ON CHARACTERIZATIONS AND UNDECIDABILITY OF THE FIRST-ORDER FUNCTIONAL CALCULUS

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In publications [8], [9] I have presented some semantic characterizations of theses of the first-order functional calculus.

In connection with ones the present paper describes more strong characterizations of the theses:

1. By means of new proof rules which kind is different from the usual ones, we obtain an important syntactic characterization; in this way we reduce the decidability problem to consistency of families of sets of atomic formulas such that indices of variables occurring in those formulas are $\leqslant 3$ [1].
2. In connection with the syntactic characterization we obtain new semantic characterization which shows that in the decidability problem we may restrict our considerations only to families of models which domains have $\leqslant 3$ elements, s. [14] [2].
3. We explain two ways for decidabling of formulas: syntactic and a semantic one.
4. In connection with 3 we introduce an algorithmic language based on $1-1$ Turing machines and describe a simple proof of the undecidability theorem in the given language.
5. We show other ways to such results according to present conclusions.

As a simple corollary we obtain Gödel-Kalmar's theorem about decidability of theses $\Sigma a_1 \Sigma a_2 \Pi a_n \cdots \Pi a_n F$, where F is quanifier and free variable-free expression and simultaneously, the decidability of the monadic first-order functional calculus.

We shall use the terminology of [8], [9] and in particular:

(01) free individual variables: $x_1, x_2, \cdots$ / simply x /,

(02) apparent individual variables: $a_1, a_2, \cdots$ / simply a /,

(03) finite numbers of relation signs: $f_1^1, \cdots f_q^1; \cdots; f_1^t, \cdots, f_q^t$ [f_i^m – of m arguments, $m=1, \cdots, t$ and $i=1, \cdots, q$],

(04) logical constants: [negation], + [alternative], Π [general quantifier],

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1 Consistency by respective proof rules; we use here a generalized definition of sets of atomic formulas and models; therefore it is not inconsistent with the undecidability theorem.

An expression in which an apparent variable a belongs to the scope of two quantifiers Πa is not a formula; if a does not occur in E, then ΠaE is not a formula.

2 See footnote 1.

(05) expressions; $E, F, G, E_1, F_1, G_1, \cdots$

(06) $w(E)$ – the number of different free individual / $p(E)$ – apparent / variables occurring in the expression E,

(07) $\{K_m\}$ – the sequence $K_1, \cdots, K_m$; $\{K_{i_m}\}$ – the sequence $K_{i_1}, \cdots, K_{i_m}$; $\{K_q^t\}$ – the sequence $K_1^1, \cdots, K_q^1; \cdots; K_1^t, \cdots, K_q^t$,

(08) $\{i_{w(E)}\}$ or $\{j_{w(E)}\}$ – indices of all free variables occurring in E,

(09) $n(E) = \max\{w(E) + p(E), \max\{i_{w(E)}\}\}$

(010) $E(u/z)$ – the expression resulting from E by the substitution of u for each occurrence of z in E; if z is an apparent variable, then z does not belong in E to the scope of Πz; if u is an apparent variable, then z does not belong to the scope of Πu in E,

(011) $E(\{j_m\}/\{i_m\}) = E(u_{j_1}/x_{i_1}) \cdots (u_{j_m}/x_{i_m})$, $u = x, a$

(012) Sks – the set of all formulas of the form:
$\Pi a_1 \cdots \Pi a_t \Sigma a_{t+1} \cdots \Sigma a_k F$[3], where F is quantifierless expression containing no free variables, Πa_j is the sign of the universal quantifier binding a_j and $\Sigma a_j G = (\Pi a_j G')'$, $j = 1, \cdots, k$,

(013) $C(E)$ – the set of all significant parts of the formula E: $H \epsilon C(E) . \equiv$[4]$. H = E$ or there exist $E_1 \epsilon C\{E\}$ and F, G, H_1 such that:
$(H = F) \wedge (E_1 = F') \vee \{(H = F) \vee (H = G)\} \wedge (E_1 = F + G) \vee (\exists i)\{H = H_1(x_i/a)\} \wedge (E_1 = \Pi a H_1)$,
Of course, each significant part of the formula E is a formula,

(014) $M, M_1, M_2, \cdots$ – models; $T, T_1, T_2, \cdots$, – tables of given rank,

(015) $Q, Q^\circ, Q_k, \cdots$ – non-empty sets of tables of the same rank k,

(016) $A, A_1, A_2, \cdots$ – sets of formulas which indices of individual variables are $\leqslant k$; A may be empty,

(017) S_A – the set of all individual variables which occur in elements of A; if E is an expression, then S_E is the set of individual variables which occur in E; if elements of S_A are all individual variables with indices $\leqslant k$, then we shall say that A has the rank k,

(018) $B, B_1, B_2, \cdots$ – families of sets defined in (016) such that if $A \epsilon B$, then A has the rank k and for each atomic formula E and its negation if $S_E \subset S_A$, then we have; $E \epsilon A . \equiv . E' \bar{\epsilon} A$; if elements of $A's$ only are atomic formulas, and their negations, then we call them "families of the rank k",

(019) (K) – for each K; $(\exists K)$ – there exists K such that, $(\{K_m\})$ – for each $\{K_m\}$;

3 It is Skolem's normal form.

4 Dots separate more strongly than parantheses.

$(\exists \{K_m\})$ —there exists $\{K_m\}$ such that,

For brieviety we shall assume that we consider only $A's$ and $B's$ of a given rank [5].

The pair $<D, \{F_q^t\}>$ denote a model ; i. e. that the domain D is an arbitrary non-empty set and $\{F_q^t\}$ is an arbitrary finite sequence of relations such that F_k^m is a m-ary relation on $D, k=1, \cdots, q$ and $m=1, \cdots, t$. A table of the rank k is a model which domain has exactly k elements which are numbers $\leqslant k$; in the following we shall only consider models which domains are cosets of the set of natural numbers, i. e. $k \leqslant \aleph_0$.

For each model $M=<D, \{F_q^t\}>$ by $M/s_1, \cdots, s_k/$ —or briefly $M/\{s_k\}$—we shall denote a table $<D_k, \{\phi_q^t\}>$ of the rank k such that for each $r_1, \cdots, r_m \leqslant k$:

$\phi_j^m(r_1, \cdots, r_m). \equiv . F_j^m(s_{r_1}, \cdots, s_{r_m}), \quad 1 \leqslant m \leqslant t, \quad 1 \leqslant j \leqslant q$ [6].

We shall also assume that if $s_i \bar{\epsilon} D$, then:

$M/s_1, \cdots, s_k/ = M/s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_k/, i \leqslant k.$

Of course, $M/\{s_k\}$ is a submodel of M in the meaning of homomorphism.

D. 0. $E \in A/s_1, \cdots, s_k/. \equiv . (E(\{s_k\}/\{k\}) \in A) \wedge$ (E is an atomic formula or its negation)

Obviously, if $s_i \bar{\epsilon} s_A$, then:

$A/s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_k / = A / s_1, \cdots, s_k /.$

We shall assume: $A / \{s_k\} = A / s_1, \cdots, s_k /.$

In the following $X, Y, X_1, Y_1, \cdots$ denote a model or a set A, and $U, V, \cdots$ denote Q, or B.

Of course:

L. 1. $X / \{s_k\} / \{j_m\} = X / \{s_{j_m}\}.$

L. 2. If $s_1, \cdots s_i, s_{i+1}, \cdots s_j, s_{j+1}, \cdots s_m$ is a sequence of different natural numbers and $X_1 / \{s_i\} = X_2 / \{s_i\}$, then there exists X such that $X / \{s_j\} = X_1 / \{s_j\}$ and $X / \{s_i\}, s_{j+1}, \cdots, s_m / = X_2 / \{s_i\}, s_{j+1}, \cdots, s_m/.$

D. 1. $R_m X). \equiv . (\{i_m\})(\{j_m\}) \{(X/ \{i_m\} = X / \{j_m\}) \rightarrow (i_1 = j_1) \wedge \cdots \wedge (i_m = j_m)\}$

$R_m(X)$ asserts that $X / \{i_m\}$ are different for different sequences $\{i_m\}$; obviously, if $R_1(X)$, then $R_m(X)$ for all m [8].

Of course, if $R_k(Y)$, then the operations $Y / \{s_k\}$ gives only different elements.

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5 The assumption is not necessary in general.

6 i. e. $M/s_1, \cdots, s_k/ = <D_k, \{\phi_q^t\}>$: if $\{s_k\}$ is empty, then it holds for all models.

7 This assumption may be replacd by : E is a quantifierlees formula, E is an arbitrary formula and other corresponding sets.

8 Examples of $R_m(X)$ may be easily given, see [8], [9].

D. 2 $X \epsilon Y[k]. \equiv .(\exists \{s_k\})(X=Y/\{s_k\})$.

$Y[k]$ is the set of all $Y/\{s_k\}$.

By an extension of a model $<D, \{F_q^t\}>$ we shall call a triple $<D, \{F_q^t\}, F>$ such that F is a function on B and values of F are sequences of numbers 0 and 1 with one number 1 (e.g. given in (L)); such triples we shall also name models, and shall use all notations given above.

(L) Let $<B, \{F_q^t\}>$ be a model and $s_1, s_2, \cdots, s_i, \cdots$ be all different elements of B; let $F(s)=(w_1, w_2, \cdots, w_i, \cdots)$ and $w_i=1. \equiv . s=s_i$; i.e. $w_j=0$ for $j \neq i$.
Let $M=<D, \{F_q^t\}, F>$; of course $R_1(M)$ and therefore also $R_m(M)$ for all m.

Here $M[k]$ has the following property:

There exists only a finite number $\leqslant 2^{qtk^t}$ of tables belonging to $M[k]$ which differ on relations of the model $<D, \{F_q^t\}>$

By an extension of A we shall call a set A° such that:

1. $A \subset A^\circ$,
2. There exists a function $F \epsilon A^\circ$ defined on S_A with values which are sequences of numbers 0 and 1 with one number 1, see (L).
3. Only elements defined in 1. and 2. belong to A°.

If A has the rank k, then we shall say that the extension A° of A also has the rank k and we shall use all notations described above.

In view of (L)—if we replace $<D, \{F_q^t\}>$ by A and M by A°, and D by S_A:

L. 3. Each X may be extended to Y such that $R_1(Y)$; there exists only a finite number $\leqslant 2^{qtk^t}$ of elements of $Y[k]$ which differ themselves upon relations of the calculus [9].

By a coelement we shall call a coset or a submodel; $X(k), U(k), \cdots$ — we shall read "$X, U, \cdots$ —has the rank k"; $v(X), v(U), \cdots$ —denote the rank of $X, U, \cdots$; $\{<s_h\}$ —we shall read "$s_1<s_2<\cdots<s_k$".

D. 3. $UX/\{s_k\}. \equiv . \{k \leqslant v(U)\} \wedge (\exists Y)(\exists \{i_k\})\{(Y \epsilon U) \wedge (Y/\{i_k\}=X/\{s_k\})\}$ [10].

$UX/\{s_k\}$ asserts that $X/\{s_k\}$ is a coelement of some element of U in the meaning in homomorphism.

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9 They are different upon the added function F; another construction of extensions is given in [8], [9] and it replaces the function F by a sequence of monadic relations.

10 If U has the property $k(U)$ defined:
$k(U). \equiv .(X)(\{t_k\})\{(X \epsilon U) \wedge (\{t_k\}$ is a sequence of numbers $\leqslant v(U)) \rightarrow (X/\{t_k\} \epsilon U)\}$,
then $\{i_k\}$ in D. 3. may be replaced by $\{s_k\}$; if $\{s_k\}$ is empty, then $UX/\{s_k\}$

Of course:

L. 4. If $UX/\{s_k\}$ and $\{i_m\}\subset\{s_k\}, m\leqslant k$, then $UX/\{i_m\}$.

D. 4. $\underline{U}\{r\}.\equiv.U(r)\wedge(X)\{X(r)\wedge UX/\{r-1\}\wedge UX/r\rightarrow(\exists Y)(UY/\{r\}\wedge Y(r)\wedge(\{i_m\})\{(\{i_m\}\subset\{r\})\wedge\{<i_m\}\wedge UX/\{i_m\}\rightarrow(X/\{i_m\}=Y/\{i_m\})\}\})\}$

$\underline{U}\{)r\}$ asserts that for all $X(r)$, if $X/\{r-1\}$ and X/r are coelements of some elements of U, then there exists $Y(r)$ such that $Y/\{r\}$ is a coelement of some elment of U and for each $\{<i_m\}$, if $\{i_m\}\subset\{r\}$ and $X/\{i_m\}$ is a coelement [11] of some element of U, then $X/\{i_m\}=Y/\{i_m\}$.

If U only has one elemeut X, then instead of $\underline{U}\{r\}$ we write simply $\hat{\underline{X}}\{r\}$.

L. 5. If $\underline{U}\{r\}$, then:
$(k)(\{s_k\})(X)\{X(k)\wedge(\{s_k\}$ are different numbers$\leqslant k)\wedge UX/\{s_{r-1}\}\wedge UX/s_r\wedge(r\leqslant k)\rightarrow(\exists Y)(Y(k)\wedge UY/\{s_r\}\wedge(\{i_m\})\{(\{i_m\}$ are different numbers $\leqslant k)\wedge(\{i_m\}\subset\{s_r\})\wedge UX/\{i_m\}\rightarrow(X/i_1,\cdots,i_m,s_{r+1},\cdots,s_k/=Y/i_1,\cdots,i_m,s_{r+1},\cdots,s_k/)\}\})\}$.

L. 5. follows from D. 4. by using many times of L. 2; if in L. 4. we have $UX/\{s_r\}$, then $Y=X$.

Of course:

L. 6. If $R_1(X)$ anb $U=X[r]$, then $\underline{U}\{r\}$.

L. 7. If we only consider the monadic first-order functional calculus and $U=X[r]$, then $\underline{U}\{r\}$.

L. 8. If $r\geqslant 2$ and $U=X[r]$, then:

1. If $X(k), i,j\leqslant k, UY/i, UY/j$, then there exists $X_1(k)$ such that $UX_1/i,j$ and:
   $X_1/1,\cdots,i-1,i+1,\cdots,k/=Y/1,\cdots,i-1,i+1\cdots,k/$,
   $X_1/1,\cdots,j-1,j+1,\cdots,k/=Y/1,\cdots,j-1j+1,\cdots,k/$. [12]
2. $\underline{U}\{2\}$,
3. For each X wehave $\hat{\underline{X}}\ \{r\}$.
4. If X is not an extension with added function F, see $p\cdots$, then for each k there exists a coelement Y of X such that $X[k]=Y[k]$ and $v(Y)\leqslant k2^{qtk^t}$.

D. 5. $H//E.\equiv.(\exists\{F_j\})(E=F_1+\cdots+F_i+H+F_{i+1}+\cdots+F_j)\wedge(F)(G)(H\neq F+G)$.

D. 6. $UX/E.\equiv.(\exists r)((v(U)=r)\wedge(H)(\{j_r\})\{(\{j_r\}\subset\{i_{w(H)}\})\wedge H//E\rightarrow UX/\{j_r\}\}$

D. 7. $E/XY.\equiv.\{(v(X)=v(Y))\wedge(\max\{i_{w(E)}\}\leqslant\min\{v(X),v(Y)\})\}\wedge(H)\{H//E\rightarrow(X/\{i_{w(H)}\}=Y/\{i_{w(E)}\})\}$.

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11 In the meaning of homomorphism.

12 L. 8 asserts L. 5., for $r=2$, in more strong form. If $UY/i,j$, then $X_1=Y$.

The sense of D. 5.–D. 7. is clear:

We give an inductive definition of the number $e(E)$, which has an important role in the following considerations:

$e(F)=0$, if F is a quantifierless formula,
$e(F+G)=\max\{e(F), e(G)\}$,
$e(\Pi aF)=e\{F(x/a)\}$, where $x \bar{\epsilon} S_F$,
$e(\Sigma aF)=w(F)+1$, if $e\{F(x/a)\}=0$ [13],
$e(\Sigma aF)=e\{F(x/a)\}$, if $e\{F(x/a)\}\neq 0$ and $x \bar{\epsilon} S_F$,
If $e(E)$ is not defined above, then $e(E)=\max\{e(G)\}$, for each $G\epsilon C(E)$, where if $G=\Pi aH$, then $e(G)=w(H)+1, e(F')=e(F), e(F+G)=\max\{e(F), e(G)\}$.

For examle:

(1°) If $E\epsilon SkS$ and $E'=\Sigma a_1\cdots\Sigma a_i\, \Pi a_{i+1}\cdots\Pi a_k F$, for some F, then $e(E')=i$,
(2°) If $E=\{\Pi af_i^m(a, x_2, \cdots, x_m)+f_j^n(x_1, \cdots, x_n)\}$, then $e(E)=m$,
(3°) $e(E)\leqslant w(E)+p(E)\leqslant n(E)$,
(4°) $e(E)=0. \equiv . E$ is an alternative of formulas of the form $\Pi a_1\cdots\Pi a_m F$, where F is quantifierless on alternative of generalizations of such alternative sets.

For an arbitrary $T=<B_k, \{F_q^t\}>$ of the rank k, for an arbitrary formula E which indices of free variables occurring in it are $\leqslant k$ and for an arbitrary non-empty Q, we introduce the inductive definition of the functional W:

(1d) $W\{T, f_j^m(x_{r_1}, \cdots, x_{r_m}), Q\}=1. \equiv . F_j^m(r_1, \cdots, r_m)$,
(2d) $W\{T, F, Q\}=1. \equiv . \backsim W\{T, F, Q\}=1. \equiv . W\{T, F, Q\}=0$,
(3d) $W\{T, F+G, Q\}=1. \equiv . W\{T, F, Q\}=1 \vee W\{T, G, Q\}=1$,
(4d) $W\{T, \Pi aF, Q\}=1. \equiv . (i)(T_1)\{(i\leqslant k)\wedge F/TT_1 \wedge QT_1/F(x_i/a)$ [14] $\wedge\, T_1(k) \rightarrow W\{T_1, F(x_i/a), Q\}=1\}$.

D. 8. $E\epsilon P(Q, T). \equiv . T\{QT/E \rightarrow W\{T, E, Q\}=1\}$.
D. 9. $E\epsilon P(r). \equiv . (Q)(T)\{Q\{r\}\wedge T(n(E)) \rightarrow (E\epsilon P(Q, T))\}$
D.10. $E\epsilon P. \equiv . E\epsilon P(e(E))$.

We explain the meaning of ones:

1. $W\{T, E, Q\}=1$ may be read: T satisfy E relatively to Q.
2. If M is a model and $Q=M[k]$, then the elements of Q are submodels of M in the meaning of homomorphism, the number i in (4d) is a name of an arbitrary element of the domain of M and in D. 8.–D. 10. we associate to

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13 If $e\{F(x/a)\}=0$, then $e\{F(x_i/a)\}=0$, for each i.
In exact given cases $e(E)$ may be less than defined above.

14 We may replace here F/TT_1 by $T/\{i_{w(F)}\}=T_1/i_{w(F)}\}$ see [8], [9], [14.]

each formula a pair of numbers and assume that we only consider submodels of M.

3. Obviously, if E is quantifier-free, then $E\epsilon P. \equiv .E$ is true.
4. We shall prove that P is the class of all true formulas, see T. 7.

Of course:

(3d') $W\{T, F+G, Q\}=0. \equiv . W\{T, F, Q\}=0 \wedge W\{T, G, Q\}=0,$

(4d') $W\{T, \Pi aF, Q\}=0. \equiv .(\exists i)(\exists T_1)\{(i \leqslant k) \wedge F/TT_1 \wedge QT_1/F(x_i/a) \wedge T_1(k) \wedge W\{T_1, F(x_i/a), Q\}=0\},$

(4d'') $W\{T, \Sigma aF, Q\}=0. \equiv .(i)(T_1)\{(i \leqslant k) \wedge F/TT_1 \wedge QT_1/F(x_i/a) \wedge T_1(k) \rightarrow W\{T_1, F(x_i/a), Q\}=0\}.$

(5d) $E\bar{\epsilon}P. \equiv .(\exists Q)(\exists T)\{Q\{e(E)\} \wedge T(n(E)) \wedge QT/E \wedge W\{T, E, Q\}=0\}.$

L. 9. Let E° results from E by replacing free variables with indices $i_1, \cdots, i_{w(E)}$ conespondingly by free variables with indices $j_1, \cdots, j_{w(E^0)}$, $w(E)=w(E^\circ)$ [15] and T, T° tables such that for some $k \geqslant n(E)$ and some $m \geqslant n(E^\circ)$ we have: $T(k), T^\circ(m)$ and $T/\{i_{w(E)}\}=T^\circ/\{j_{w(E^0)}\}$.

Then:

$$W\{T, E, Q\}=1. \equiv . W\{T^\circ, E^\circ, Q\}=1.$$

The proof of L.9. is easy and is inductivel on the length of the formula E, see [8], [9].

D. 11. $M(A, Q). \equiv .(E)\{(E\epsilon A) \rightarrow W\{M, E, Q\}=1\}.$

$M(A, Q)$ may be read "M is a Q-model of A".

Let $M=<\mathrm{D}, \{F_q^t\}, F>$ and $v(M) \geqslant v(A)$; we shall write $M\{A\}$ if and only if for each $m_1, \cdots, m_j \leqslant v(A)$ and $j \leqslant t, i \leqslant q$ we have:

1. $F_i^j(m_1, \cdots, m_j). \equiv .f_i^j(x_{m_1}, \cdots, x_{m_j})\epsilon A.$
2. If $f_i^j(x_{m_1}, \cdots, x_{m_j})\bar{\epsilon}A$, then $f_i^{j\prime}(x_{m_1}, \cdots, x_{m_j})\epsilon A$.
3. $F(r)=F(x_r)$.

L. 10. For each $k \geqslant 1$:

(1) for each M, if $k \leqslant v(M)$, then the model M determine A such that $v(A)=k$ and $M\{A\}$.

(2) for each A, if $k \geqslant v(A)$, then the set A determine M such that $v(M)=k$ and $M\{A\}$.

(3) If $M\{A\}$, see both cases (1) and (2), then $M(A, Q)$ for each non-empty Q.

L. 11. If $A(k)$ and $T=M/1, \cdots, k/$, then $T\{A\}. \equiv . M\{A\}$.

L. 12. If $v(M_1)=v(M_2), v(A_1)=v(A_2), M_1\{A_1\}$ and $M_2\{A_2\}$, then: If $M_1/\{i_j\}=M_2/\{i_j\}$, then $A_1/\{i_j\}=A_2/\{i_j\}$; if also $i_1, \cdots, i_j \leqslant v(A_1)$, then the last implication may be replaced by the equivalence.

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15 Then E results from E° by an inverse substitution.

D. 12. $Q\{B\}. \equiv .(A)(\exists M)\{(A\epsilon B)\rightarrow(M\epsilon Q)\wedge M\{A\}\}\wedge(M)(\exists A)\{(M\epsilon Q)\rightarrow(A\epsilon B)$ $\wedge M\{A\}\}$.

L. 13. If $Q(B)$ and $r=v(B)=v(Q)$, then: $\underline{B}\{r\}. \equiv .\underline{Q}(r)$,

The proof of the above lemmas is immediately.

For an arbitrary number k we shall assume that the sequence of different elements

(K) $A_1, A_2, \cdots, A_t, \cdots$

includes all sets A of the rank k[16]; i.e. we assume that we enumerated all sets of the rank k which we have a denumerable number.

Because we only consider the enumeration (K), therefore we shall sometimes omit indices of $A's$.

For an arbitrary A and B; and for each formula E we introduce the symbol $A, B \vdash E$ which we read. "the formula E is a thesis of A respectively to B"

(11) $A, B \vdash F$, for each formula $F\epsilon A$,

(12) $A, B \vdash F+F'$. for each formula F,

(13) If $A, B \vdash F_1 + \vdots + F_m$ and $k_1, \cdots, k_m$ is an arbitrary permutation of numbers $\leqslant m$, then $A, B \vdash F_{k_1}+\cdots+F_{k_m}$[17].

(14) If $A, B \vdash F$ and G is a formula, then $A, B \vdash F+G$,

(15) If the following conditions are satisfied:

(1°) $A, B \vdash F+G, A, B \vdash F+G', G'$ occurs in F or $w(G') \leqslant e(F)$,

(2°) If B is non-empty, and G'' does uot occur in F, then for all A°, if $v(A^\circ) = v(A), F/A^\circ A$ and $BA^\circ/F+G'$, then $A^\circ, B \vdash F+G$ and $A^\circ, B \vdash F+G'$,

then $A, B \vdash F$[18],

(16) If the following conditions are satisfied:

(1°) $A, B \vdash F+G, x_m \bar{\epsilon} S_F$;

(2°) If B is non-empty, then for all A°, if $v(A^\circ)=v(A), F+\Pi aG(a/x_m)/A^\circ A$ and $BA^\circ/F+G$, then $A^\circ, B \vdash F+G$,

(3°) If B is non-empty, then for each $x\epsilon S_{G(a/x_m)}$ we have $A, B \vdash F+G(x/x_m)$,

then $A, B \vdash F+\Pi aG(a/x_m)$.

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16 See footnote 19; for an arbitrary B we may assume that we only consider such $A's$ that $(\exists s)(BA/s)$.

17 By a simple modification of D. 13. the rule (13) may be replaced by two usual laws, see [8].

18 In the first reading the reader may assume less strong form of the rules.
Here also $A^\circ, B \vdash F$.
In (17) we may also assume only $w(G) \leqslant c(F)$ instead of $w(G) < c(F)$.
All considerations remain true if we replace in all rules the number $c(F)$ by the number $c(E)$ which occur in D. 13.

(17) If the following conditions are satisfied:

(1°) $A, B \vdash F+\Pi aG$ and $w(G) < e(F)$,

(2°) If B is non-empty, then for all A°, if $v(A^{\circ})=v(A), F+G(x_m/a) / A^{\circ}A$ and $BA^{\circ} / F+\Pi aG$, then $A^{\circ}, B \vdash F+\Pi aG$,

then $A, B \vdash F+G(x_m/a)$,

(18) If the following conditions are satisfied:

(1°) $S_F+G(x_m/a) \subset S_A$,

(2°) If B is non-empty, then there exists A° such that $v(A^{\circ})=v(A)$, $F+\Sigma aG / A^{\circ}A, BA^{\circ}/F+G(x_m/a)$ aud $A, B \vdash F+G(x_m/a)$,

then $A, B \vdash F+\Sigma aG$.

If B is empty. then:

(1′) The rules (12)–(15) are proof rules of the propositional calculus.

(2′) The rules (12)–(17) are proof rules of the first-order functional calculus.

D. 13. The doubly sequence $E_{i1}, \cdots, E_{in_i}, i=1,2,\cdots$ is a formal proof of the formula E in A_j respectively to B if and only if $E=E_{jn_j}, j=1,2.\cdots, e(E) \geqslant e(H)$ for each H occurring in the formal proof, and for each $i=1,2,\cdots$ and $k=1,\cdots,n_i$ one of the following conditions is satisfied:

1. E_{ik} is an element of A_i, see (K), or $E_{ik}=F+F'$, for some F,
2. There exists $l<k$ such that E_{ik} reshlts from E_{il} by meaus of one of the rules (13), (14).
3. There exist $l, n<k$ such that E_{ik} results from E_{jl} and E_{cn} by means of the rule (15), $c=1,2,\cdots$
4. E_{ik} results from the doubly sequence $E_{c1},\cdots,E_{ck-1}, c=1,2,\cdots$ by means of the rule (16) or (17).
5. There exists $l<k$ and n such that E_{ik} results from E_{nl} by means of the rule (18).

D. 14. The formula E is a thesis of A_i respectively to B–in symbols: A_i, $B \vdash E$ – if and only if there exists a formal proof of E in A_i respectively to B.

The usual defiuition of the thesis we shall obtain by:

D. 15. The formula E is a thesis–in symbols: $\vdash E$–if and only if E is a thesis of A_i respectively to B, for some i, and A_i, B are empty.

(For the propositional calculus, B in D. 15. may be arbitrary.)

By an interval E_{is} of the doubly sequence $E_{j1},\cdots,E_{jn_j} j=1,2,\cdots$, we shall call the sequence $E_{i1},\cdots,E_{is-1}$.

D. 16 The formula E is a B–thesis–in symbols: $B \vdash E$–if and only if

19 Therefore it suffices to consider in (K) only A_j with the condition given in D. 16. and footnote 16.

for all $j=1,2,\cdots$, if $v(A_j)=n(E)$ and BA_j/E[19], then E is a thesis of A_j respectively to B.

D. 17. The formula E is a r–thesis if and only if for each B, if $B\{r\}$, then E is a B–thesis.

D. 18. The formula E is a e–thesis–in symbols: $\vdash eE$–if and only if E is a $e(E)$–thesis.

Of course:

T. 1. If $\vdash E$, then for each $j=1,2,\cdots$, we have: $A_j, B\vdash E$; therefore also $\vdash eE$.

The basis of a converse theorem is:

T. 2. If $A_j, B\vdash E$, $T(A_j, Q)$, $Q\{B\}$, $\underline{Q}\{e(E)\}$, $B(e(E))$ and $v(T)\geqslant n(E)$, then $E\epsilon P(Q, T)$.

T. 2. may be read: If T is a Q–model of A_j, $Q\{B\}$, $\underline{Q}\{e(E)\}$, $B(e(E))$ and $v(T)\geqslant n(E)$ and QT/E, then T is a Q–model for the thesis E of A_j respectively to B.

Proof: Let the assumptions of T. 2. hold; therefore in view of L. 13. also $\underline{B}\{eE)\}$.

Let $E_{i1},\cdots,E_{in_i}$, $i=1,2,\cdots$ be a formal proof of the formula E in A_j respectively to B.

In view of D. 13. we may assume that $e(E)\geqslant e(H)$, for each H occurring in the formal proof given above.

First of all we consider the case $r=e(E)>0$; in this case we shall prove T. 2. by induction on $s=n_i$–simultaneously for all $i=1,2,\cdots$

Of course, if $n_i\leqslant s$ and $E_{in_i}=F+F'$ or $E_{in_i}\epsilon A_i$, then in view of the assumption, T. 2. holds.

In view of D. 13. it suffices to verify:

(1′) If E_{is} results from E_{ik}, $k<s$, by means of the rule (13) or (14) and T. 2. holds for E_{ik}, then it holds for E_{is}.

(2′) If $E_{im}=F+G$, $E_{ik}=F+G'$, G' occurs in F or $w(G')\leqslant e(F)\leqslant r$, $E_{is}=F$, $m, k<s$, and for for all $l=1,2,\cdots$, if F/A_lA_i, $BA_l/F+G'$ and G' does not occur in F, then $E_{lm}=F+G$ and $E_{lk}=F+G'$, and T. 2. holds for all $n<s$, then it also holds for s.

(3′) If $E_{ik}=F+G$, $x_m\bar{\epsilon}S_F$, $k<s$, $E_{is}=F+\Pi aG(a/x_m)$, formulas $F+G(x/x_m)$ occur in the interval E_{is} for all $x\epsilon S_{G(a/x_m)}$, and for each $l=1,2,\cdots$, if $F+\Pi aG(a/x_m)/A_lA_i$ and $BA_l/F+G$, then $F+G$ occurs in the interval E_{ls} and T. 2. holds for

$n<s$, then it also holds for s.

(4′) If $E_{ik}=F+\Pi aG$, $w(G)<e(F)$, $E_{is}=F+G(x_m/a)$, $k<s$, and for each $l=1,2,\cdots$, if $F+G(x_m/a)/A_l A_l$ and $BA_l/F+\Pi aG$, then $E_{lk}=F+\Pi aG$, and T. 2. holds for $n<s$, then it also holds for s.

(5′) If $S_{F+G(x_m/a)}\subset S_{A_l}$, $E_{is}=F+\Sigma aG$ and there exists l such that $F+\Sigma aG/A_l A_l$, $BA_l/F+G(x_m/a)$ and $E_{lk}=F+G(x_m/a)$, for some $k<s$, and T. 2. holds for $n<s$, then it also for s.

First of all it is obviously that (1′) is true.

2′. Let the assumptions of (2′) hold and $W\{T^\circ, F, Q\}=0$, for some T° such that QT°/F and $v(T^\circ)\geqslant n(F)$; hence $T^\circ\{A_l\}$.

The case G' occurs in F is immediately.

Therefore it suffices to consider $w(G')\leqslant e(F)\leqslant r=e(E)$.

In view of the above and L. 2 there exists T such that F/TT°, QT/F, $v(T)\geqslant n(F+G)=n(F+G')$, QT/c for all $c\leqslant v(T)$ and $v(T)\geqslant v(A_l)$.

Because QT/F and QT/c for all $c\leqslant v(T)$, and $\underline{Q}\{e(E)\}$, therefore using L. 5. we conclude that there exists T_1 such that $v(T_1)=v(T)$, QT_1/G' and F/T_1T.

Hence by virtue of the above, (3d') L. 3. and L. 9. we obtain:

$W\{T_1, F, Q\}=0$, $QT_1/F+G+G'$ and therefore either $W\{T_1, F+G, Q\}=0$ or $W\{T_1, F+G', Q\}=0$; i. e. either $F+G\,\bar{\epsilon}\,P(Q, T_2)$ or $F+G'\,\bar{\epsilon}\,P(Q_1 T_1)$

Because in view of L. 10. there exists l such that $T_1\{A_l\}$, therefore by virtue of L. 12, $Q\{B\}$, $T^\circ\{A_l\}$ and $v(Q)=v(B)$ we have F/A_lA_l and $BA_l/F+G+G'$; because in view of the above and assumptions of (2′) formulas $F+G$ and $F+G'$ occur in the interval E_{ls} and T. 2 holds for ones therefore we have a contradiction with the above conclusions.

3′. Let the assumptions of (3′) hold and $W\{T, F+\Pi aG(a/x_m), Q\}=0$, for some T such that $QT/F+\Pi aG(a/x_m)$ and $v(T)\geqslant n(F+\Pi aG(a/x_m))$; hence $T\{A_l\}$.

In view of L. 9. we may assume $v(T)\geqslant n(F+G)$, $v(T)\geqslant m$ and $v(T)\geqslant v(A_l)$

By virtue of (3d') also $W\{T, F, Q\}=0$, $W\{T, \Pi aG(a/x_m), Q\}=0$.

Therefore by virtue of (4d') there exist $i\leqslant v(T)$ and T_1 such that $v(T_1)=v(T)$, $G(a/x_m)/TT_1$, $QT_1/G(x_i/x_m)$ and $W\{T_1, Q, G(x_i/x_m)\}=0$. Hence, if $x_i\,\epsilon\, S_{G(a/x_m)}$, then in view of L. 9. and (3d'): $W\{T, G(x_i/x_m), Q\}=0$ and $W\{T, F+G(x_i/x_m), Q\}=0$; therefore in view of the above $F+G(x_i/x_m)\,\bar{\epsilon}\,P(Q, T)$ and this is inconsistent with the assumption of (3′), because $F+G(x_i/x_m)$ occurs in the interval E_{is}.

Therefore it remains to consider the case $x_i\,\bar{\epsilon}\,S_{G(a/x_m)}$; in view of L. 2., we may assume $G'(a/x_m)/TT_1$, s. footnote 14.

Let

$$T_2=\begin{cases} T_1, \text{ if } i=m \\ T_1/1,\cdots,i-1,m,i+1,\cdots,m-1,i,m+1,\cdots v(T)/, \text{ if } i\leqslant m. \end{cases}$$

(We note that $i\geqslant m$, then we permute i and m.)

Hence by L. 1. $T_2/\{i_{w(G)}\}=T_1/\{i_{w(G(a/x_m))}\},i/$, where $\{i_{w(G)}\}=\{i_{w(G(a/x_m))}\},m$. Therefore in view of L. 2. there exists T° of the rank $v(T)$ such that $T^\circ/\{i_{w(G)}\}=T_2/\{i_{w(G)}\}$ and $T^\circ/\{j_{w(F)}\}=T/\{j_{w(F)}\}$' Because $QT_1/G(x_i/x_m)$, therefore the above proves $QT^\circ/F+G$ and by L. 9. and (3d') $W\{T^\circ,F+G,Q\}=0$; hence $F+G\bar{\epsilon}P(Q,T^\circ)$.

Because in view of L. 10. there exists l such that $T^\circ\{A_l\}$, therefore by virtue of L. 12, $Q\{B\}$, $T\{A_i\}$ and $v(Q)=v(B)$ we obtain $F+\Pi aG(a/x)/A_t A_i$ and $BA_1/F+G$; hence $F+G$ occurs in the interval E_{ls} and analogously to (2′) we have a contradiction.

4′. Let the assumptions of (4′) hold and $W\{T,F+G(x_m/a),Q\}=0$, for some T such that $QT/F+G(x_m/a)$ and $v(T)\geqslant n(F+G(x_m/a))$.

In view of L. 9. we may assume $v(T)\geqslant n(F+\Pi aG)$ and $v(T)\geqslant v(A_i)$,

Hence by (3d') $W\{T,F,Q\}=0$ and $W\{T,G(x_m/a),Q\}=0$.

Therefore in view of the above and (4d'), $W\{T,\Pi aG,Q\}=0$ and by (3d') $W\{T,F+\Pi ag,Q\}=0$.

If G is not an alternative of some formulas, then in view of the above and L. 4. also $QT/F+\Pi aG$ and therefore $F+\Pi aG\bar{\epsilon}P(Q,T)$ which contradicts with the assumption—because $F+\Pi aG$ occurs in the interval E_{is}.

If G is an alternative of some formulas, then by virtue of $w(G)<r$ and L. 5. we obtain that there exists T_1 such that $v(T)=v(T_1)$, $QT_1/F+\Pi aG$ and $F+G(x_m/a)/TT_1$, and also $QT_1/G(x_m/a)$.

Therefore in view of (3d'), L. 9. and the above $W\{T_1,F+G(x_m/a),Q\}=0$, and by (3d), (4d') $W\{T_1,F+\Pi aG,Q\}=0$; therefore also $F+\Pi aG\bar{\epsilon}P(Q,T_1)$. By virtue of L. 10. there exists l such that $T_1\{A_l\}$; therefore by virtue of L. 12, $Q\{B\}$, $T\{A_l\}$, $v(Q)=v(B)$ and the above $F+G(x_m/a)/A_lA_t$ and $QA_l/F+\Pi aG$. Therefore in view of the assumptions of (4′), $F+\Pi aG$ occurs in the interval E_{ls} which give a contradiction with the above conclusions.

5′. Let the assumption of (5′) hold and $W\{T,F+\Sigma aG,Q\}=0$, for some T such that $QT/F+\Sigma aG$ and $v(T)\geqslant n(F+\Sigma aG)$; hence $T\{A_i\}$.

In view of L. 9. we may assume $v(T)\geqslant n(F+G(x_m/a)$ and $v(T)\geqslant v(A_l)$.

By virtue of (5′) there exists l sueh that $v(A_l)=v(A_l)$, $F+\Sigma aG/A_lA_1$, $BA_l/F+G(x_m/a)$ and $F+G(x_m/a)$ occurs in the interval E_{ls}.

In vitew of L. 10. there exists T° such that $v(T^\circ)=v(T)$ and $T^\circ\{A_l\}$.

Because $S_{F+G(a_m/x)}\subset S_{A_i}$, therefore in view of L. 12. $Q\{B\}$, $T\{A_i\}$, $v(Q)=v(B)$ and the above we have $F+\Sigma aG/T^\circ T$, $QT^\circ/F+G(x_m/a)$. Hence and in view of

L. 9, (3d') and the above $W\{T^\circ, F, Q\}=0$ and $W\{T^\circ, \Sigma aG, Q\}=0$; therefore by (4$d''$) $W\{T^\circ, G(x_m/a), Q\}=0$. and by (3$d'$) also $W\{T^\circ, F+G(x_m/a), Q\}=0$ and $F+G(x_m/a)\bar{\epsilon} P(Q, T^\circ)$.

Because $F+G(x_m/a)$ occurs in the interval E_{ls}, therefore in view of $T^\circ\{A_l\}$ we have a contradiction with the assumptions of (5').

In view of the above and (4°), p. 50, the case $e(E)=0$ is obvious; q. e. d.

T. 3. If E' is not a thesis, then there exists a set J of formulas such that $E\epsilon J$ and:

(1) $F\epsilon J. \equiv . F'\bar{\epsilon}J$, for each F'.

(2) $F+G\epsilon J. \equiv . F\epsilon J\vee G\epsilon J$, for each $F+G$.

(3) $\Pi aF\epsilon J. \equiv . (i)\{F(x_i/a)\epsilon J\}$, for each ΠaF.

The proof of T. 3. is given in [5] and [7].

T. 4. Let J be the set defined in T. 3. and $E\epsilon J$; let $E\epsilon Sks, F\epsilon C(E), k=n(E)\geqslant n(F)$ and $B=J[e(E')]$; then:

(1) If $k=v(A), J/\{s_{t_{w(F)}}\}=A/\{i_{w(F)}\}$ and $F(\{s_{t_{w(F)}}\}/\{i_{w(F)}\})\epsilon J$, then $A, B\vdash F$.

(2) $B\vdash E$ and E' is not a 2-thesis.

(3) If $R_1(J)$, then it is not true that $\vdash eE'$.

Proof: — First of all we note that (2) follows from assumptions, T. 2, (1) and L. 8.;, we note here that each $B(k)$ determine $Q(k)$ such that $Q\{B\}$.

However (3) follows from (2), L. 6. and T. 2.

We shall prove (1) by induction on the number of quantifiers occurring in F. If $F\epsilon C(E)$ and F is quantifierless, then (1) holds; we note here that $A(k)$ and if it is not true that $A, B\vdash F$, then in view of the deduction theorem for the propositional calculus and T. 3. (1), (2) we obtain a contradiction.

Therefore it suffices to verify that if (1) holds for $F(x_i/a)\epsilon C(E)$ then it also holds for formulas belonging to $C(E)$ of the form:

(1') ΣaF,

(2') ΠaF.

In the case (1') by virtue of the assumption, T. 3. (3), L. 1. and definitions we obtain:

If $J/\{s_{t_{w(F)}}\}=A/\{i_{w(F)}\}$ and $\Sigma aF(\{s_{i_{w(F)}}\}/\{i_{w(F)}\})\epsilon J$, then $(\exists i)(\exists s_i)\{(i\leqslant k)\wedge (x_i\bar{\epsilon}S_F)\wedge F(x_i/a)(\{s_{i_{w(F)}}\}/\{i_{w(F)}\})(x_{s_t}/x_t)\epsilon J\}$, then $(\exists i)(\exists s_i)(\exists A^\circ)\{(i\leqslant k)\wedge (x_i\bar{\epsilon}S_F)\wedge (J/\{s_{i_{w(F)}}\}, s_i/=A^\circ/\{i_{w(F)}\}, i/)\wedge (F(x_i/a)(\{s_{i_{w(F)}}\}/\{i_{w(F)}\})(x_{s_t}/x_t)\epsilon J)\wedge BA^\circ/F(x_i/a)\}$, then $(\exists i)(\exists A^\circ)\{(i\leqslant k)\wedge (x_i\bar{\epsilon}S_F)\wedge \Sigma aF/A^\circ A\wedge BA^\circ/F(x_i/a)\wedge A^\circ, B\vdash F(x_i/a)\}$; therefore in view of (18): $A, B\vdash \Sigma aF$.

20 i. e. here $BA^\circ/\{i_{w(F)}\}, i$ and $A^\circ/\{i_{w(F)}\}, i/\epsilon B$; we consider here E' as Skolem's normal form for theses.

In the case (2') by virtue of $\Pi aF \epsilon C(E), E \epsilon Sks$, T. 3. (3), $E \epsilon J, B = J[e(E')]$ and of the assumption we obtain that for an arbitrary $i \leqslant k$ and each A° such that $v(A^\circ) = v(A) = k$ and $BA^\circ / F(x_i/a)$ [20] we have $A^\circ, B \vdash F(x_i / a)$; therefore in view of (16): $A, B \vdash \Pi aF$ for all A such that $BA / \Pi aF$; q. e. d.

T. 5. If E is a $r-$thesis, $r \geqslant e(E) > 0$, then it is a $r+1-$thesis.

Proof: – Let $\underline{B}\{r+1\}$ holds and let

$$A \epsilon B_1 . \equiv . (\exists A^\circ)(\exists \{t_r\}) \{(A^\circ \epsilon B) \wedge (t_1, \cdots, t_r \leqslant r+1) \wedge (A = A^\circ / t_1, \cdots t_r /)\}.$$

It is easy to see that $\underline{B}_1\{r\}$ and in view of D. 4. if $B_1 \vdash E$, then $B \vdash E$; q. e. d.

L. 14. For each formula E which has no free variables may be written down a formula $F = G'$, for some $G \epsilon Sks$, such that $E' + F$ is a thesis and E is a thesis if and only if F is a thesis (it is possible to assume $e(F) \geqslant e(E)$), see [2].

T. 6. $\vdash E . \equiv . \vdash eE$.

Proof; – In view of T. 1. it suffices to verify:

(1) If $\vdash eE$, then $\vdash E$.

In view of (4°), $p \cdots$, the case $e(E) = 0$ is obvious.

Therefore we shall only consider the second case $e(E) > 0$.

The sign " $\nvdash$ " we shall read: not $\vdash$.

We are giving the proof of (I) for sentences:

Let $\vdash eE$ holds and let $\nvdash E$; let F be the Skolem's normal form for E determined by L. 14. and such that $e(F) \geqslant e(E)$; therefore $\nvdash F$. Hence and in view of L. 3, L. 6. and T. 3.–4. there exists B such that $\underline{B}\{e(F)\}$ and $B \nvdash F$. Therefore in view of L. 14. and T. 5: $\nvdash eE$, what is impossible.

The general case we shall obtain by simple modyfication of the above considerations; we also note above that $B \vdash F'$ and use here T. 2. and the following lemma:

If $T/1, \cdots, j-1, i, j+1, \cdots, k/ = T^\circ$, then:

$$W\{T, E(x_i/x_j), Q\} = 1 . \equiv . W\{T^\circ, E, Q\} = 1.$$

The easy inductive proof of the lemma and all other details we remain for the readers; q. e. d.

In view of L. 10. and L. 13. if $v(Q) = v(B)$, then each $\underline{Q}\{r\}$ determine $\underline{B}\{r\}$ and each $\underline{B}\{r\}$ determine some $\underline{Q}\{r\}$ such that $Q\{B\}$; Therefore from T. 3. T. 4. and T. 6. follows.

T. 7. $\vdash E . \equiv . E \epsilon P$.

T. 7. may be also proved analogic to [8], [9].

It is easy to show:

1. T. 6–7. remain true if we only consider $U's$ with property given in L. 3. for $Y[k]$.
2. If $E \epsilon Sks$ and $E' = \Sigma a_1 \cdots \Sigma a_i \Pi a_{i+1} \cdots \Pi a_k F$, then E' is a thesis if and only if: (1°) it is a i-thesis; (2°) $E \epsilon P(i)$.
3. P (1)$\subset$P (2)$\subset \cdots$/ analogously to T. 5. /
4. The monadic first-order functional calculus and 1 - theses, 2 - theses–i. e. the classes P (1) and P (2)–are decidable; it follows from L. 7, L. 8. and T. 6–7.

In view of L. 8. we obtain that in 4. we may only consider U oneelementong such that $v(U) \leqslant k2^{qtk^t}$, where $k = e(E)$; one's are analogical results to Godel's [1].

Because all theses of the form $\Sigma a_1 \Sigma a_2 \Pi a_3 \cdots \Pi a_k F$, where F is quantifier and free veriable–free, are 2 - theses / belong to P (2) / therefore 4. and T. 6–7 are generalizations of Godel's and Kalmar's results [1], [3].

It is known, [1], [2], that the decidability of 3 - theses (of P (3)) is equivalent with the decidability of the first-order functional calculus and therefore we may restrict our consideration to $B\{3\}$ and $Q\{3\}$; and by virtue of the undecidability theorem; which we shall describe, the function W is not computable (B and Q may be infinite, see L. 3, L. 6, T. 4)

The above considerations give two possibilities for decidabling of formulas: the syntactic way described in T. 6. and the semantic way given in T. 7., see [13], [14]; both ways may be described in λ_0 propositional calculus.

The undecidability theorem is based on theory of algorithms, [15].

By an algorithmic calculation we do the following elementary operations:

1. We choose an algorithm: an alphabet, expressions and algoritmic rules of transformation.
2. We distinguish different signs.
3. We indicate a definite sign and replace one by another sign.
4. We remember the obtained results.

Alphabet of the considered algorithmic language is composed on three signs: o, l, s.

We explain that by means of s we shall only indicate an expression and a sign.[21]

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21 The considered algorithmic language is based on 1/1 Turing machines, see [3] : it is easy to describe another one.
In my other paper T. 4. and other conclusions are proved for the first-order functional calulus with termes and on such basis we have a new way to the algorithmic theory.

Expressions are finite sequence of signs: o, l, s; for example: olso, loosllos,$\cdots$

Expressions we shall denote by : p, r, p_1, r_1, $\cdots$

Numbers are : 0=1, 1=11, 2=111, $\cdots$; e.g. 111111 denote the number 5.

For an arbitrary n, arbitrary expressions p, r, and i, $j=1, 2, \cdots, n$ we consider the following algoritmic rules:

(A1)	ispor $\rightarrow$ jspsr	,		
	isposr $\rightarrow$ jsplsr	,	isplsr $\rightarrow$ jsposr,	
	isposr $\rightarrow$ jspsor	,	isplsr $\rightarrow$ jspslr,	
	isposr $\rightarrow$ jspslr	,	isplsr $\rightarrow$ jspsor,	
	isposxr $\rightarrow$ jspoxsr	,	isplsxr $\rightarrow$ jsplxsr	$x=o$, 1
	isposxr $\rightarrow$ jsplxsr	,	isplsxr $\rightarrow$ jspoxsr	
	ispos $\rightarrow$ jspoos	,	ispos $\rightarrow$ jsps	

p, r may be empty; exception the first, third, fourth line and the last rule for p.

The rules (A1) assert that the antecedent of each rule may be replaced by it succedent; th rules (A1) are not independent /e.g. the first rule and third line depend from others/, but they describe all possibilities of a working Turing machine with two signs: 0, 1; therefore it may be proved:

(A11) isps $\rightarrow$ jsrps, isprws $\rightarrow$ jspwrs, ispos $\rightarrow$ jspls, ispls $\rightarrow$ jspos, isps $\rightarrow$ jspos, ispos $\rightarrow$ jsps, $i, j=1, \cdots, n$.

Of course, the rules (A11) enables all intuitive algorithmic transformation of an expression composed on 0 and 1; therefore it may be used (A11 instead[22] of (A1).

Each rule we shall call a sequent and denote by: $S_1, S_2, \cdots$.

A finite sequence $\{S_n\}$ of consistent sequents given in (A 1), we shall call an algorithm, and shall denote by: $f, g. f_1, g_1, \cdots$

A finite sequence $p_1, \cdots, p_k$ of expressions is an algorithmic transformation in the algorithm $f=\{S_n\}$ if and only if $p_1=$1sps, $p_k=$0srs for some p. r. and for each i=1, $\cdots$, k, there exists $j \leqslant n$ such that $S_j=p_i \rightarrow p_{i+1}$.

An algorithm f transformate p in r—in symbols: $r=f(p)$—if and only if there exists an alorithmic transformation $p_1, \cdots, p_k$ such that $p_1=$1sps and $p_k=$ 0srs; then r we call the value of f in p.

It is easy to show:

If $r_1=f(p)$ and $r_2=f(p)$, then $r_1=r_2$.

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22 By means of (All) it is very easy to define addition, multiplication, $\cdots$ and to prove their properties.

By the definition domain D_f of the algorithm f we shall call the set of all p such that $f(p)=r$, for some r.

If elements of D_f has the form $om_1o\cdots m_{j-1}om_j$, where $m_1,\cdots,m_j$ are numbers, then f we call a partial rekursive function of j-arguments.

If elements of D_f are all expressions composed on o and 1, then we shall say that f is computable; if D_f is here the set of all j—tuples $om_1o\cdots m_{j-1}om_j$ of all natural numbers, then f we call general rekursive function of j—arguments.

Algorithmic relation R is defined by an equation $f=0$, where f is an algorithm; if f is computable, partial or general rekursive, then R is respectively computable, partial or general rekursive.

Each theory of functional calculi may be developed in the language of algorithms:

Let us read:

1. 2, 3, 5—respectively as: s, o and 1.
2. 5, 55 (=111111111111), 555, ⋯—numbers 0, 1, ⋯; we shall denote $\bar{0}=5, \bar{1}=55,\cdots$
3. 7, 11, 13—respectively as negation, alternative and quantifier,
4. 17o1, 17o11, ⋯—free variables.
5. 19o1, 19o11, ⋯—apparent variables.
6. 23omo1, 23omo11, ⋯—signs of relations of m—arguments, $m<17$.

Constant terms are:

1. 2,3 and numbers $\bar{1}, \bar{2}, \cdots$
2. If p, q are constant terms, then pooq are constant terms.

(*) Terms are constant terms and all expressions resulting from one of (A1) by replacing[23] $i, j, s, o, 1, p, r$ respectively by $\bar{i}, \bar{j}, 2, 3, 5, u, z$, where u, z are variables, and writing between each two last signs the expression oo; e.g. ispsr we replace by $\bar{\text{i}}$oo2oouoo2ooz, ispos by $\bar{\text{i}}$oo2oouoo3oo2, jspslr by $\bar{\text{j}}$oo2oouoo2oo5oou, ⋯

Atomic expressions are of the form Rooo$t_1$ooo$t_2\cdots$ooot_m, where R is a relation sign of m-arguments and $t_1,\cdots,t_m$ are terms defined above.

If E, F are expressions, then 7ooooE, 11ooooEooooF and 13ooooaooooF are expressions (a-apparent variable).

The further description of the above first-order functional calculus with defined terms x—simply: Ct—is identical with the usuel one, but in the above description we may only substitute variables and constant terms for variables.

23 Using (AII) we obtain simpler terms; we are omitting parantheses by using the Lukasiewicz's notations.

We see that all expressions of Ct are composed on o and 1, and therefore are expressions of the algorithmic language.

Hence, the metamathematical notions may be consider in this language (a sequence $E_1, \cdots, E_k$ of formulas we may write in the form $E_1ooooo E_2 \cdots ooooo E_k$).

The completeness theorem of Ct may be proved analogic to [5], [7]; we note here that if the semantic interpretation of x, y are respectively x°, y°, then the semantic interpretation of xy is $x^\circ y^\circ$.

The theory is decidable - according to A. Church - if and only if the set of its all theorems is computable.

It is easy to reduce the decidability of the first-order functional calculus to the decidability of Ct; therefore it suffices to prove the undecidability theorem for Ct:

The algorithmic rules (A1) may be described in Ct:

Let E be relation of two arguments and let $\bar{p}$ be the expression resulting from p by transformation described in (*).

By replacing each sequent $p \rightarrow r$ of (A1) by $E(\bar{p}, \bar{r})$ we obtain a finite sequence $\overline{A1}$ of atomic formulas.

Therefore each algorithm f determine a consistent subsequence $\bar{f}$ of $\overline{A1}$. Let F be the conjuction of all formulas of $\bar{f}$ and K_f the conjuctien of two formulas[24]: $\Pi a_1 \Pi a_2 \Pi a_3 \{E(a_1, a_2) E(a_2, a_3) \supset E(a_1, a_3)\}$ and $\Pi a_1 \cdots \Pi a_n F(\{a_n\}/\{x_n\})$, where $\{x_n\}$ are all variables occurring in F.

Of course:

$$r = f(p). \equiv . K_f \vdash E(\overline{1sps}, \overline{0srs},)$$

and in view of the deduction theorem:

$$r = f(p). \equiv . \vdash K_f \supset E(\overline{1sps}, \overline{0srs})$$

Because s is an indicate letter, therefore in the following we shall omit s; therefore the last equality we shall write in the form:

(0) $$r = f(p). \equiv . \vdash K_f \supset E(\overline{1p}, \overline{0r}).$$

T. 8. The calculus Ct is undecidable.

Proof:—Let Ct be decidable; therefore there exists a computable algotithm f with two values 0 and 1 such that:

(00) $$f(p) = 0. \equiv . \vdash p, \; f(p) = 1. \equiv . \not\vdash p.$$

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24 For simplification we use the usual symbols which are names of the respective signs; conjuction and implication are defined in the usual way.

Let $g(x)$ be an algorithm such that if p is an expression, then $g(p)=p(\overline{p/x})$; it is easy to see that g is computable.

Let fg be the supperposition of two algorithmes f and g; i.e. fg results from f and g by replacing each sequent $ip \to 0r$ of g by $ip \to (k+1)r$ and each sequent $ip \to jr$ of f by $(i+k)p \to (j+k)r, j \neq 0$, and $ip \to 0r$ by $(i+k)p \to 0r$, where k is the number of sequents of g. Therefore fg also is computable and by virtue of (0):

(000) $f\{g(p)\}=0. \equiv . \vdash K_{fg} \supset E(\overline{1p}, \overline{00})$,

$f\{g(p)\}=1. \equiv . \vdash K_{fg} \supset E(\overline{1p}, \overline{01})$.

Let $m=K_{fg} \supset E(\overline{1x}, \overline{01})$; therefore $g(m)=K_{fg} \supset E(\overline{1m}, \overline{01})$.

We note that the last formula asserts that it is not a thesis.

If $f\{g(m)\}=1$, then in view of (00): $\not\vdash g(m)$, i.e. $\not\vdash K_{fg} \supset E(\overline{1m}, \overline{01})$.

Hence, in view of (000): $f\{g(p)\} \neq 1$, which is impossible.

If $f\{g(m)\}=0$, then $\vdash g(m)$, i.e. $\vdash K_{fg} \supset E(\overline{1m}, \overline{01})$.

Therefore in view of (000): $f\{g(m)\}=1$, which is imposible; q.e.d.[25]

The proved undecidability theorem of the first-order functional calculus shows that the process of extending of a set B to $B_1\{r\}$, where $r=e(E) \geqslant 3$ see D.1., such that if $B \vdash E$, then $B_1 \vdash E$ is not computable; but if we have such extension then E' is not a thesis, see T.6.

The last note is connected with a choosing of an algorithm, see $p \cdots$ and [15] – [16].

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25 It is easy to see the connection of T.8. with the non-computability of the set of all K_f, where f-computable and with the problem of words. In notion of the algorithmic language the given proof may be reduced to one line. s. [15].

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