GENERALIZATION OF THE MIXED TOPOLOGY

By

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Wiweger [3] has introduced a new topology, in a linear space X with two topologies τ^* and τ, by the form of neighbourhood $\bigcup_{n=1}^{\infty}(\sum_{i=1}^{n}(U_i^* \cap iU))$ where U_i^* and U are neighbourhoods of 0 in the topologies τ^* and τ respectively, and it is, in the case of linear normed space, identical with τ^* on the τ-bounded subsets, and is called the mixed topology associated with τ^* and τ.

Persson [2] has made an attempt to generalize it in a locally convex linear topological space by saying that it is the finest locally convex topology on X which is identical with τ^* on the τ-bounded subsets of X, without giving the form of its neighbourhood.

We shall give, in this paper, the form of neighbourhood for the generalized mixed topology and the general method to investigate various mixed topologies determined correspondingly to a certain topology μ and a system $\mathfrak{A}$.

We see that, if $\mathfrak{A}$ consists of all τ-bounde (or, τ-totally bounded, τ-compact, equicontinuous) convex circled subsets of X, then the mixed topology is the finest topology which is identical with μ on the τ-bounded (or τ-totally bounded, τ-compact, equicontinuous respectively) subsets.

We shall give more detailed discussion of them in another paper.

§ 1. Definition of the general mixed topology.

By a linear topological space we understand any linear space X over the field of real numbers R with a linear topology τ defined in such a way that addition and multiplication by scalars are continuous in both variables; there exists a fundamental neighbourhoods system $\mathfrak{U}_\tau$ satisying the following conditions;

(0_1) *if* $U \in \mathfrak{U}_\tau$ *and* $\lambda \in R$, $\lambda \neq 0$, *then* $\lambda U \in \mathfrak{U}_\tau$,

(0_2) *if* $U \in \mathfrak{U}_\tau$ *and* $\lambda \in R$, $|\lambda| \leq 1$, *then* $\lambda U \subset U$,

(0_3) *if* $U \in \mathfrak{U}_\tau$, *then for every* $x \in X$, *there exists* $\lambda \in R$, $\lambda \neq 0$ *such that* $\lambda x \in U$,

(0_4) *if* $U \in \mathfrak{U}_\tau$ *and* $V \in \mathfrak{U}_\tau$, *then there exists* $W \in \mathfrak{U}_\tau$ *such that* $W \subset U \cap V$,

(0_5) *if* $U \in \mathfrak{U}_\tau$, *then there exists* $V \in \mathfrak{U}_\tau$ *such that* $V + V \subset U$.

Definition 1. *We say that a family* $\mathfrak{A}$ *of subsets of* X *is* **primitive** *if it satisfies the following conditions*;

(P_1) *if* $A \in \mathfrak{A}$, $\lambda \in R$, $\lambda \neq 0$, *then* $\lambda A \in \mathfrak{A}$,

(P_2) *if* $A \in \mathfrak{A}$, $\lambda \in R$, $|\lambda| \leq 1$, *then* $\lambda A \subset A$,

(P_3) *if* $x \in X$, *then there exists an element* A *of* $\mathfrak{A}$ *such as* $x \in A$.

The conditions (P_1), (P_2) and (P_3) imply that;

(1.1) The family $\mathfrak{A}$ is frequently absorbing, that is, if $x \in X$ then there exists $\lambda_0 \in R$, $\lambda_0 > 0$ such that $\lambda x \in A$ for $|\lambda| \leq \lambda_0$ and for some member A in $\mathfrak{A}$.

The condition (P_3) is rephrased as the following;

(1.2) if $\mathfrak{A}$ consists of the set $\{A_\iota, \iota \in I\}$, then $\bigcup_{\iota \in I} A_\iota = X$.

Suppose that a locally convex linear Hausdorff topology μ is defined in a linear space X. For the primitive system $\mathfrak{A} = \{A_\iota, \iota \in I\}$, and subcollection $\mathfrak{S} = \{U_\iota, \iota \in I\}$ of neighbourhoods of 0 for the topology μ we put

$$U^\alpha = k\{\bigcup_{\iota \in I} (U_\iota \cap A_\iota)\} \tag{1}$$

where $k(A)$ denotes the convex envelope of A.

Proposition. *Let $\mathfrak{U}^\alpha$ denote the family of all the sets (1) which are determined correspondingly to $\mathfrak{S}$. Then, the family $\mathfrak{U}^\alpha$ defines the new topology, and X is the locally convex linear Hausdorff topological space by it.*

Proof. i) For any non-zero $\lambda \in R$ and $U^\alpha \in \mathfrak{U}^\alpha$,

$\lambda U^\alpha = \lambda k\{\bigcup_{\iota \in I}(U_\iota \cap A_\iota)\} = k\{\lambda \bigcup_{\iota \in I}(U_\iota \cap A_\iota)\} = k\{\bigcup_{\iota \in I}(\lambda U_\iota \cap \lambda A_\iota)\}$ where $\lambda U_\iota \in \mathfrak{A}_\mu$ and $\lambda A_\iota \in \mathfrak{A}$ for each $\iota \in I$ by (P_1), So, $\lambda U^\alpha \in \mathfrak{U}^\alpha$, hence $\mathfrak{U}^\alpha$ satisfies (0_1).

ii) For $\lambda \in R$, $|\lambda| \leq 1$ and $U^\alpha \in \mathfrak{A}^\alpha$,

$\lambda U^\alpha = k\{\bigcup(\lambda U_\iota \cap \lambda A_\iota)\}$, where $\lambda U_\iota \subset U_\iota$ and $\lambda A_\iota \subset A_\iota$ by (P_2) for each $\iota \in I$, so $\lambda U^\alpha \subset U^\alpha$, hence $\mathfrak{U}^\alpha$ satisfies (0_2).

iii) For any $x \in X$, there exist λ_1 and λ_2 such that for at least one $\iota_0 \in I$, $\lambda_1 x \in A_{\iota_0}$, $\lambda_2 x \in U_{\iota_0}$ by (P_3). Let $\lambda_0 = \min(\lambda_1, \lambda_2)$, since U_{ι_0} and A_{ι_0} are both circled, $\lambda_0 x \in U_{\iota_0} \cap A_{\iota_0} \subset k\{\bigcup_{\iota \in I}(U_\iota \cap A_\iota)\} = U^\alpha$, so (0_3) is satisfied.

iv) If $U^\alpha \in \mathfrak{U}^\alpha$ and $V^\alpha \in \mathfrak{U}^\alpha$,

$$U^\alpha = k\{\bigcup_{\iota \in I}(U_\iota \cap A_\iota)\} \text{ and } V^\alpha = k\{\bigcup_{\iota \in I}(V_\iota \cap A_\iota)\}$$

for $U_\iota \in \mathfrak{S} \subset \mathfrak{U}_\mu$ and $V_\iota \in \mathfrak{S}' \subset \mathfrak{U}_\mu$. Then, there exists $W_\iota \subset U_\iota \cap V_\iota$ for every $\iota \in I$, so

$$\bigcup_{\iota \in I}(W_\iota \cap A_\iota) \subset \bigcup_{\iota \in I}\{(U_\iota \cap A_\iota) \cap (V_\iota \cap A_\iota)\} \subset \{\bigcup_{\iota \in I}(U_\iota \cap A_\iota)\} \cap \{\bigcup_{\iota \in I}(V_\iota \cap A_\iota)\}$$

Therefore

$$k\{\bigcup_{\iota \in I}(W_\iota \cap A_\iota)\} \subset k[\{\bigcup_{\iota \in I}(U_\iota \cap A_\iota)\} \cap \{\bigcup_{\iota \in I}(V_\iota \cap A_\iota)\}]$$
$$\subset k\{\bigcup_{\iota \in I}(U_\iota \cap A_\iota)\} \cap k\{\bigcup_{\iota \in I}(V_\iota \cap A_\iota)\}$$

hence, there exists $W^\alpha \in \mathfrak{U}^\alpha$ such that $W^\alpha \subset U^\alpha \cap V^\alpha$, so (0_4) is satisfied.

v) $U^\alpha \in \mathfrak{U}^\alpha$ is convex and circled, so $U^\alpha/2 + U^\alpha/2 \subset U^\alpha$ and $U^\alpha/2 \in \mathfrak{U}^\alpha$.

Thus, (0_5) is also satisfied.

Consequently, the family $\mathfrak{U}^\alpha$ is a basis of neighbourhoods of 0 in a new locally convex linear topology in X.

Finally, we shall show that if the topology μ is Hausdorff, so is the new topology determined by the family $\mathfrak{U}^\alpha$. In fact, choosing only a neighbourhood as a subcollection $\mathfrak{S} = \{U_\iota = U, \iota \in I\}$, we see by (1.2)

$$k\{\bigcup_{\iota\in I}(U_\iota\cap A_\iota)\}=k\{\bigcup_{\iota\in I}(U\cap A_\iota)\}=k\{U\cap(\bigcup_{\iota\in I}A_\iota)\}=k(U\cap X)=U$$

hence we have the inclusion relation

(1.3) $$\mathfrak{U}_\mu\subset\mathfrak{U}^\alpha$$

From this, for $x\neq 0$ in X, there exists a set $U^\alpha\in\mathfrak{U}^\alpha$ such that $x\bar{\in} U^\alpha$ if the topology μ is Hausdorff.

We call this topology of which neighbourhood basis is the family $\mathfrak{U}^\alpha$ in proposition 1.1, the ***general mixed topology*** (g. m. topology) determined by the primitive family $\mathfrak{A}$ and the topology μ. We denote it by $\alpha(\mu,\mathfrak{A})$ or shortly μ^α.

Definition 2. *We denote $\mathfrak{E}(\mathfrak{A})$ the family of all subsets in X satisfying the following conditions:*

(E_1) *if $A\in\mathfrak{E}(\mathfrak{A})$, $\lambda\in R$, $\lambda\neq 0$, then $\lambda A\in\mathfrak{E}(\mathfrak{A})$,*

(E_2) *if $A\in\mathfrak{E}(\mathfrak{A})$, the circled envelope of A, belongs to $\mathfrak{A}$.*

(E_3) $\mathfrak{A}\subset\mathfrak{E}(\mathfrak{A})$.

§ 2. properties of the general mixed topology.

Lemma 2.1 *For each $U\in\mathfrak{U}_\mu$ there exists a $U^\alpha\in\mathfrak{U}^\alpha$ such as $U^\alpha\subset U$.*

Proof. At first, we notice that for each $U\in\mathfrak{U}_\mu$, there exists a subcollection of μ-neighbourhoods, $\{U_\iota, \iota\in I\}$, such that $\bigcup_{\iota\in I} U_\iota\subset U$. In fact, let $\{V_\iota, \iota\in I\}$ be a subcollection of μ-neighbourhoods, then for each $\iota\in I$, there exists U_ι such that $U_\iota\subset U\cap V_\iota$, so $\bigcup_{\iota\in I} U_\iota\subset U$.

For its subcollection $\{U_\iota, \iota\in I\}$, we have a neighbourhood $U^\alpha\in\mathfrak{U}^\alpha$ such that

$$U^\alpha=k\{\bigcup_{\iota\in I}(U_\iota\cap A_\iota)\}\subset k(\bigcup_{\iota\in I}U_\iota)\subset\bigcup_{\iota\in I}k(U_\iota)=\bigcup_{\iota\in I}U_\iota\subset U.$$

We denote $\nu|A$ the topology induced on A by the topology ν.

Lemma 2.2. *Let the primitive system $\mathfrak{A}$ satisfy the following condition;*

(K) *every $A\in\mathfrak{A}$ is convex.*

Then, for every $A\in\mathfrak{E}(\mathfrak{A})$, $\mu|A=\mu^\alpha|A$.

Proof. The neighbourhoods of an $x_0\in A$ in the topologies $\mu|A$ and $\mu^\alpha|A$ are of the forms $(x_0+U)\cap A$ and $(x+U^\alpha)\cap A$ respectively.

By the conditions (P_1) and (E_2), for everv $A\in\mathfrak{E}(A)$ there exist $A_{\iota_1}\in\mathfrak{A}$ and $A_{\iota_2}\in\mathfrak{A}$ such that $A_{\iota_1}\supset A$ and $A_{\iota_2}\supset 2A_{\iota_1}$.

For $(x_0+U^\alpha)\cap A$, take $U\in\mathfrak{U}_\mu$ such as $U\subset U_{\iota_2}$, then

$$(x_0+U)\cap A\subset(x_0+U)\cap A_{\iota_1}\subset x_0+U\cap A_{\iota_2}$$

In fact, if $x\in(x_0+U)\cap A_{\iota_1}$, then $x=x_0+y$, where $x_0\in A_{\iota_1}$, $y\in U$ and $x_0+y\in A_{\iota_1}$, by the condition (K) and (P_2). $(x_0+y-x_0)/2=y/2\in A_{\iota_1}$, that is, $y\in 2A_{\iota_1}\subset A_{\iota_2}$. so, $x_0+y\in x_0+U\cap A_{\iota_2}$.

Moreover,

$$x_0+U\cap A_{\iota_2}\subset x_0+U_{\iota_2}\cap A_{\iota_2}\subset x_0+k\{\bigcup_{\iota\in I}(U_\iota\cap A_\iota)\}=x_0+U^\alpha.$$

$(x_0+U)\cap A\subset(x_0+U^\alpha)\cap A$ follows from $(x_0+U)\cap A\cap A\subset(X_0+U^\alpha)\cap A$. Thus, $\mu|A\geq\mu^\alpha|A$.

On the other hand, since $\mu\leq\mu^\alpha$ by lemma 2.1, $\mu|A\leq\mu^\alpha|A$. Hence, $\mu|A=\mu^\alpha|A$.

Let τ' denote a locally convex linear topology defined in X, and let τ'^α be the g. m. topology determined by the topology τ' and $\mathfrak{A}$, where $\mathfrak{A}$ is the same that determines μ^α.

Henceforth, we postulate that (i) the primitive system $\mathfrak{A}$ satisfies (K) and (ii) if $A\in\mathfrak{E}(\mathfrak{A})$, $k(A)$ belongs to $\mathfrak{E}(\mathfrak{A})$. Thus we obtain,

Theorem 1. *For any $A\in\mathfrak{E}(\mathfrak{A})$, the following conditions are equivalent*; (i) $\mu|A=\tau'|A$ (ii) $\tau'\leq\mu^\alpha$ and $\mu\leq\tau'^\alpha$ (iii) $\mu^\alpha=\tau'^\alpha$

Proof. If $\mu|A=\tau'|A$ for any $A\in\mathfrak{E}(\mathfrak{A})$, then obviously $\mu|A_\iota=\tau'|A_\iota$ for each $A_\iota\in\mathfrak{A}$, and for every $U'\in\mathfrak{U}'$, where $\mathfrak{U}'$ is a basis of neighbourhoods of 0 in the topology τ', we have a subcollection of neighbourhoods of $\mathfrak{U}'$, $\{U'_\iota, \iota\in I\}$ such that $\bigcup_{\iota\in I}U'_\iota\subset U'$, moreover for each U'_ι, we have $U_\iota\in\mathfrak{A}_\mu$ such that

$$U_\iota\cap A_\iota\subset U'_\iota\cap A_\iota\subset U'_\iota,$$

so, $\bigcup_{\iota\in I}(U_\iota\cap A_\iota)\subset\bigcup_{\iota\in I}(U'_\iota\cap A_\iota)\subset\bigcup_{\iota\in I}U'_\iota\subset U'$, hence

$$\tau'\leqq\tau'^\alpha\leqq\mu^\alpha$$

On the other hand, it follows from $\mu|A_\iota=\tau'|A_\iota$ that for every $U\in\mathfrak{U}$, we have $\{U_\iota, \iota\in I\}$ such that $\bigcup_{\iota\in I}U_\iota\subset U$ and for each U_ι we have $U'_\iota\in\mathfrak{U}'$ such that

$$U'_\iota\cap A\subset U_\iota\cap A_\iota\subset U_\iota$$

so $$\bigcup_{\iota\in I}(U'_\iota\cap A_\iota)\subset\bigcup_{\iota\in I}(U_\iota\cap V_\iota)\subset\bigcup_{\iota\in I}U_\iota\subset U$$

Hence $$\mu\leq\mu^\alpha\leq\mu'^\alpha$$

Two inequalities show that (i) implies (ii) and (iii).

By lemma 2.1, (iii) implies (ii).

If $\tau'\leq\tau^\alpha$ and $\mu\leq\tau'^\alpha$ simultaneously, then for any $A\in\mathfrak{E}(\mathfrak{A})$. $\tau'|A\leq\mu^\alpha|A$ and $\mu|A\leq\tau'^\alpha|A$. By lemma 2.2, $\mu^\alpha|A=\mu|A$ and $\tau'^\alpha|A=\tau'|A$, so $\tau'|A\leq\mu|A$ and $\mu|A\leq\tau'|A$, that is $\tau'|A=\tau^*|A$. Hence (ii) implies (i).

We denote the g. m. topology determined by $\mathfrak{A}$ and μ^α where μ^α is the g. m. topology determined by $\mathfrak{A}$ and μ, by $\mu^{\alpha\alpha}$.

In particular, taking μ^α as τ' in theorem 1, we obtain

Corollary 2.1. $\mu^{\alpha\alpha}=\mu^\alpha$

In fact, by lemma 2.1, for any $A\in\mathfrak{E}(\mathfrak{A})$, $\mu^\alpha|A=\mu^{\alpha\alpha}|A$, while $\mu|A=\mu^\alpha|A$, so $\mu|A=\mu^{\alpha\alpha}|A$. Hence (i) in theorem 1 is satisfied. So, $\mu^\alpha=\mu^{\alpha\alpha}$.

Corollary 2.2. *The g. m. topology μ^α is characterized as follow; The finest* *locally convex topology on X which is identical with the topology μ on any member in $\mathfrak{E}(\mathfrak{A})$* is the ***g.m.** topology determined by μ and $\mathfrak{A}$.*

Theorem 2. *Let f be a linear operator from X into another linear space Y with a locally convex topology τ'. Then, f is (μ, τ')-continuous on every $A \in \mathfrak{E}(\mathfrak{A})$ if and only if f is (μ^a, τ') continuous on X.*

Proof. Sufficiency: If f is (μ^a, τ')-continuous, then f A, the restriction of f on A, is $(\mu^a | A, \tau')$-continuous, hence by lemma 2.2, $f|A$ is $(\mu^a\ A, \tau')$-continuous, that is, (μ, τ')-continuous on A.

Necessity: If f is (μ, τ')-continuous on every $A \in \mathfrak{E}(\mathfrak{A})$, then for every $A_\iota \in \mathfrak{A}$, $f|A_\iota$ is $(\mu | A, \tau')$-continuous at the point 0.

Let W be an arbitrary convex neighbourhood for 0 in the topology τ', and take a collection $\{W_\iota, \iota \in I\}$ such that $\bigcup_{\iota \in I} W_\iota \subset W$.

It follows from the $(\mu | A, \tau')$-continuity of f A_ι at the point 0 that for each $A_\iota \in \mathfrak{A}$ there exists $U_\iota \in \mathfrak{U}$ such that

$$f(U_\iota \cap A_\iota) \subset W_\iota$$

hence

$$f(\bigcup_{\iota \in I}(U_\iota \cap A_\iota)) \subset \bigcup_{\iota \in I} W_\iota \subset W.$$

that is, there exists $U^a \in \mathfrak{U}^a$ such that $f(U^a) \subset W$.

Corollary 2.3. *For every $A \in \mathfrak{A}$, the closure of A is the same for the topology μ and μ^a*

In fact, the sets of all linear continuous functionals on X with μ^a and those on each A with μ are the same by Theorem 2. So, by separation theorem (see [1] p. 22, Theorem 5, Cor. 2) the μ^a-closure of A coincides with the μ-closure of A.

§ 3. The g. m. topology in various cases.

Some examples of the primitive systems in a linear space X with two locally convex linear Hausdorff topologies μ and τ are:

(i) *the set $\mathfrak{B}_\tau$ of all τ-bounded convex circled subsets of X.*

(ii) *the set $\mathfrak{T}_\tau$ of all τ-totally bounded convex circled subests of X.*

(iii) *the set $\mathfrak{K}_\tau$ of all τ-compact convex circled subsets of X.*

(iv) *the set $\mathfrak{E}_\tau$ of all convex circled equicontinuous subsets of X.*

(v) $\mathfrak{U}_\tau$; *a basis of neighbourhoods at 0 in the τ-topology.*

We call the g. m. topology determinde by the topology μ and $\mathfrak{B}_\tau$ (or $\mathfrak{T}_\tau$, $\mathfrak{K}_\tau$, $\mathfrak{E}_\tau$, $\mathfrak{U}_\tau$ respectively) γ-(or t-, c-, e-, u-) mixed topology, which is charaterized as below:

Theorem 3. *The γ-(or t-, c-, -e) mixed topology is the finest locally eonvex linear Hausdorff topology which is identical with μ on the τ-bounded (or τ-totally* *bounded, τ-compact, equicontinuous) subsets of X.*

Proof. It is an immediate consequence by corollary 2.2.

Corollary 3.1. *In a locally convex linear topological space, the γ-mixed topology coincides with Persson's mixed topology.*

In fact, Persson [2] has defined the mixed topology by saying that it is the finest locally convex linear Hausdorff topology which is identical with μ on the τ-bounded subsets of X. So the γ-mixed topology coincides with it according to the theorem 3.

Corollary 3.2. *In the case of two-norm spaces, the γ-mixed topology is identical with Wiweger's mixed topology.*

In fact, Persson's mixed topology coincides with the topology introduced by Wiweger in the case of two-norm spaces. (see [3])

Corresponding to some kinds of conditions which are added to the primitive system, the linear space or the topologies μ and τ, we shall obtain various mixed topologies. For instance, we see that the topology $\beta(E',E)$ which is defined in the dual space E' of E by the uniform convergence on each τ-precompact subset of a locally convex linear Hausdorff space E is an e-mixed topology, and is a c-mixed topology if E is a Mackey space, and is a γ-mixed topology if E is a barrelled space. In particular $\beta(E',E)$ determined by the weak*-topology and $\mathfrak{E}$ in E', coincides with the almost weak*-topology. (see [1])

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