A CHARACTERIZATION OF THE INCREASING MAPPINGS

by

SADAYUKI YAMAMURO.

Let E be a real Banach space, G be an open set in E and $\bar{G}$ be the closure of G. In [3] (cf. [4], [5]), we have given the following definitions:

A mapping $f : \bar{G} \to E$ *is said to be* (δ)*-increasing at* $a \in G$ *if the following two conditions are satisfied*:

1°. $\|x\| < \delta$ *implies* $a+x \in G$;

2°. $f(a+x)-f(a) \neq \alpha x$ *if* $\alpha \leqq 0$ *and* $0 < \|x\| < \delta$.

A mapping $f : \bar{G} \to E$ *is said to be* (ε, δ)*-uniformly increasing at* $a \in G$ *if the following two conditions are satisfied*:

1°. $\|x\| < \delta$ *implies* $a+x \in G$;

3°. $\|f(a+x)-f(a)-\alpha x\| \geqq \varepsilon \|x\|$ *if* $\alpha \leqq 0$ *and* $0 < \|x\| < \delta$.

On the other hand, G. Minty and F. Browder have written a series of papers about *the monotonic mappings* (for example, [1] and [2]) which, in the case of real Banach spaces E, can be defined as follows: *A mapping* $f : \bar{G} \to E^*$ (the conjugate space of E) *is said to be monotonic on* $\bar{G}$ *if*

$$(f(x)-f(y), x-y) \geqq 0 \qquad \text{for any } x, y \in \bar{G}.$$

(The left-hand side of this inequality means the value of $f(x)-f(y)$ at $x-y$.)

The purpose of this paper is to study about the relations among these monotonicities. For this purpose, we assume that the space E is a real Hilbert space, and we shall characterize the increasingnesses defined above in terms of $(f_a(x), x)$, where $f_a(x) \equiv f(a+x)-f(a)$ and $(f_a(x), x)$ is the inner product of $f_a(x)$ and x.

Theorem. Let E be a real Hilbert space, G be an open subset and $\bar{G}$ *be the closure of G. Then,*

(1) *A mapping* $f : \bar{G} \to E$ *is* (δ)*-increasing at* $a \in G$ *if and only if f satisfies the following two conditions*:

4° $f_a(x) \neq 0$ *if* $0 < \|x\| < \delta$;

5° $(f_a(x), x) > -\|f_a(x)\| \cdot \|x\|$ *if* $0 < \|x\| < \delta$.

(2) *A mapping* $f : \bar{G} \to E$ *is* (ε, δ)*-uuiformly increasing at* $a \in G$ *if and only if f satisfies the following two conditions*:

6° $\|f_a(x)\| \geqq \varepsilon \|x\|$ *if* $\|x\| < \delta$;

7° $$(f_a(x), x) \geqq -\|x\|(\|f_a(x)\|^2 - \varepsilon^2\|x\|^2)^{\frac{1}{2}} \quad \textit{if } \|x\| < \delta.$$

Proof. Let f be (δ)-increasing at $a \epsilon G$. We get 4° from 2° by putting $\alpha=0$. To prove 5°, we assume that there exists an element $x \epsilon E$ such that

$$(f_a(x), x) = -\|f_a(x)\| \cdot \|x\| \quad \text{and} \quad 0 < \|x\| < \delta.$$

Then, for $\alpha_0 = -\|f_a(x)\| / \|x\|$, we get

$$\|f_a(x) - \alpha_0 x\|^2 = \|f_a(x)\|^2 - 2\alpha_0 (f_a(x), x) + \alpha_0^2 \|x\|^2$$
$$= \|f_a(x)\|^2 + 2\alpha_0 \|f_a(x)\| \cdot \|x\| + \alpha_0^2 \|x\|^2 = (\|f_a(x)\| + \alpha_0 \|x\|)^2 = 0,$$

from which it follows that $f_a(x) = \alpha_0 x$, which contradicts the condition 2°.

Next, suppose that the condition 4° and 5° are satisfied and that there exist $\alpha \leqq 0$ and $x \epsilon E$ such that

$$f_a(x) = \alpha x \quad \text{and} \quad 0 < \|x\| < \delta.$$

Then, by 4° we get $\alpha \neq 0$ and by 5° we have that

$$\alpha \|x\|^2 = (\alpha x, x) = (f_a(x), x) > -\|f_a(x)\| \|x\| = -|\alpha| \cdot \|x\|^2,$$

hence it follows that $\alpha > 0$, which contradicts our assumption.

(2) Let $f : \bar{G} \to E$ be (ε, δ)-uniformly increasing at $a \epsilon G$. We get 6° from 3° by putting $\alpha=0$. To prove 7°, suppose that there exists $x \epsilon E$ such that

$$(f_a(x), x) < -\|x\|(\|f_a(x)\|^2 - \varepsilon^2\|x\|^2)^{\frac{1}{2}} \text{ and } 0 < \|x\| < \delta.$$

Then, for $\alpha_0 = -(\|f_a(x)\|^2 - \varepsilon^2\|x\|^2)^{\frac{1}{2}} / \|x\|$, if $\alpha_0 < 0$, we have

$$\|f_a(x) - \alpha_0 x\|^2 = \|f_a(x)\|^2 - 2\alpha_0 (f_a(x), x) + \alpha_0^2 \|x\|^2$$
$$< \|f_a(x)\|^2 + 2\alpha_0 \|x\| (\|f_a(x)\|^2 - \varepsilon^2\|x\|^2)^{\frac{1}{2}} + \alpha_0^2 \|x\|^2$$
$$= [\alpha_0 \|x\|^2 + (\|f_a(x)\|^2 - \varepsilon^2\|x\|^2)^{\frac{1}{2}}]^2 + \varepsilon^2 \|x\|^2 = \varepsilon^2 \|x\|^2,$$

which contradicts the condition 3°. If $\alpha_0 = 0$, then $\|f_a(x)\| = \varepsilon \|x\|$, hence and from 3° it follows that

$$\alpha^2 \|x\|^2 - 2\alpha (f_a(x), x) = \alpha^2 \|x\|^2 - 2\alpha (f_a(x), x) + \|f_a(x)\|^2 - \varepsilon^2 \|x\|^2 \geqq 0$$

for any $\alpha < 0$, namely,

$$2\alpha (f_a(x), x) \leqq \alpha^2 \|x\|^2 \quad \text{for any} \quad \alpha < 0,$$

or equivalently

$$2(f_a(x), x) \geqq \alpha \|x\|^2 \quad \text{for any} \quad \alpha < 0,$$

which means that $(f_a(x), x) \geqq 0$. This is a contradiction.

Conversely, suppose that the conditions 6° and 7° are satisfid and that there exist a number α and an element $x \epsilon E$ such that

$$\|f_a(x) - \alpha x\| < \varepsilon \|x\| \quad \text{and} \quad 0 < \|x\| < \delta.$$

From 6° it follows that $\alpha \neq 0$. If $\alpha < 0$, we have

$$[\alpha \|x\| + (\|f_a(x)\|^2 - \varepsilon^2 \|x\|^2)^{\frac{1}{2}}]^2$$

$$=\alpha^2\|x\|^2+2\alpha\|x\|(\|f_a(x)\|^2-\varepsilon^2\|x\|^2)^{\frac{1}{2}}+\|f_a(x)\|^2-\varepsilon^2\|x\|^2$$
$$\leqq\alpha^2\|x\|^2-2\alpha(f_a(x),x)+\|f_a(x)\|^2-\varepsilon^2\|x\|^2$$
$$=\|f_a(x)-\alpha x\|^2-\varepsilon^2\|x\|^2<0,$$

which is impossible.

Corollary. (1) *If a mapping* $f:\bar{G}\to E$ *satifies the condition* 4° *and*

8° $\quad (f_a(x),x)\geqq 0 \qquad$ *if* $0<\|x\|<\delta$,

then $(f$ *is* (δ)*-increasing at* $a\epsilon G$.

(2) *If a mapping* $f:\bar{G}\to E$ *satisfies the conditions* 6° *and* 8°, *then* f *is* (ε,δ)*-uniformly increasing at* $a\epsilon G$. *Therefore, if* $(f_a(x),x)\geqq\varepsilon\|x\|^2$ $(\|x\|<\delta)$, f *is* (ε,δ)*-uniformly increasing at* $a\epsilon G$.

Example 1. Let E be a real Hilbert space with a complete normalized orthogonal sequence $\{e_i\}$. Then, the linear mapping

$$f(x)=\sum_{i=1}^{\infty}\lambda_i(x,e_i),\quad\text{where } \lambda_i>0 \text{ and } \lim_{i\to\infty}\lambda_i=0,$$

is (∞)-increasing at 0, but it is not (ε,δ)-uniformly increasing for any $\varepsilon>0$ and $\delta>0$.

Example 2. In the 3-dimensional Euclidean space E, let us consider the following mapping:

$$f:(\xi_1,\xi_2,\xi_3)\to(\xi_3,\xi_1,\xi_2).$$

This linear mapping is $\left(\frac{\sqrt{3}}{2},\infty\right)$-uniformly increasing at 0, but it does not satisfy the condition 8°.

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Institute of Advanced Studies,

Australian National University.

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