4 by

$$\overline{\xi}^l_h, \quad (l=1, 2, \cdots, r\,; h, p, q=1, 2, \cdots, n\,; r \geqq n)$$

and consider the extended Pfaffians

$$(25.4)\qquad \omega^l=\bar{\xi}^l_h(x,\dot{x},\cdots,\overset{(m)}{x})\,dx^h,\ \Big(\bar{\xi}^l_h(x,\dot{x},\cdots\overset{(m)}{x})=\Big(\frac{\partial f^i(x;a(x,\dot{x},\cdots,\overset{(m)}{x})}{\partial a^h}\Big)_{a^h=a^h_0}\Big),$$

which are assumed to be of rank r, so that the condition

$$(25.5)\qquad \|\bar{\xi}^l_h x,\dot{x},\cdots,\overset{(m)}{x})\|^2\neq 0 \text{ in } M$$

is satisfied.

We define the connection parameter Λ^i_{pq} by

$$(25.6)\qquad \Lambda^i_{pq}\overset{\text{def}}{=}\xi^i_l\Big(\frac{\partial}{\partial x^q}+\frac{\ddot{x}^s}{\dot{x}^q}\frac{\partial}{\partial \dot{x}^s}+\cdots+\frac{\overset{(m+1)}{x^s}}{\dot{x}^i}\frac{\partial}{\partial \overset{(m)}{x^s}}\Big)\bar{\xi}^l_p(x,\dot{x},\cdots,\overset{(m)}{x})$$

$$\equiv-\bar{\xi}^l_p\Big(\frac{\partial}{\partial x^q}+\frac{\ddot{x}^s}{\dot{x}^q}\frac{\partial}{\partial \dot{x}^s}+\cdots+\frac{\overset{(m+1)}{x^s}}{\dot{x}^q}\frac{\partial}{\partial \overset{(m)}{x^s}}\Big)\xi^i_l(x,\dot{x},\cdots,\overset{(m)}{x})$$

as in the case of (4.7), the last identity arising from (25.2).

Consider a parametrized curve $x^i=x^i(t)$, $(i=1,2,\cdots,n)$.

We can easily prove the identity

$$(25.7)\qquad \frac{d}{dt}\frac{\omega^l}{dt}=\bar{\xi}^l_i(x,\dot{x},\cdots,\overset{(m)}{x})\Big\{\frac{d^2x^i}{dt^2}+\Lambda^i_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}\Big\}.$$

We consider the combined manifold ($\{x^i\}$, $\{\bar{\xi}^l_j(x,\dot{x},\cdots,\overset{(m)}{x})\}$), the

$$\{\bar{\xi}^i(x,\dot{x},\cdots,\overset{(m)}{x})\}=\{\xi^i_l(x,\dot{x},\cdots,\overset{(m)}{x})\}$$

forming the *structure group*.

We can convert (25.7) into

$$(25.8)\qquad \xi^i_l\frac{d}{dt}\frac{\omega^l}{dt}\equiv\frac{d^2x^i}{dt^2}+\Lambda^i_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}.$$

From (25.7) and (25.8), we see that

$$(25.9)\qquad \frac{d}{dt}\frac{\omega^l}{dt}=0 \qquad\Longleftrightarrow\qquad \frac{d^2x^i}{dt^2}+\Lambda^i_{pq}\frac{dx^p}{dt}\frac{dx^q}{dt}=0$$

as in the case (4.10).

The left-hand side of (25.9) may be integrated readily:

$$(25.10)\qquad \omega^l=c^l\,dt,\qquad (c^l=\text{const.}),$$

$$(25.11)\qquad \int\frac{\omega^l}{dt}dt=c^lt+d^l,\qquad (d^l=\text{const.}),$$

the (25.11) being guided by the simple character of the right-hand side of (25.10). Noticing again the simple clear character of the right-hand side of (25.11), we set

$$\xi^l=c^lt+d^l,$$

so that

$$\xi^l = \int \frac{\omega^l}{dt}\, dt = c^l t + d^l. \tag{25.12}$$

This means that we adopt such curves as ξ^l-axes in the r-dimensional space containing subspace $\{x^i\}$.

From (25.12), we see that *the curves represented by* (25.12) *behave as for meet and join like straight lines in the large.* We will call such curves *non-locally line-elemented* II-*geodesic curves.*

The expression (25.12) tells us that, for the given $\bar{\xi}^l_i(x, \dot{x}, \cdots \overset{(m)}{x})\, dx^i$, there exists $x^i(t)$, whose line-elements $\{dx^i\}$ with direction $\{c^l\}$ is given by the differential $d\xi^l$. This is the case for all the directions $\{c^l\}$. Thus in (25.4), we may omit t and write it down as follows;

$$d\xi = \bar{\xi}^l_i(x, \dot{x}, \cdots, \overset{(m)}{x})\, dx^i. \tag{25.13}$$

The first differential equation (25.9) may be rewritten as follows:

$$\frac{d^2\xi^l}{dt^2} = 0.$$

Multiplying (25.10) with $\xi^i_l(x, \dot{x}, \cdots, \overset{(m)}{x})$ and taking (25.1) into account, we see that *the relations*

$$\frac{dx^i}{dt} = c^l\, \xi^i_l \tag{25.14}$$

hold along the non-locally line-elemented II-*geodesic line-elements.*

We will call $\{\xi^l\}$ the *non-locall* II-*geodesic parallel coordinates corresponding to* $\bar{\xi}^l_i$ referred to the ξ^l-axes. The $\{\xi^l\}$ are *global.*

From (25.3), we obtain

$$\xi^l = \int \bar{\xi}^l_i(x, \dot{x}, \cdots, \overset{(m)}{x})\, dx^i = \bar{\xi}^l_i(x, \dot{x}, \cdots, \overset{(m)}{x})\, x^i - \int x^i\, d\bar{\xi}^l_i(x, \dot{x}, \cdots, \overset{(m)}{x}),$$

$$\xi^l = \bar{\xi}^l_i(x, \dot{x}, \cdots, \overset{(m)}{x})\, x^i + \bar{\xi}^l_0, \qquad (\bar{\xi}^l_0 = \text{const.}) \tag{25.15}$$

as in the case of (5.8), the differential equations to the non-locally line-elemented II-geodesic curves being

$$d\bar{\xi}^l_i(x, \dot{x}, \cdots, \overset{(m)}{x})\, dx^i = 0 \tag{25.16}$$

or

$$d\bar{\xi}^l_i(x, \dot{x}, \cdots, \overset{(m)}{x})\, x^i = 0 \tag{25.17}$$

as in the case of (5.2) and (5.5).

26. To prove $\dfrac{\partial}{\partial \xi^l}=\xi^i_l\dfrac{\partial}{\partial x^i}$ **and** $\dfrac{\partial}{\partial \alpha^l}=\alpha^j_l\dfrac{\partial}{\partial a^j}$.

$$\tau^l=\bar{\xi}^l_k\,dx^k,\qquad dx^i=\xi^i_h\,\tau^h.$$

$$\frac{\partial\psi(x\,;a(x,\dot{x},\cdots,\overset{(m)}{x}))}{\partial \xi^l}=\lim_{dx^i=0}\frac{\dfrac{\partial\psi(x\,;a)}{\partial x^i}dx^i}{\bar{\xi}^l_k\,dx^k}$$

$$=\lim_{dx^i=0}\frac{\dfrac{\partial\psi(x\,;a)}{\partial x^i}\xi^i_h\tau^h}{\bar{\xi}^l_h\,\xi^h_j\,\tau^j}$$

$$=\lim_{dx^i=0}\frac{\tau^h}{\delta^l_j\,\tau^j}\xi^i_h\frac{\partial\psi(x\,;a)}{\partial x^i}$$

$$=\lim_{dx^i=0}\frac{\tau^h}{\tau^l}\xi^i_h\frac{\partial\psi(x\,;a)}{\partial x^i}$$

$$=\delta^h_l\,\xi^i_h\frac{\partial\psi(x\,;a)}{\partial x^i}=\xi^i_l\frac{\partial\psi(x\,;a)}{\partial x^i}.$$

$$\alpha^j=\beta^j_l\,da^l,\qquad da^j=\alpha^j_k\,d\alpha^k.$$

$$\frac{\partial\psi(x\,;a(x,\dot{x},\cdots,\overset{(m)}{x}))}{\partial \alpha^l}=\lim_{da^j=0}\frac{\dfrac{\partial\psi(x\,;a)}{\partial a^j}da^j}{\beta^l_j\,da^j}$$

$$=\lim_{da^j=0}\frac{\dfrac{\partial\psi(x\,;a)}{\partial a^j}\alpha^j_k\,d\alpha^k}{\beta^l_j\,\alpha^j_h\,\alpha^h}$$

$$=\lim_{da^j=0}\frac{\alpha^k}{\delta^l_h\,\alpha^h}\alpha^j_k\frac{\partial\psi(x\,;a)}{\partial a^j}$$

$$=\lim_{da^j=0}\frac{\alpha^k}{\alpha^l}\alpha^j_k\frac{\partial\psi(x\,;a)}{\partial a^j}$$

$$=\delta^k_l\,\alpha^j_k\frac{\partial\psi(x\,;a)}{\partial a^j}=\alpha^j_l\frac{\partial\psi(x\,;a)}{\partial a^j}.$$

Hence

$$(26.1)\qquad \frac{\partial}{\partial \xi^l}=\xi^i_l\frac{\partial}{\partial x^i}.\qquad\Big|\qquad \frac{\partial}{\partial \alpha^l}=\alpha^j_l\frac{\partial}{\partial a^j}.$$

27. Simplification of the First Fundamental Theorem on the Doubly Extended Lie Transformation Group by means of the Non-Local Parallel Coordinates. The First Fundamental Theorem of the Theory of the Doubly Extended Lie Transformation Groups has been stated in the form of Cor. 2⁰ of Art. 20. Now, by virtue of the last Art. as well as of (4.12), it may be simplified and made global as follows.

THE FIRST FUNDAMENTAL THEOREM (The simplified form). *In the doubly extended Lie transformation group G as doubly extended parameter group, the* $f^k(\xi\,;\eta(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi}))$, (cf. (27.6)), $(k=1,2,\cdots,n)$ *are n independent solutions of the completely integrable simultaneous linear partial differential equations*

$$(27.1)\qquad \frac{\partial f}{\partial \eta^l}=\frac{\partial f}{\partial \xi^l},\quad (j,l=1,2,\cdots,r\,;i,k=1,2\cdots,n)$$

such that

$$(27.2)\qquad \xi^i=f^i(\xi\,;0),\qquad (i=1,2,\cdots,n).$$

Conversely, when an r-dimensional doubly extended Lie gronp is given, the (27.1) *is completely integrable, their solutions* $f^l(\xi\,;\eta(\xi,\dot{\xi},\cdots\overset{(m)}{\xi}))$, $(l=1,2,\cdots,r)$ *satisfying* (27.2), *determine a doubly extended Lie transformation group having G as* *doubly extended parameter group.*

SOLUTION OF (27.1). The Lagrange's auxiliary differential equations of (27.1) are

(27.3) $d\xi^l = -d\eta^l$, ((20.4), (20.5), (20.)). (cf. (27.6)).

The (20.22) gives in this case:

(27.4) $$\bar{X} = e^j \xi_j^i \frac{\partial}{\partial x^i} - e^j \alpha_j^l \frac{\partial}{\partial a^l} = e^j \left(-\frac{\partial}{\partial a^j} + \frac{\partial}{\partial \eta^j} \right).$$

Cousider

(27.5) $$-\bar{X} f = 0.$$

The Lagrange's auxiliary differential equations become

$$\frac{d\xi^l}{e^j \delta_j^l} = \frac{d\eta^l}{-e^j \delta_j^l}, \qquad (j, l = 1, 2, \cdots, r)$$

$$= \frac{\bar{\xi}_i^l \, dx^i}{e^j \bar{\xi}_i^l \xi^i} = \frac{\alpha_k^l \, da^k}{-e^j \alpha_k^l \alpha_j^k}$$

$$= \frac{dx^i}{e^j \xi_j^i} = \frac{da^k}{-e^j \alpha_j^k} = dt, \qquad (i, k = 1, 2, \cdots, n),$$

so that

(27.6) $$d\xi^l = d\eta^l = e^l \, dt$$

in conformity with (4.12) aud (27.3), whence follows:

(27.7) $\xi^l = \eta_0^l - \eta^l(x, \dot{x}, \cdots, \overset{(m)}{x}) = e^l(t - t_0)$, (cf. (27.12)) ($\eta_0^l$, $t_0 = \text{const.}$),

which represents a non-locally line-elemented II-*geodesic curve*

$\xi^i = e^i(t - t_0)$	$\eta^l = \eta_0^l - e^l(t - t_0)$
corresponding to	
ξ_j^i	α_j^l
in the differentiable base manifolds.	*group manifold.*

The *complete integral* consists of (27.7) and the *general integral* is

(27.8) $\chi(\xi^1 + \eta^1(x, \dot{x}, \cdots, \overset{(m)}{x}), \cdots, \xi^r + \eta^r(x, \dot{x}, \cdots, \overset{(m)}{x}))$, (cf. (27.12)),

where χ is an arbitrary function.

Comparing (27.7) with

(27.9) $\xi^l = \bar{\xi}_i^l(x, \dot{x}, \cdots, \overset{(m)}{x}) x^i + \xi_0^l$, ($\bar{\xi}_0^l = \text{const.}$, $i = 1, 2, \cdots, n$; $l = 1, 2, \cdots, r$),

we see that

$$\bar{\xi}^l_i(x,\dot{x},\cdots,\overset{(m)}{x})x^i+\bar{\xi}^l_o=\eta^l_o-\eta^l(x,\dot{x},\cdots,\overset{(m)}{x}), \tag{27.10}$$

so that

$$\eta^l(x,\dot{x},\cdots,\overset{(m)}{x})=\eta^l_o-\bar{\xi}^l_i(x,\dot{x},\cdots,\overset{(m)}{x})-\bar{\xi}^l_o=\eta^l_o-\xi^l(x,\dot{x},\cdots,\overset{(m)}{x}),$$

i. e.

$$\eta^l(x,\dot{x},\cdots,\overset{(m)}{x})=\eta^l_o-\xi^l(x,\dot{x},\cdots,\overset{(m)}{x}). \tag{27.11}$$

The inverse transformation of (27.9) was

$$x^i=\xi^i_j(\xi,\dot{\xi},\cdots,\overset{(m)}{\xi})\xi^j+\xi^i_0, \qquad (\xi^i_0=\text{const.}). \tag{27.12}$$

N. B. The differential equation (20.19) reduce to (27.1).

28. Simplification of the Second Fundamental Theorem. *When a given r-dimensional doubly extended Lie group G as a doubly extended parameter group has the structure constants*

$$C^k_{ij}, \qquad (i,j,k=1,2,\cdots,r)$$

the necessary and sufficient condition for that (27.1) *may be completely integrable, is that the relations*

$$C^k_{jl}=0, \qquad (h,j,l=1,2,\cdots,r), \tag{28.1}$$

holds.

PROOF. In (20.32), we have

$$\begin{aligned}(X_j,X_l)&=\left\{\xi^h_j(x,\dot{x},\cdots,\overset{(m)}{x})\frac{\partial}{\partial x^h}\right\}\left\{\xi^k_l(x,\dot{x},\cdots,\overset{(m)}{x})\frac{\partial}{\partial x^k}\right\}\\&\quad-\left\{\xi^k_l(x,\dot{x},\cdots,\overset{(m)}{x})\frac{\partial}{\partial x^k}\right\}\left\{\xi^h_j(x,\dot{x},\cdots,\overset{(m)}{x})\frac{\partial}{\partial x^h}\right\}\\&=\frac{\partial^2}{\partial\xi^j\,\partial\xi^l}-\frac{\partial^2}{\partial\xi^l\,\partial\xi^j}=0\end{aligned}$$

and X_j are linearly independent.

29. Simplification of the Third Fundamental Theorem. *When r linearly independent differential operators*

$$X_j=\xi^i_j(x,\dot{x},\cdots\overset{(m)}{x})\frac{\partial}{\partial x^i}=\frac{\partial}{\partial\xi^j}, \qquad (i=1,2,\cdots n;j=1,2,\cdots,r), \tag{29.1}$$

$$(\xi^i_j(x,\dot{x},\cdots,\overset{(m)}{x})\in C^2)$$

are given, the necessary and sufficient condition for that they are fundamental differential operators for a doubly extended Lie transformation group, is that the *following relations hold*:

$$C^h_{jk}=0, \qquad (h, j, k=1, 2, \cdots, r). \tag{29.2}$$

30. Simplification of the Fourth Fundamental Theorem. *The r^3 constants*

$$C^h_{jl}=0, \qquad (h, j, l=1, 2, \cdots, r)$$

for the fundamental operators

$$\frac{\partial}{\partial\xi^1}, \frac{\partial}{\partial\xi^2}, \cdots, \frac{\partial}{\partial\xi^r}$$

of a doubly extended Lie transformation group make the following three conditions identities:

$$(X_j, X_l)=C^h_{jl}\, X_h, \qquad (h, k, j, l=1, 2, \cdots, r), \tag{30.1}$$

$$C^h_{jl}=-C^h_{lj}, \tag{30.2}$$

$$C^h_{ij}\, C^l_{hk}+C^h_{jk}C^l_{hi}+C^h_{ki}C^l_{hj}=0, \qquad (i, j, k, l=1, 2, \cdots, r), \tag{30.3}$$

so that the Fourth Fundamental Theorem of Art. 20 *holds.*

N. B. Thus the Second, the Third and the Fourth Fundamental Theorems reduce extremely, the First and the Second Fundamental Theorems only remain effectively.

The following relations are readily deducible and are note-worthy:

$$\begin{aligned} e^l=\dot{\xi}^l&=\omega^l_\mu(x, \dot{x}, \cdots, \overset{(m)}{x})\,\dot{x}^\mu \\ &=-\dot{\eta}^l=-\alpha^l_k(a, \dot{a}, \cdots, \overset{(\bar{m})}{a})\,\dot{a}^k(x, \dot{x}, \cdots, \overset{(m)}{x}), \end{aligned} \tag{30.4}$$

$$\begin{aligned} \dot{a}^l=(x, \dot{x}, \cdots, \overset{(m)}{x})&=-\beta^l_h(a, \dot{a}, \cdots, \overset{(\bar{m})}{a})\,\omega^h_\mu(x, \dot{x}, \cdots, \overset{(m)}{x})\,\dot{x}^\mu \\ &=\beta^l_h(a, \dot{a}, \cdots, \overset{(\bar{m})}{a})\,\dot{\eta}^h=-e^h\,\beta^l_h(a, \dot{a}, \cdots, \overset{(m)}{a}), \end{aligned} \tag{30.5}$$

$$\begin{aligned} \dot{x}^\lambda&=-\Omega^\lambda_l(x, \dot{x}, \cdots, \overset{(m)}{x})\,\alpha^l_h(a, \dot{a}, \cdots, \overset{(\bar{m})}{a})\,\dot{a}^h(x, \dot{x}, \cdots, \overset{(m)}{x}) \\ &=\Omega^\lambda_l(x, \dot{x}, \cdots, \overset{(m)}{x})\,\dot{\xi}^l=e^l\,\Omega^\lambda_l(x, \dot{x}, \cdots, \overset{(m)}{x}). \end{aligned} \tag{30.6}$$

§ 10. Adjoint Doubly Extended Lie Transformation Groups.

31. Adjoint Doubly Extended Group of Doubly Extended Lie Transformation Groups. In Art. 13, we have extended the concept of adjoint groups ([16], p. 450) of a Lie transformation group doubly to the case of the *adjoint doubly extended group of a doubly extended Lie transformation group G.*

I. We shall first study the adjoint doubly extended transformations

$$\tilde{e}^i=\xi^i_l(c, \dot{c}, \cdots, \overset{(m)}{c})\,e^l,$$

where the e^l are those, which we have considered in (27.4).

Since

$$x^i(t)=\tilde{e}^i t+\tilde{c}^i_0, \qquad (\tilde{c}^i_0=\text{const.}) \tag{31.1}$$

for the II-*geodesic curves in the manifold*, the (27.6) and (27.7) may be rewritten as follows:

$$\begin{aligned} d\xi^l&=\bar{\xi}^l_i(x,\dot{x},\cdots\overset{(m)}{x})\,\tilde{e}^i\,dt \\ &=e^l\,dt, \end{aligned} \quad\Bigg|\quad \begin{aligned} \xi^l&=\bar{\xi}^l_i(x,\dot{x},\cdots\overset{(m)}{x})(\tilde{e}^i t+\tilde{c}^i_0)+\bar{c}^l \\ &=e^l t+c^l, \qquad (c^l=\text{const.},\ \bar{c}^l=\bar{\xi}^l_0), \end{aligned} \tag{31.2}$$

so that

$$\bar{\xi}^l_i(x,\dot{x},\cdots,\overset{(m)}{x})\,\tilde{e}^i=e^l, \quad\Big|\quad \bar{\xi}^l_i(x,\dot{x},\cdots,\overset{(m)}{x})\,\tilde{c}^i_0+\bar{c}^l=c^l, \tag{31.3}$$

whose inverse transformation is

$$\tilde{e}^i=\xi^i_l(x,\dot{x},\cdots,\overset{(m)}{x})\,e^l. \quad\Big|\quad \tilde{c}^i_0=\xi^i_l(x,\dot{x},\cdots,\overset{(m)}{x})\,c^l+\xi^i_0,\ (\xi^i_0=-\xi^i_l\,\bar{\xi}^l_0). \tag{31.4}$$

Thus

e^l	$\tilde{e}^i$	c^l	$\tilde{c}^i_0$

undergo the doubly extended affine transformations

(31.4).	(31.3).	(31.4).	(31.3).

II. Next we will consider the general case. Let us denote the operator corresponding to

$$x'^i=f^i(x\,;a(x,\dot{x},\cdots,\overset{(m)}{x})) \tag{31.5}$$

by $X'_i f$. Then we shall have

$$e^h X_h f=e'^h X'_h f, \tag{31.6}$$

where e'^h are certain functions of

$$a^1, a^2, \cdots, a^r, e^1, e^2, \cdots, e^r$$

by virtue of (31.4).

If we set $f=x^i$ in (31.6), then it results that

$$e^l\,\xi^i_l(x,\dot{x},\cdots,\overset{(m)}{x})=e'^l\,X'_l x^i, \qquad (i=1,2,\cdots,n). \tag{31.7}$$

If we give n determinate values $\overset{(p)}{x^i}, (p=1,2,\cdots,n)$ to x^i, then x'^i becomes functions of $a^1, a^2, \cdots, a^r$. Thus we obtain

$$e^l\,\xi^i_l(\overset{(p)}{x},\overset{(p)}{\dot{x}},\cdots,\overset{(m)}{\overset{(p)}{x}})=e'^l\,\xi'^k_l(\overset{(p)}{x},\overset{(p)}{\dot{x}},\cdots,\overset{(m)}{\overset{(p)}{x}})\left[\frac{\partial x^i}{\partial x'^k}\right]_{x^i=\overset{(p)}{x^i}}. \tag{31.8}$$

Thereby we assume that r values $(p=1, 2, \cdots, r)$ of $\overset{(p)}{x}{}^i$ have been so chosen that

(31.9) $$|\xi^i_l(\overset{(p)}{x}, \overset{(\dot{p})}{x}, \cdots, \overset{\overset{(m)}{(p)}}{x})| \neq 0, \qquad (i=1, 2, \cdots, n).$$

Let $\bar{\xi}^j_i(\overset{(p)}{x})$ be a matrix such that

(31.10) $$\xi^i_l(\overset{(p)}{x})\bar{\xi}^j_i(\overset{(p)}{x})=\delta^j_l \qquad (p: \text{summed}; i: \text{not summed}),$$

and multiply (31.8) with $\bar{\xi}^j_i(\overset{(p)}{x})$ and sum the result with respect to p. Then we obtain

$$e^j=e^l\,\delta^j_l=e'^l\,\xi'^k_l(x', \dot{x}', \cdots, \overset{(m)}{x'})\,\bar{\xi}^j_i(\overset{(p)}{x})\left[\frac{\partial x^i}{\partial x'^k}\right]_{x'^i=x^i}, \qquad (p: \text{ summed})$$

i. e.

(31.11) $$e^j=\rho^j_l(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\,e'^l, \qquad (|\rho^j_l(a(x, \dot{x}, \cdots, \overset{(m)}{x}))| \neq 0),$$

where

(31.12) $$\rho^j_l(a(x, \dot{x}, \cdots, \overset{(m)}{x}))=\xi'^k_l(x, \dot{x}, \cdots \overset{(m)}{x})\,\bar{\xi}^j_i(\overset{(p)}{x}, \overset{(\dot{p})}{x} \cdots \overset{\overset{(m)}{(p)}}{x})\left[\frac{\partial x^i}{\partial x'^k}\right]_{x^i=\overset{(p)}{x}{}^i}, \qquad (p: \text{ summed}).$$

If we denote the inverse transformation of (31.12) by $\bar{\rho}^l_j(a(x, \dot{x}, \cdots, \overset{(m)}{x}))$, then we have

(31.13) $$\begin{aligned}&\rho^h_l(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\,\bar{\rho}^l_k(a(x, \dot{x}, \cdots \overset{(m)}{x}))=\delta^h_k,\\ &\rho^l_k(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\,\bar{\rho}^h_l(a(x, \dot{x}, \cdots \overset{(m)}{x}))=\delta^h_k,\end{aligned}$$

and

(31.4) $$e'^l=\rho^l_j(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\,e^j.$$

That (31.11) *forms a group*, may be proved as in the case of ([16], p. 452).

32. Adjoint Doubly Extended Transformation Group in terms of the Non-Local II-Geodesic Parallel Coordinates. The (31.6) becomes

(32.1) $$e^l\,X_l f \equiv e^l\frac{\partial f}{\partial \xi^l}=e'^l\,X'_l f \equiv e'^l\frac{\partial f}{\partial \xi'^l},$$

when ξ^l and ξ'^l are respective doubly extended II-geodesic parallel coordinates, such that

(32.2) $$\xi'^l=\bar{\xi}^l_j(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})\,\xi^j, \qquad \xi^l=\xi^l_j(\xi', \dot{\xi}', \cdots, \overset{(m)}{\xi'})\,\xi'^j.$$

If we set $f=\xi^l$, then we obtain

$$e^l=e^j\,\delta^l_j=e'^j\frac{\partial \xi^l}{\partial \xi'^j}=e'^j\,\xi^l_j\,(\xi', \dot{\xi}' \cdots, \overset{(m)}{\xi'})$$

from (31.7), i. e.

(32.3) $$e^l=e'^j\,\xi^l_j(\xi', \dot{\xi}', \cdots, \overset{(m)}{\xi'}), \qquad e'^l=\bar{\xi}^l_k(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})\,e^k.$$

Thus $\xi^l_j(\xi', \dot{\xi}', \cdots, \overset{(m)}{\xi'})$ *and* $\bar{\xi}^l_j(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})$ *themselves play the rôles of* $\rho^l_j(a(\xi', \dot{\xi}', \cdots, \overset{(m)}{\xi'}))$ *and* $\bar{\rho}^l_j(a(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi}))$ *in* (31.11) *and* (31.14) ***respectively.***

33. The Canonical Equations of r-Dimensional Doubly Extended Lie Transformation Gronp. The following theorem is an extension of a Theorem ([16], p. 454, Theorem 32) of Sophus Lie.

THEOREM. *If*

$$X'^i = x^i + e^l X_l x^i + \frac{1}{2!} e^j e^l X_j X_l x^i + \cdots \tag{33.1}$$

be the canonical equations of an r-dimensional doubly extended Lie tranformtion group $X_1 f, X_2 f, \cdots, X_r f$ in n variables $x^1, x^2, \cdots, x^n$ and if we apply the transformation (31.4), *then the transformations $(e^1, e^2, \cdots, e^r)$ are transformed into $(e'^1, e'^2, \cdots, e'^r)$ by the transformations*

(31.14),	(32.3),

where

$\lvert(\rho^l_j(a(x, \dot{x}, \cdots, \overset{(m)}{x}))\rvert \neq 0.$	$\lvert\bar{\xi}^l_k(\xi, \dot{\xi}, \cdots, \overset{(m)}{\xi})\rvert \neq 0.$

The transformations

(31.14)	(32.3)

constitute a group and the relation

(31.6)	(32.1)

holds.

The part concerning (33.1) may be proved quite as in the case of ([16], p. 454, Theorem 32).

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