ON THE DERIVATIVES OF INCREASING MAPPING

by

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If a real-valued function $f(x)$ of a real variable is differentiable at a point a and the derivative $f'(a)$ is not zero, then the function $f(x)$ is either increasing or decreasing at a.

The purpose of this paper is to generalize this fact into the case when the function f is a mapping acting on a space of an arbitrary dimension.

1. Throughout this paper, let E be a real Banach space, G be an open subset and $\bar{G}$ be its closure.

We need the following three definitions:

Definition 1. *A mapping* $f: \bar{G} \rightarrow E$ *is said to be* (δ)*-increasing at* $a \epsilon G$ *if there exists a number* $\delta > 0$ *such that*

(1) $$a+x \epsilon G \qquad \text{if} \quad \|x\| < \delta;$$

(2) $$f(a+x)-f(a) \neq \alpha x \quad \text{if} \quad \alpha \leqq 0 \quad \text{and} \quad 0 < \|x\| < \delta.$$

A mapping $f: \bar{G} \rightarrow E$ *is said to be* (ε, δ)*-uniformly increasing at* $a \epsilon G$ *if there exist numbers* $\varepsilon > 0$ *and* $\delta > 0$ *such that*

(1) $$a+x \epsilon G \qquad \text{if} \quad \|x\| < \delta;$$

(3) $$\|f(a+x)-f(a)-\alpha x\| \geqq \varepsilon \|x\| \qquad \text{if } \alpha \leqq 0 \quad \text{and} \quad 0 < \|x\| < \delta.$$

Definition 2. *A mapping* $F: \bar{G} \rightarrow E$ *is said to be completely continuous on* $\bar{G}$ *if it is continuous and, for any bounded subset* $B \subset \bar{G}$, $F(B)$ *is contained in a compact set. A mapping* $f: \bar{G} \rightarrow E$ *is said to be a completely continuous vector field on* $\bar{G}$ *if the mapping* $F(x) = x - f(x)$ *is completely continuous on* $\bar{G}$.

Definition 3. *A mapping* $f: \bar{G} \rightarrow E$ *is said to be Fréchet-differentiable at* $a \epsilon G$ *if there exists a continuous linear mapping* $D_a: E \rightarrow E$ *such that*

(4) $$f(a+x)-f(a) = D_a(x) + r(a, x) \qquad \text{for every} \quad x \epsilon E,$$

where

(5) $$\lim_{\|x\| \rightarrow 0} \frac{\|r(a, x)\|}{\|x\|} = 0.$$

The linear mapping D_a is called *the Fréchet-derivative of f at a* and is denoted by $f'(a)$.

Now, we can state our theorem in the following form.

Theorem. Let E *be a real Hilbert space and* $\pm f: \bar{G} \rightarrow E$ *be completely continuous vector fields on* $\bar{G}$. *If* f *is Fréchet-differentiable at* $a \epsilon G$ *and the Fréchet-derivative* $f'(a)$ *satisfies the following condition*:

(6) $$(f'(a)(x), x) \neq 0 \qquad \text{if} \quad x \neq 0,$$

then f or $-f$ is (ε, δ)-uniformly increasing at $a \in G$ for some $\varepsilon > 0$ and $\delta > 0$.

Remark 1. We denote the value of the mapping $f'(a)$ at x by $f'(a)(x)$.

Remark 2. We denote the inner product of x and y by (x, y).

Remark 3. $-f$ is the mapping defined by $(-f)(x) = -f(x)$.

Remark 4. Definition 1 was given first in [2] (cf. [3], [4] and [5]).

2. In this section, we give three lemmas. The space E is assumed to be a real Banach space.

Lemma 1. *The Fréchet-derivative of a completely continuous mapping is a completely continuons linear mapping.*

A proof of this well-known fact can be found, for example, in [1, p. 51, Theorem 4.7].

Lemma 2. *Every proper value of the linear mapping $f'(a)$ is positive if and only if the Fréchet-differentiable completely continuous vector field f is (ε, δ)-uniformly increasing at a for some $\varepsilon > 0$ and $\delta > 0$.*

Proof. Assume that every proper value of $f'(a)$ is positive and that f is not (ε, δ)-uniformly increasing at $a \in G$ for any $\varepsilon > 0$ and $\delta > 0$. Then, by (3), we can find elements $x_n (n=1, 2, \cdots)$ and numbers $\alpha_n (n=1, 2, \cdots)$ such that

$$\|f_a(x_n) - \alpha_n x_n\| < \frac{1}{n}\|x_n\|, \ \alpha_n \leqq 0 \text{ and } 0 < \|x_n\| < \frac{1}{n}, \tag{7}$$

where $f_a(x_n) = f(a + x_n) - f(a)$.

This sequence (α_n) is bounded, because

$$\begin{aligned} |\alpha_n| &= \|\alpha_n x_n\| / \|x_n\| \\ &\leqq \frac{1}{\|x_n\|}(\|f_a(x_n) - \alpha_n x_n\| + \|f_a(x_n)\|) \\ &\leqq \frac{1}{n} + \frac{\|f_a(x_n)\|}{\|x_n\|} && \text{(by (7))} \\ &\leqq \frac{1}{n} + \left\|f'(a)\left(\frac{x_n}{\|x_n\|}\right)\right\| + \frac{\|r(a, x_n)\|}{\|x_n\|} && \text{(by (4))} \\ &\leqq \frac{1}{n} + \|f'(a)\| + \frac{\|r(a, x_n)\|}{\|x_n\|}, \end{aligned}$$

where the right-hand side is bounded because of (5). ($\|f'(a)\|$ is the norm of the linear mapping $f'(a)$. Since it is a continuous linear mapping, this is a finite number.) Therefore, there exists subsequence $(\alpha_m) \subset (\alpha_n)$ such that

$$\lim_{m \to \infty} \alpha_m = \alpha_0 \tag{8}$$

for some non-positive number α_0.

Next, put $x'_m = x_m / \|x_m\|$. Then, since $\|x'_m\| = 1$, it follows from Lemma 1 that there exists a subsequence $(x'_k) \subset (x'_m)$ such that

$$\lim_{k\to\infty} F'(a)(x'_k) = x_0 \tag{9}$$

for some element x_0. ($F'(a)$ is the Fréchet-derivative of $F(x)$ at a. It exists because of the differentiability of $f(x)$ at a and we have $F'(a) = I - f'(a)$, where I is the identity mapping.)

On the other hand, we have, by (4) and (7), that

$$\begin{aligned}
&\lim_{k\to\infty} [(1-\alpha_k)x'_k - F'(a)(x'_k)] \\
&= \lim_{k\to\infty} [(x'_k - F'(a)(x'_k)) - \alpha_k x'_k] \\
&= \lim_{k\to\infty} (f'(a)(x'_k) - \alpha_k x'_k) \qquad (10)\\
&= \lim_{k\to\infty} \frac{1}{\|x_k\|}(f'(a)(x_k) - \alpha_k x_k) \\
&= \lim_{k\to\infty} \frac{1}{\|x_k\|}(f'(a)(x_k) - f_a(x_k)) + \lim_{k\to\infty} \frac{1}{\|x_k\|}(f_a(x_k) - \alpha_k x_k) \\
&= 0.
\end{aligned}$$

Therefore, it follows from (9) that

$$\lim_{k\to\infty} (1-\alpha_k)x'_k = x_0 ,$$

and, by (8), we have

$$\lim_{k\to\infty} x'_k = \frac{1}{1-\alpha_0} x_0 ,$$

which implies that

$$\|x_0\| = 1 - \alpha_0 ,$$

and, by (10), we have that

$$f'(a)\left(\frac{x_0}{1-\alpha_0}\right) = \lim_{k\to\infty} f'(a)(x'_k) = \lim_{k\to\infty} \alpha_k x'_k = \alpha_0 \frac{x_0}{1-\alpha_0} .$$

This means that α_0 is a non-positive proper value of $f'(a)$.

Conversely, let us assume that f is (ε, δ)−uniformly increasing at a for some $\varepsilon > 0$ and $\delta > 0$. Suppose that α is a proper value of $f'(a)$, namely, suppose that there exists an element x_0 such that $\|x_0\| = 1$ and

$$f'(a)(x_0) = \alpha x_0 .$$

Then, since $f'(a)$ is linear,

$$f'(a)(\xi x_0) = \alpha \xi x_0 \qquad \text{for any number } \xi. \tag{11}$$

It follows from (5) that there exists $\delta_1 > 0$ such that $\delta_1 < \delta$ and

$$\|r(a, \xi x_0)\| < \varepsilon |\xi| \qquad \text{if } |\xi| < \delta_1.$$

Then, we have, if $|\xi| < \delta_1$,

$$\|f_a(\xi x_0)-\alpha\xi x_0\|$$
$$=\|f'(a)(\xi x_0)+r(a,\xi x_0)-\alpha\xi x_0\| \qquad \text{(by (4))}$$
$$=\|r(a,\xi x_0)\| \qquad \text{(by (11))}$$
$$<\varepsilon|\xi|=\varepsilon\|\xi x_0\|,$$

which, by (3), implies that $\alpha>0$.

Lemma 3. *Assnme that $-f$ is a completely continuous vector field. Then, every proper value of the linear mapping $f'(a)$ is negative if and only if the mapping $-f$ is (ε,δ)-uniformly increasing at a for some $\varepsilon>0$ and $\delta>0$.*

Proof. This lemma is equivalent to Lemma 2, because a number is a positive proper value of $f'(a)$ if and only if it is the absolute value of a negative proper value of $-f'(a)=(-f)'(a)$.

3. *Proof of Theorem.* Assume that neither f nor $-f$ is (ε,δ)-uniformly increasing at a for any $\varepsilon>0$ and $\delta>0$. Then, from Lemma 2 and Lemma 3, it follows that there exist numbers $\alpha_i\,(i=1,2)$ and elements $x_i\,(i=1,2)$ such that

$$f'(a)(x_i)=\alpha_i x_i\ (i=1,2),\ \alpha_1\geqq 0,\ \alpha_2\leqq 0 \text{ and } \|x_i\|=1\ (i=1,2).$$

By (6), α_i are not zero, namely,

$$\alpha_1>0 \quad\text{and}\quad \alpha_2<0.$$

Since x_1 and x_2 are linearly independent, we have

$$z(t)\equiv(1-t)x_1+tx_2\neq 0 \qquad (0\leqq t\leqq 1). \tag{12}$$

Now, consider the following continuous function

$$\phi(t)\equiv(f'(a)(z(t)),z(t)) \qquad (0\leqq t\leqq 1).$$

Then, since $\phi(0)=\alpha_1>0$ and $\phi(1)=\alpha_2<0$, there exists $t_0\epsilon(0,1)$ such that $\phi(t_0)=0$, namely,

$$(f'(a)(z(t_0)),\ z(t_0))=0.$$

Since $z(t_0)\neq 0$ by (12), this contradicts the condition (6).

4. In [2; Theorem 2], we have proved the following fact:

Let E be a real Banach space and $f:\bar{G}\to E$ be a completely continuous vector field on G. If f is (δ)-increasing at $a\epsilon G$, then, for any $\delta_1>0$ such that $\delta_1<\delta$, we have

$$d(0,B(0,\delta_1),f_a)=1,$$

where the left-hand side is the mapping degree of $B(0,\delta_1)=\{x\,|\,\|x\|<\delta_1\}$ at 0 by f_a.

Since an (ε,δ)-uniformly increasing mapping at a is (δ)-increasing at a, we have the following corollary:

Corollary. Let E be a real Hilbert space and $\pm f: \bar{G} \to E$ be completely continuous vector fields on $\bar{G}$. If f is Fréchet-differentiable at $a \in G$ and the Fréchet-derivative $f'(a)$ satisfies the condition (6), *there exists $\delta > 0$ such that*

$$d(0, B(0, \delta_1), f_a) = 1 \qquad \text{for any } \delta_1 \text{ such that } 0 < \delta_1 < \delta,$$

or

$$d(0, B(0, \delta_1), -f_a) = 1 \qquad \text{for any } \delta_1 \text{ such that } 0 < \delta_1 < \delta.$$

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