SINGLE QUEUE WITH ERLANGIAN INPUT AND HOLDING TIME

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1. Introduction

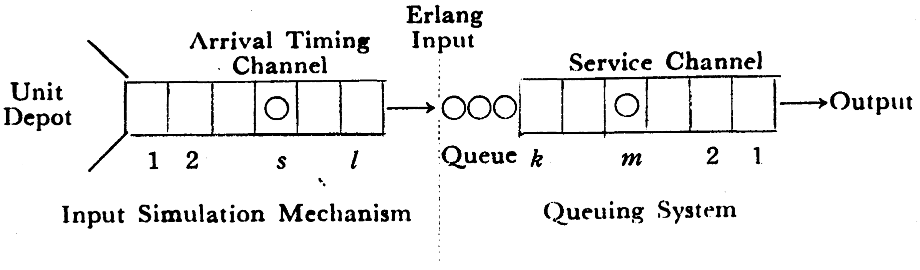
Many authors have discussed on the queuing systems with various kinds of interarrival and service time distributions under the conditions that infinite queue is allowed and queue discipline is "first come, first served". But a great part of their papers were concerned with the queuing systems with Poisson arrival or exponential service distribution. The results on the queuing systems with non-Poisson arrival and non-exponential service distributions are few because of difficulties of their analytical treatment.

In this paper we shall discuss on the single server queuing system with Erlang arrival and Erlang service distributions. P. M. Mosre [1] has dealt with such a queue in the special case where the number l of the input phases equals to the number k of the service phases. However his differential equation method is also applicable to treat the present case and enables us to derive some formulas on state probabilities, queue length and waiting time in equilibrium state.

D. M. G. Wishart [2] dealt with the queuing system $(GI/E_k/1)$ by the Kendall's imbedded Markov method. He also touched on the system $(D/E_k/1)$ on which D. V. Lindley [3] had discussed Many other authors [1, 4]–[7] have published their papers on the systems $(E_l/M/1)$, $(M/E_k/1)$ and other systems to which we shall relate in the later sections.

In order to derive the formulas on the queuing system $(E_l/E_k/1)$, we consider a simulation mechanism which we show in the following figure.

Fig.



As is shown in above figure, departing from unit depot, each unit passes successively through arrival timing channel with l phases numbered $1, 2, \cdots\cdots, l$ in order, in which it takes a time depending on an identical exponential time distribution with mean rate $l\lambda$ to pass through. Supposing that the storage depot is always not emyty and a unit enters into the 1-st phase as soon as the preceeding unit departs from l-th phase. So that only one unit is allowed to stay somewhere in l phases. As is wellknown, the sum of n identically distributed exponential time intervals $I_j\,(j=1,2,\cdots\cdots,n)$ with mean rate $n\eta$ has n-Erlang distribution with mean η. Hence the interdeparture interval of units from the last phase has l-Erlang distribution with mean rate λ.

We consider the situation that units departing from the last phase of arrival timing channel enter into the queuing system. Then the input of this system obeys l-Erlang distribution.

The unit which departs from the arrival timing channel is served immediately when the service channel is free, or otherwise (when the service channel is busy) it waits in queue and will be served at the service channel in order of arrivals. The service channel consists of a series of k phases, the passing time through each of which has an identical exponential time distribution with mean rate $k\mu$. In the service channel only one unit is allowed to be served. The total service time for one unit in the service channel has k-Erlang distribution with mean rate μ. The fraction $\rho=\dfrac{\lambda}{\mu}$ is called "relative intensity". Immediately after the service is finished, the unit leaves the queuing system.

We shall discuss the behavior of such queuing system in equilibrium state.

2. Balance Equations and Existence of Their Solutions

In this section we shall find the balance equations of this system in equilibrium state and their solutions.

If no unit is in queuing system, then the index s of the phase at which a unit stays in the arrival timing channel and the number of units in the queuing system, that is, 0 in our case characterizes the state $(s, 0)$ of the whole system indicated in the figure. If n units are in the queuing system, then the number s, the index m of the phase at which a unit is being served and the number n characterize the state (s, m, n) of the system. Thus

$$1\leqq s\leqq l,\quad 1\leqq m\leqq k,\quad \text{and}\quad n\geqq 1.$$

Let $P_{s,0}(t)$ and $P_{s,m,n}(t)$ be the probabilities that, supposing when $t=0$ the system is in a certain state E_0, the system is at time t in the states $(s,0)$ and (s,m,n) respectively. Since the occupying time in each phase in the arrival and the service channel has exponential distribution, the processes $P_{s,0}(t)$ and $P_{s,m,n}(t)$ are Markovian. Therefore we can easily get a set of differential difference equations

$$P'_{1,0}(t)=-l\lambda\,P_{1,0}(t)+k\mu\,P_{1,1,1}(t),$$

$$P'_{s,0}(t)=-l\lambda\,P_{s,0}(t)+l\lambda\,P_{s-1,0}(t)+k\mu\,P_{s,1,1}(t) \qquad (2\leqq s\leqq l),$$

$$P'_{1,m,1}(t) = -(l\lambda+k\mu)P_{1,m,1}(t) + k\mu P_{1,m+1,1}(t) \qquad (1 \leqq m \leqq k-1),$$
$$P'_{1,k,1}(t) = -(l\lambda+k\mu)P_{1,k,1}(t) + l\lambda P_{l,0}(t) + k\mu P_{1,1,2}(t),$$
$$P'_{1,m,n}(t) = -(l\lambda+k\mu)P_{1,m,n}(t) + l\lambda P_{l,m,n-1}(t) + k\mu P_{1,m+1,n}(t) \quad (1 \leqq m \leqq k-1, n \geqq 2),$$
$$P'_{1,k,n}(t) = -(l\lambda+k\mu)P_{1,k,n}(t) + l\lambda P_{l,k,n-1}(t) + k\mu P_{1,1,n+1}(t) \quad (n \geqq 2),$$
$$P'_{s,m,n}(t) = -(l\lambda+k\mu)P_{s,m,n}(t) + l\lambda P_{s-1,m,n}(t) + k\mu P_{s,m+1,n}(t)$$
$$(1 \leqq m \leqq k-1, 2 \leqq s \leqq l, n \geqq 1),$$
$$P'_{s,k,n}(t) = -(l\lambda+k\mu)P_{s,k,n}(t) + l\lambda P_{s-1,k,n}(t) + k\mu P_{s,1,n+1}(t) \quad (2 \leqq s \leqq l, n \geqq 1).$$

If the intensity ρ is less than unity, then the equilibrium of this system will be obtained no matter what the initial state E_0 of the system may be, and all the derivatives in the above equations tend to zero [8].

Let $P_{s,0}$ and $P_{s,m,n}$ be the probabilities that in equilibrium state the system is in state $(s, 0)$ and (s, m, n) respectively, then it results

$$P_{s,0}(t) \to P_{s,0}, \qquad P_{s,m,n}(t) \to P_{s,m,n} \qquad (t \to +\infty)$$

and from the above differential equations we obtain the following difference equations

$$(1) \qquad P_{1,1,1} = \phi P_{1,0},$$
$$(2) \qquad \phi P_{s-1,0} + P_{s,1,1} = \phi P_{s,0}, \qquad (2 \leqq s \leqq l)$$
$$(3) \qquad P_{1,m+1,1} = (1+\phi)P_{1,m,1} \qquad (1 \leqq m \leqq k-1),$$
$$(4) \qquad \phi P_{l,0} + P_{1,1,2} = (1+\phi)P_{1,k,1},$$
$$(5) \qquad \phi P_{l,m,n-1} + P_{1,m+1,n} = (1+\phi)P_{1,m,m} \qquad (1 \leqq m \leqq k-1, n \geqq 2),$$
$$(6) \qquad \phi P_{l,k,n-1} + P_{1,1,n+1} = (1+\phi)P_{1,k,n} \qquad (n \geqq 2),$$
$$(7) \qquad \phi P_{s-1,m,n} + P_{s,m+1,n} = (1+\phi)P_{s,m,n} \qquad (1 \leqq m \leqq k-1, 2 \leqq s \leqq l, n \geqq 1),$$
$$(8) \qquad \phi P_{s-1,k,n} + P_{s,1,n+1} = (1+\phi)P_{s,k,n} \qquad (2 \leqq s \leqq l, n \geqq 1),$$

where

$$(9) \qquad \phi = \frac{l}{k}\rho, \qquad \rho = \frac{\lambda}{\mu} \qquad (\rho: \text{relative intensity}).$$

To solve these equations we first deal with the equations (4)–(8). Using the quantities u, v and w which satisfy the relations

$$(10) \qquad \begin{cases} \phi + uv - (1+\phi)v = 0, \\ w = u^k = v^l \end{cases}$$

and

$$(11) \qquad 0 < |u|, \ |v|, \ |w| < 1,$$

we can easily see that

$$P_{s,m,n} = Cu^{m-1}v^{s-1}w^{n-1},$$
$$P_{l,0} = Cv^{l-1}/u$$

satisfy the equations (4)–(8), where C is any constant. If there are many solutions u_j, v_j and w_j $(j = 1, 2, \cdots\cdots)$ of (10) inside the unit circle,

$$P_{s,m,n} = \sum_j C_j u_j^{m-1} v_j^{s-1} w_j^{n-1},$$

$$P_{l,0}=\sum_j C_j v_j^{l-1}/u_j$$

also satisfy the equations (4)–(8), where $C_j\,(j=1,2,\cdots\cdots)$ are any constants.

Now we consider how many different solutions u_j, v_j and w_j of the equations (10) there are inside the unit circle. To find the number of these solutions, we transform the equations (10) into

$$v=\frac{\phi}{1+\phi-u}$$

and

$$\left(\frac{\phi}{1+\phi-u}\right)^l=u^k. \tag{12}$$

We can easily prove that, if $0<\rho<1$, the equation (12) has $l+k$ different roots, including the root $u=1$, and among them exactly k roots have absolute values less than unity.[9]

Let $u_1, u_2, \cdots\cdots\cdots, u_k$ be the roots of (12) lying inside the unit circle and let $u_{k+1}, u_{k+2}, \cdots\cdots\cdots, u_{k+l-1}, u_{k+l}$ be the others and u_{k+l} be 1, then it can be easily seen that $u_{k+1}, \cdots\cdots, u_{k+l-1}$ can not have absolute value 1. Therefore it follows

$$0<|u_j|<1 \quad (j=1,2,\cdots,k), \tag{13}$$

$$|u_j|>1 \quad (j=k+1,\cdots,k+l-1), \tag{13'}$$

$$u_{k+k}=1. \tag{14}$$

Setting

$$v_j=\frac{\phi}{1+\phi-u_j} \qquad (j=1,2,\cdots\cdots,k+l), \tag{15}$$

we obtain $u_j^k=v_j^l$. Accordingly, if we put

$$w_j=u_j^k=v_j^l, \tag{16}$$

then u_j, v_j and w_j satisfy the equations (10), where

$$\begin{aligned} &0<|v_j|<1,\ 0<|w_j|<1 \qquad (j=1,2,\cdots\cdots,k),\\ &|v_j|>1, \qquad |w_j|>1 \qquad (j=k+1,k+2,\cdots\cdots,k+l-1) \end{aligned} \tag{17}$$

and

$$v_{k+l}=1, \quad w_{k+l}=1. \tag{17'}$$

Then we obtain the general solution of the difference equations (4)–(8):

$$\begin{aligned} &P_{s,m,n}=\sum_{j=1}^{k} C_j\, u_j^{m-1}\, v_j^{s-1}\, w_j^{n-1}, \quad (1\leqq s\leqq l,\ 1\leqq m\leqq k,\ n\geqq 1),\\ &P_{l,0}\ =\sum_{j=1}^{k} C_j \frac{v_j^{l-1}}{u_j}=\sum_{j=1}^{k} C_j \frac{u_j^{k-1}}{v_j}, \end{aligned} \tag{18}$$

where $C_j\,(j=1,2,\cdots\cdots\cdots,k)$ are any constans.

In fact, under the conditions (13) and (15), we may easily verify that the solutions (18) satisfy the equations (4)–(8).

Next, we shall find the solutions of the system of all the equations (1)–(8).

As the particular case of (18), it results

$$(19) \qquad P_{1,m,1}=\sum_{j=1}^{k} C_j u_j^{m-1}.$$

In order that (18) satisfy (3), the equations

$$(20) \qquad \begin{cases} \sum_{j=1}^{k} C_j = P_{1,1,1} \\ \sum_{j=1}^{k} C_j u_j = (1+\phi) P_{1,1,1} \\ \sum_{j=1}^{k} C_j u_j^2 = (1+\phi)^2 P_{1,1,1} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{j=1}^{k} C_j u_j^{k-1} = (1+\phi)^{k-1} P_{1,1,1} \end{cases}$$

should hold, from which we get

$$(21) \qquad C_j = \frac{\begin{vmatrix} 1 & \cdots\cdots & 1 & 1 & 1 & \cdots\cdots & 1 \\ u_1 & \cdots\cdots & u_{j-1} & (1+\phi) & u_{j+1} & \cdots\cdots & u_k \\ \cdots & & & \cdots & & & \cdots \\ u_1^{k-1} & \cdots\cdots & u_{j-1}^{k-1} & (1+\phi)^{k-1} & u_{j+1}^{k-1} & \cdots\cdots & u_k^{k-1} \end{vmatrix}}{A} P_{1,1,1}$$

$$(j=1, 2, \cdots\cdots, k),$$

where

$$(22) \qquad A=\begin{vmatrix} 1 & 1 & \cdots\cdots & 1 \\ u_1 & u_2 & \cdots\cdots & u_k \\ \cdots & & & \cdots \\ u_1^{k-1} & u_2^{k-1} & \cdots\cdots & u_k^{k-1} \end{vmatrix}.$$

From the equation (2), using $C_1, C_2, \cdots\cdots, C_k$ which have been determined by (21) and (22), we get

$$(23) \qquad P_{s,0}=\frac{-1}{\phi}\sum_{j=1}^{k} C_j (v_j^s+\cdots+v_j^{l-1})+\sum_{j=1}^{k} C_j \frac{v_j^{l-1}}{u_j} \qquad (s=1, 2, \cdots\cdots, l-1),$$

and in particular we have

$$(24) \qquad P_{1,0}=\frac{1}{\phi}\sum_{j=1}^{k} C_j \left(\frac{\phi v_j^l}{u_j v_j}-\frac{v_j-v_j^l}{1-v_j}\right).$$

Using (24), (15) and (19), we can easily see that

$$(25) \qquad \phi P_{1,0}=\sum_{j=1}^{k} C_j \left[\frac{\phi u_j^k}{\frac{\phi u_j}{1+\phi-u_j}}-\frac{\frac{\phi}{1+\phi-u_j}-u_j^k}{\frac{1-u_j}{1+\phi-u_j}}\right]$$

$$=\sum_{j=1}^{k} C_j \frac{u_j^{k-1}-u_j^k-\phi(1-u_j^{k-1})}{1-u_j}$$

$$=\sum_{j=1}^{k} C_j \{u_j^{k-1}-\phi(1+u_j+\cdots\cdots+u_j^{k-2})\}$$

$$=P_{1,k,1}-\phi\,(P_{1,1,1}+\cdots\cdots+P_{1,k-1,1})$$
$$=[(1+\phi)^{k-1}-\phi\,\{1+(1+\phi)+\cdots\cdots+(1+\phi)^{k-2}\}]\,P_{1,1,1}$$
$$=P_{1,1,1}.$$

This means that the solutions which satisfy the equations (2)–(8) necessarily satisfy the equation (1).

By (25) the coefficients C_j turned to

$$(26)\qquad C_j=\frac{\begin{vmatrix} 1 & \cdots\cdots & 1 & 1 & 1 & \cdots\cdots & 1 \\ u_1 & \cdots\cdots & u_{j-1} & 1+\phi & u_{j+1} & \cdots\cdots & u_k \\ \cdots & & & \cdots & & & \cdots \\ u_1^{k-1} & \cdots & u_{j-1}^{k-1} & (1+\phi)^{k-1} & u_{j+1}^{k-1} & \cdots\cdots & u_k^{k-1} \end{vmatrix}}{A}\,\phi P_{1,0}.$$

Then the solutions of the equations (1)–(8) are expressed by (18) and (23) with coefficients (26), where $P_{1,0}$ is arbitrary.

The phase probability $P_{1,0}$ should be determined by the relation

$$(27)\qquad \sum_{s=1}^{l} P_{s,0}+\sum_{n=1}^{\infty}\sum_{m=1}^{k}\sum_{s=1}^{l} P_{s,m,n}=1.$$

Using this relation we shall have later the formulas (45) and (48) of the phase probabilities $P_{s,0}$ and $P_{s,m,n}$ $(1\leqq s\leqq l,\ 1\leqq m\leqq k,\ n\geqq 1)$.

3. Phase Probabilities and State Probabilities

In the present section we shall find the phase and state probabilities.

Putting

$$Q_{s,0}=P_{s,0}\quad\text{and}\quad Q_{s,(n-1)k+m}=P_{s,m,n},$$

from (1)–(8) we get

$$\begin{array}{lllll}
(1') & & Q_{1,1} & =\ \phi\, Q_{1,0}, & \\
(2') & \phi Q_{s-1,0} & +Q_{s,1} & =\ \phi\, Q_{s,0} & (2\leqq s\leqq l), \\
(3') & & Q_{1,m+1} & =(1+\phi)\,Q_{1,m} & (1\leqq m\leqq k-1), \\
(4') & \phi Q_{l,0} & +Q_{1,k+1} & =(1+\phi)\,Q_{1,k}, & \\
(5') & \phi Q_{l,(n-2)k+m} & +Q_{1,(n-1)k+m+1} & =(1+\phi)\,Q_{1,(n+1)k+m} & (1\leqq m\leqq k-1,\ n\geqq 2), \\
(6') & \phi Q_{l,(n-1)k} & +Q_{1,nk+1} & =(1+\phi)\,Q_{1,nk} & (n\geqq 2), \\
(7') & \phi Q_{s-1,(n-1)k+m} & +Q_{s,(n-1)k+m+1} & =(1+\phi)\,Q_{s,(n-1)k+m} & (1\leqq m\leqq k-1,\ 2\leqq s\leqq l, n\geqq 1), \\
(8') & \phi Q_{s-1,nk} & +Q_{s,nk+1} & =(1+\phi)\,Q_{s,nk} & (2\leqq s\leqq l,\ n\geqq 1).
\end{array}$$

Multiplying these equations by $x, x^s, xy^m, xy^k, xy^{(n-1)k+m}, xy^{nk}, x^s y^{(n-1)k+m}$ and $x^s y^{nk}$ respectively in order and adding them all, we obtain, after some calculations,

$$(28)\qquad \sum_{s=1}^{l} Q_{s,0}\,x^s+\phi xy^k\sum_{\nu=0}^{\infty} Q_{l,\nu}\,y^\nu+\phi x\sum_{\nu=0}^{\infty}\sum_{s=1}^{l-1} Q_{s,\nu}\,x^s y^\nu+\frac{1}{y}\sum_{\nu=1}^{\infty}\sum_{s=1}^{l} Q_{s,\nu}\,x^s y^\nu$$
$$=(1+\phi)\sum_{\nu=0}^{\infty}\sum_{s=1}^{l} Q_{s,\nu}\,x^s y^\nu.$$

Setting the generating function

$$(29)\qquad F(x,y)=\sum_{\nu=0}^{\infty}\sum_{s=1}^{l}Q_{s,\nu}x^s y^\nu,$$

we get the relation

$$(30)\qquad \left(1+\phi-\phi x-\frac{1}{y}\right)F(x,y)=\left(1-\frac{1}{y}\right)\sum_{s=1}^{l}Q_{s,0}x^s+\phi x(y^k-x^l)\sum_{\nu=0}^{\infty}Q_{l,\nu}y^\nu.$$

Substituting $x=1$ into this relation, we obtain

$$\left(1-\frac{1}{y}\right)F(1,y)=\left(1-\frac{1}{y}\right)P_0+\phi(y^k-1)\sum_{\nu=0}^{\infty}Q_{l,\nu}y^\nu$$

and

$$(31)\qquad F(1,y)=P_0+\phi(y+y^2+\cdots\cdots+y^k)\sum_{\nu=0}^{\infty}Q_{l,\nu}y^\nu,$$

where $P_0=\sum_{s=1}^{l}F_{s,0}$ which is the probability that the service channel is free.

Hence we see

$$(32)\qquad 1=F(1,1)=P_0+k\phi\sum_{\nu=0}^{\infty}Q_{l,\nu}.$$

Similary, substituting $y=1$ into (30), we obtain

$$(33)\qquad F(x,1)=(x+x^2+\cdots\cdots+x^l)\sum_{\nu=0}^{\infty}Q_{l,\nu}.$$

Therefore, subtituting $x=1$ into this equation, we further obtain

$$(34)\qquad \sum_{\nu=0}^{\infty}Q_{l,\nu}=\frac{1}{l}\quad\text{or}\quad P_{l,0}+\sum_{n=1}^{\infty}\sum_{m=1}^{k}P_{l,m,n}=\frac{1}{l}.$$

Substituting this value into (32), we obtain

$$(35)\qquad P_0=1-\rho$$

and, from (29) and (33), it follows

$$F(x,1)=\sum_{s=1}^{l}\left(\sum_{\nu=0}^{\infty}Q_{s,\nu}\right)x^s=\frac{1}{l}(x+x^2+\cdots\cdots+x^l),$$

$$(36)\qquad \sum_{\nu=0}^{\infty}Q_{s,\nu}=\frac{1}{l}\quad\text{or}\quad P_{s,0}+\sum_{n=1}^{\infty}\sum_{m=1}^{k}P_{s,m,n}=\frac{1}{l}\qquad(1\leqq s\leqq l).$$

Similarly from (29) and (31) we obtain

$$\sum_{\nu=0}^{\infty}\left(\sum_{s=1}^{l}Q_{s,\nu}\right)y^\nu=(1-\rho)+\phi(y+y^2+\cdots\cdots+y^k)\sum_{\nu=0}^{\infty}Q_{l,\nu}y^\nu$$

$$=(1-\rho)+\phi\sum_{\nu=1}^{k}\sum_{j=0}^{\nu-1}Q_{l,j}y^\nu+\phi\sum_{\nu=k+1}^{\infty}\sum_{j=\nu-k}^{\nu-1}Q_{l,j}y^\nu.$$

Hence, comparing the coefficients of same powers of y on both sides of this equation, we find

$$(37)\qquad \sum_{s=1}^{l} Q_{s,\nu}=\phi\sum_{j=0}^{\nu-1} Q_{l,j} \qquad (1\leqq\nu\leqq k),$$

$$\sum_{s=1}^{l} Q_{s,\nu}=\phi\sum_{j=\nu-k}^{\nu-1} Q_{l,j} \qquad (\nu\geqq k),$$

and in particular we get

$$(37')\qquad \sum_{s=1}^{l} P_{s,1,1}=\phi\, P_{l,0} \quad\text{and}\quad \sum_{s=1}^{l} P_{s,1,n+1}=\phi\sum_{m=1}^{k} P_{l,m,n} \qquad (n\geqq 1).$$

Putting

$$R_{0,s}=P_{s,0}, \qquad P_{m,(n-1)l+s}=P_{s,m,n},$$

$$(38)\qquad G(x,y)=\sum_{\nu=1}^{\infty}\sum_{m=1}^{k} R_{m,\nu}\,x^m y^\nu,$$

we obtain the equation

$$(39)\qquad \left(1+\phi-\phi y-\frac{1}{x}\right)G(x,y)=-\phi\sum_{s=1}^{l} R_{0,s}\,y^s+\left(\frac{x^k}{y^l}-1\right)\sum_{\nu=l+1}^{\infty} R_{1,\nu}\,y^\nu$$
$$+\phi y\left(\sum_{s=1}^{l-1} R_{0,s}\,y^s+R_{0,l}\,x^k\right).$$

If we substitute $x=1$ into this equation, then we get in turn

$$\phi(1-y)\sum_{m=1}^{k}\left(\sum_{\nu=1}^{\infty} R_{m,\nu}\,y^\nu\right)=-\phi\sum_{s=1}^{l} R_{0,s}\,y^s+\left(\frac{1}{y^l}-1\right)\sum_{\nu=l+1}^{\infty} R_{1,\nu}\,y^\nu$$
$$+\phi y\left(\sum_{s=1}^{l-1} R_{0,s}\,y^s+R_{0,l}\right),$$

$$\phi(1-y)\sum_{s=1}^{l} R_{0,s}\,y^s+\phi(1-y)\sum_{m=1}^{k}\left(\sum_{\nu=1}^{\infty} R_{m,\nu}\,y^\nu\right)=\frac{1-y^l}{y^l}\sum_{\nu=l+1}^{\infty} R_{1,\nu}\,y^\nu$$
$$+\phi y(1-y^l)R_{0,l}$$

and

$$\phi\sum_{s=1}^{l} R_{0,s}\,y^s=\frac{1+y+\cdots\cdots+y^{l-1}}{y^l}\sum_{\nu=l+1}^{\infty} R_{1,\nu}\,y^\nu+\phi(y+y^2+\cdots\cdots+y^l)R_{0,l}$$
$$-\phi\sum_{m=1}^{k}\left(\sum_{\nu=1}^{\infty} R_{m,\nu}\,y^\nu\right).$$

As $y\to 1$, the last expression tends to

$$\phi(1-\rho)=l\sum_{\nu=l+1}^{\infty} R_{1,\nu}+\phi l\, R_{0,l}-\phi\rho,$$

and accordingly, using (37′), we get

$$\sum_{\nu=1}^{\infty} R_{1,\nu}=\frac{\rho}{k}.$$

Similarly, if we substitute $y=1$ into (39), then we find

$$\left(1-\frac{1}{x}\right)\sum_{m=1}^{k}\sum_{\nu=1}^{\infty} R_{m,\nu}\, x^m = -\phi\,(1-\rho)+(x^k-1)\sum_{\nu=l+1}^{\infty} R_{1,\nu}+\phi\left(\sum_{s=1}^{l-1} R_{0,s}+R_{0,l}\, x^k\right)$$

and, using (37′), we have

$$\sum_{m=1}^{k}\left(\sum_{\nu=1}^{\infty} R_{m,\nu}\right) x^m=(x+x^2+\cdots\cdots+x^k)\sum_{\nu=1}^{\infty} R_{1,\nu}$$

and, comparing the coefficients of the same powers of x on both sides of this equation, we get

$$\sum_{\nu=1}^{\infty} R_{1,\nu}=\sum_{\nu=1}^{\infty} R_{2,\nu}=\cdots\cdots=\sum_{\nu=1}^{\infty} R_{k,\nu}=\frac{\rho}{k} \tag{40}$$

or

$$\sum_{n=1}^{\infty}\sum_{s=1}^{l} P_{s,m,n}=\frac{\rho}{k}\quad (1\leq m\leq k). \tag{40′}$$

Setting $y=\dfrac{1}{1-\phi-\phi x}$ into (30), we have

$$(1-x)\sum_{s=1}^{l} Q_{s,0}\, x^s = x\left\{\left(\frac{1}{1+\phi-\phi x}\right)^k - x^l\right\}\sum_{\nu=0}^{\infty} Q_{l,\nu}\left(\frac{1}{1+\phi-\phi x}\right)^\nu. \tag{41}$$

Further, putting $x=\dfrac{1}{v_{k+\nu}}$ $(\nu=1, 2, \cdots\cdots, l-1)$ into this equation, we see that

$$\frac{1}{1+\phi-\phi x}=\frac{1}{1+\phi-\dfrac{\phi}{v_{k+\nu}}}=\frac{v_{k+\nu}}{(1+\phi)\,v_{k+\nu}-\phi}=\frac{1}{u_{k+\nu}},$$

$$\left(\frac{1}{1+\phi-\phi x}\right)^k - x^l=\left(\frac{1}{u_{k+\nu}}\right)^k-\left(\frac{1}{v_{k+\nu}}\right)^l=0,$$

$$|u_{k+\nu}|,\ |v_{k+\nu}|>1 \qquad (\nu=1, 2, \cdots\cdots\cdots, l-1),$$

and that the series

$$\sum_{\nu=0}^{\infty} Q_{l,\nu}\left(\frac{1}{1+\phi-\phi x}\right)^\nu=\sum_{\nu=0}^{\infty} Q_{l,\nu}\left(\frac{1}{u_{k+\nu}}\right)^\nu$$

converges. Hence from the equation

$$Q_{1,0}+Q_{2,0}+\cdots\cdots\cdots+Q_{l,0}=1-\rho$$

and $l-1$ equations

$$\begin{aligned}
&\frac{1}{v_{k+1}}\, Q_{1,0}+\left(\frac{1}{v_{k+1}}\right)^2 Q_{2,0}+\cdots\cdots\cdots+\left(\frac{1}{v_{k+1}}\right)^l Q_{l,0}=0\\
&\frac{1}{v_{k+2}}\, Q_{1,0}+\left(\frac{1}{v_{k+2}}\right)^2 Q_{2,0}+\cdots\cdots\cdots+\left(\frac{1}{v_{k+2}}\right)^l Q_{l,0}=0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\frac{1}{v_{k+l-1}}\, Q_{1,0}+\left(\frac{1}{v_{k+l-1}}\right)^2 Q_{2,0}+\cdots\cdots\cdots+\left(\frac{1}{v_{k+l-1}}\right)^l Q_{l,0}=0,
\end{aligned} \tag{42}$$

which we shall obtain by substitution

$$x=\frac{1}{v_{k+\nu}} \quad (\nu=1,2,\cdots\cdots\cdots,l-1)$$

into (41), we can determine the values of $Q_{s,0}\,(=P_{s,0})\,(s=1,2,\cdots,l)$:

$$(43)\qquad P_{s,0}=\frac{\begin{vmatrix} 1 & \cdots\cdots & 1 & 1-\rho & 1 & \cdots\cdots & 1 \\ 1 & \cdots\cdots & v_{k+1}^{l-s-1} & 0 & v_{k+1}^{l-s+1} & \cdots\cdots & v_{k+1}^{l-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots\cdots & v_{k+l-1}^{l-s-1} & 0 & v_{k+l-1}^{l-s+1} & \cdots\cdots & v_{k+l-1}^{l-1} \end{vmatrix}}{\mathrm{B}} \qquad (s=1,2,\cdots\cdots,l),$$

where

$$(44)\qquad \mathrm{B}=\begin{vmatrix} 1 & 1 & \cdots\cdots\cdots & 1 \\ 1 & v_{k+1} & \cdots\cdots\cdots & v_{k+1}^{l-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & v_{k+l-1} & \cdots\cdots\cdots & v_{k+l-1}^{l-1} \end{vmatrix}$$

We can simplify the expression (43) as

$$(45)\qquad P_{s,0}=\frac{(-1)^{s-1}(1-\rho)\,\sigma_{s-1}}{(1-v_{k+1})(1-v_{k+2})\cdots\cdots\cdots(1-v_{k+l-1})}, \quad *$$

where

$$(45')\qquad \sigma_s=\sum v_{i_1}v_{i_2}\cdots\cdots\cdots v_{i_s} \quad (k+1\leqq i_1,i_2,\cdots\cdots\cdots,i_s\leqq k+l-1)$$

in which the summation is taken over all s different integers $i_1,i_2,\cdots\cdots,i_s$, and we have particularly

$$(46)\qquad P_{1,0}=\frac{1-\rho}{(1-v_{k+1})(1-v_{k+2})\cdots\cdots\cdots(1-v_{k+l-1})}.$$

Also it can be easily verified that

$$P_0=\sum_{s=1}^{l}P_{s,0}=\frac{1-\rho}{(1-v_{k+1})(1-v_{k+2})\cdots\cdots(1-v_{k+l-1})}\sum_{s=1}^{l}(-1)^{s-1}\sigma_{s-1}=1-\rho.$$

Hence the coefficients C_j and all phase probabilities are completely determined as follows:

$$(47)\qquad C_j=\frac{\phi(1-\rho)}{\mathrm{A}(1-v_{k+1})(1-v_{k+2})\cdots\cdots(1-v_{k+l-1})}\begin{vmatrix} 1 & \cdots\cdots & 1 & 1 & 1 & \cdots\cdots & 1 \\ u_1 & \cdots\cdots & u_{j-1} & 1+\phi & u_{j+1} & \cdots\cdots & u_k \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_1^{k-1} & \cdots\cdots & u_{j-1}^{k-1} & (1+\phi)^{k-1} & u_{j+1}^{k-1} & \cdots\cdots & u_k^{k-1} \end{vmatrix}$$

$$(j=1,2,\cdots\cdots\cdots,k),$$

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* From the fact that an algebraic equation with real coefficients has pairs of conjegate roots, it can be easily seen that $P_{s,0}\,(s=1,2\cdots\cdots,l)$ are positive.

(48) $$P_{s,m,n} = \frac{-\phi(1-\rho)}{A(1-v_{k+1})(1-v_{k+2})\cdots\cdots(1-v_{k+l-1})}$$

$$\times \begin{vmatrix} 1 & 1 & \cdots\cdots 1 & 1 \\ u_1 & u_2 & \cdots\cdots u_k & 1+\phi \\ \cdots & \cdots & \cdots & \cdots \\ u_1^{k-1} & u_2^{k-1} & \cdots\cdots u_k^{k-1} & (1+\phi)^{k-1} \\ U_1^{(s,m,n)} & U_2^{(s,m,n)} & \cdots\cdots U_k^{(s,m,n)} & 0 \end{vmatrix} *$$

where $U_j^{(s,m,n)} = u_j^{m-1} v_j^{s-1} w_j^{n-1} \qquad (j=1,2,\cdots\cdots,k).$

Then the state probabilities can be obtained by the equality

(48′) $$P_n = \sum_{s=1}^{l} \sum_{m=1}^{k} P_{s,m,n}.$$

4. Queue Length

Insert $1+\phi-\phi x-\frac{1}{y}=0$ or $x=1+\frac{1}{\phi}-\frac{1}{\phi y}$ into (30), then it becomes

(49) $$\left(1-\frac{1}{y}\right)\sum_{s=1}^{l} Q_{s,0}\,x^s+\phi x\,(y^k-x^l)\sum_{\nu=0}^{\infty} Q_{l,\nu}\,y^\nu=0$$

and if we divide both sides of this equation by $1-\frac{1}{y}$, getting

$$\sum_{s=1}^{l} Q_{s,0}\,x^s+\phi\left(1-\frac{1}{y}\right)^{-1}\left\{y^k-\left(1+\frac{1}{\phi}-\frac{1}{\phi y}\right)^l\right\}\sum_{\nu=0}^{\infty} Q_{l,\nu}\,y^\nu$$

$$+(y^k-x^l)\sum_{\nu=0}^{\infty} Q_{l,\nu}\,y^\nu=0,$$

and we differentiate by y, then we obtain

$$\frac{1}{\phi y^2}\sum_{s=1}^{l} sQ_{s,0}\,x^{s-1}-\frac{\phi\left\{y^k-\left(1+\frac{1}{y}-\frac{1}{\phi y}\right)^l\right\}}{1-\frac{1}{y}}+(y^k-x^l)\sum_{\nu=1}^{\infty}\nu Q_{l,\nu}\,y^{\nu-1}$$

$$+\ \phi\,\frac{\left(1-\frac{1}{y}\right)\left\{ky^{k-1}-lx^{l-1}-\frac{1}{\phi y^2}\right\}-\left\{y^k-\left(1+\frac{1}{\phi}-\frac{1}{\phi y}\right)^l\right\}\frac{1}{y^2}}{\left(1-\frac{1}{y}\right)^2}$$

$$+\left\{ky^{k-1}-lx^{l-1}\frac{1}{\phi y^2}\right\}\sum_{\nu=0}^{\infty} Q_{l,\nu}\,y^\nu=0.$$

Using the relations

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* It can be proved that $P_{s,m,n} \geqq 0$. See foot note (p. 10).

$$\lim_{y\to 1}\frac{y^k-\left(1+\frac{1}{\phi}-\frac{1}{\phi y}\right)^l}{1-\frac{1}{y}}=k-\frac{l}{\phi}=k\left(1-\frac{1}{\rho}\right)$$

and

$$\lim_{y\to 1}\frac{\left(1-\frac{1}{y}\right)\left\{ky^{k-1}-l\left(1+\frac{1}{\phi}-\frac{1}{\phi y}\right)^{l-1}\frac{1}{\phi y^2}\right\}-\left\{y^k-\left(1+\frac{1}{\phi}-\frac{1}{\phi y}\right)^l\right\}\frac{1}{y^2}}{\left(1-\frac{1}{y}\right)^2}$$

$$=\frac{1}{2}\left\{k(k+1)-\frac{l(l-1)}{\phi^2}\right\},$$

we get

$$\frac{1}{\phi}\sum_{s=1}^{l} sQ_{s,0}+\phi\left(k-\frac{l}{\phi}\right)\sum_{\nu=1}^{\infty}\nu Q_{l,\nu}+\frac{1}{l}\left\{\frac{k(k+1)}{2}\phi-\frac{l(l-1)}{2\phi}+\left(k-\frac{l}{\phi}\right)\right\}, \tag{50}$$

which leads the following simple relation between

$$\sum_{\nu=1}^{\infty}\nu Q_{l,\nu} \quad \text{and} \quad S_0=\sum_{s=1}^{l} sQ_{s,0},$$

that is,

$$(1-\rho)\sum_{\nu=1}^{\infty}\nu Q_{l,\nu}=\frac{1}{\phi}\left\{\frac{k+1}{2k}\rho^2-\frac{l-1}{2l}+\frac{S_0-(1-\rho)}{l}\right\}. \tag{51}$$

Multiplying (1)–(8) by $x, x^s, xy^m, xy^k, xy^m z^{n-1}, xy^k z^{n-1}, x^s y^m z^{n-1}$ and $x^s y^k z^{n-1}$ respectively in order, adding all, and setting

$$H(x,y,z)=\sum_{s=1}^{l} P_{s,0}x^s+\sum_{n=1}^{\infty}\sum_{m=1}^{k}\sum_{s=1}^{l} P_{s,m,n}x^s y^m z^{n-1}, \tag{52}$$

we obtain

$$\left(1+\phi-\phi x-\frac{1}{y}\right)H(x,y,z)=\left(1-\frac{1}{y}\right)\sum_{s=1}^{l} P_{s,0}x^s+\phi x(y^k-x^l)P_{l,0} \tag{53}$$

$$+\phi x(z-x^l)\sum_{n=1}^{\infty}\sum_{m=1}^{k} P_{l,m,n}y^m z^{n-1}-\left(1-\frac{y^k}{z}\right)\sum_{n=2}^{\infty}\sum_{s=1}^{l} P_{s,1,n}x^s z^{n-1}.$$

Using (37′), we have

$$H(1,y,z)=(1-\rho)+\phi(y+y^2+\cdots\cdots+y^k)P_{l,0} \tag{54}$$

$$+\phi z\sum_{n=1}^{\infty}\sum_{m=1}^{k}(y+y^2+\cdots\cdots+y^m)P_{l,m,n}z^{n-1}$$

$$+\phi\sum_{n=1}^{\infty}\sum_{m=1}^{k} y^m(y+y^2+\cdots\cdots+y^{k-m})P_{l,m,n}z^{n-1}$$

and

(55) $$H(1,1,z)=(1-\rho)+l\rho P_{l,0}+k\phi\sum_{n=0}^{\infty}\sum_{m=1}^{k}P_{l,m,n+1}z^{n}$$
$$+\phi(z-1)\sum_{n=0}^{\infty}\sum_{m=1}^{k}mP_{l,m,n+1}z^{n}.$$

Substitute $x=1+\frac{1}{\phi}-\frac{1}{\phi y}$ into (53) and differentiate it partially by y, and let both of x and y be 1, then we get

$$\frac{1}{\phi y^2}\left(1-\frac{1}{y}\right)\sum_{s=1}^{l}sP_{s,0}x^{s-1}+\frac{1}{y^2}\sum_{s=1}^{l}P_{s,0}x^{s}+\left\{\phi x\left(ky^{k-1}-lx^{l-1}\frac{1}{\phi y^2}\right)\right.$$
$$\left.+\phi(y^k-x^l)\frac{1}{\phi y^2}\right\}P_{l,0}+\phi x(z-x^l)\sum_{n=1}^{\infty}\sum_{m=1}^{k}mP_{l,m,n}y^{m-1}z^{n-1}$$
$$+\{\phi z-\phi(l+1)x\}\frac{1}{\phi y^2}\sum_{n=1}^{\infty}\sum_{m=1}^{k}P_{l,m,n}y^{m}z^{n-1}$$
$$-\left(1-\frac{y^k}{z}\right)\frac{1}{\phi y^2}\sum_{n=2}^{\infty}\sum_{s=1}^{l}sP_{s,1,n}x^{s-1}z^{n-1}+\frac{ky^{k-1}}{z}\sum_{n=2}^{\infty}\sum_{s=1}^{l}P_{s,1,n}x^{s}z^{n-1}=0,$$

which, as $x\to 1, y\to 1$, becomes

$$P_0+(\phi k-l)P_{l,0}+\phi(z-1)\sum_{n=1}^{\infty}\sum_{m=1}^{k}mP_{l,m,n}z^{n-1}$$
$$+\{z-(l+1)\}\sum_{n=1}^{\infty}\sum_{m=1}^{k}P_{l,m,n}z^{n-1}-\frac{1}{\phi}\left(1-\frac{1}{z}\right)\sum_{n=2}^{\infty}\sum_{s=1}^{l}sP_{s,1,n}z^{n-1}$$
$$+\frac{k}{z}\sum_{n=2}^{\infty}\sum_{s=1}^{l}P_{l,1,n}z^{n-1}=0.$$

After some calculations, we get

$$\frac{1}{\phi}(1-z)\sum_{n=1}^{\infty}\sum_{s=1}^{l}sP_{s,1,n+1}z^{n}-\phi(1-z)\sum_{n=1}^{\infty}\sum_{m=1}^{k}mP_{l,m,n}z^{n}$$
$$-(1-z)\sum_{n=1}^{\infty}\sum_{m=1}^{k}P_{l,m,n}z^{n}+k\sum_{n=1}^{\infty}\sum_{s=1}^{l}P_{s,1,n}z^{n-1}$$
$$-l\sum_{n=1}^{\infty}\sum_{m=1}^{k}P_{l,m,n}z^{n}+P_0z-lP_{l,0}z-\phi k(1-z)P_{l,0}=0.$$

If we differentiate the left side of this equation by z, then we get

$$\frac{1}{\phi}(1-z)\sum_{n=1}^{\infty}\sum_{s=1}^{l}nsP_{s,1,n+1}z^{n-1}-\phi(1-z)\sum_{n=1}^{\infty}\sum_{m=1}^{k}mnP_{l,m,n}z^{n-1}$$
$$-(1-z)\sum_{n=1}^{\infty}\sum_{m=1}^{k}nP_{l,m,n}z^{n}-\frac{1}{\phi}\sum_{n=1}^{\infty}\sum_{s=1}^{l}sP_{s,1,n+1}z^{n}+\phi\sum_{n=1}^{\infty}\sum_{m=1}^{k}mP_{l,m,n}z^{n}$$
$$+\sum_{n=1}^{\infty}\sum_{m=1}^{k}P_{l,m,n}z^{n}+\phi kP_{l,0}+k\sum_{n=2}^{\infty}\sum_{s=1}^{l}(n-1)P_{s,1,n}z^{n-2}$$

$$+P_0-lP_{l,0}-l\sum_{n=1}^{\infty}\sum_{m=1}^{k}nP_{l,m,n}z^{n-1}=0.$$

Purting $z=1$ into this equality, we obtain

$$\phi\sum_{n=1}^{\infty}\sum_{m=1}^{k}mP_{l,m,n}+(\phi k-l)\sum_{n=1}^{\infty}\sum_{m=1}^{k}nP_{l,m,n}\tag{56}$$

$$-\frac{1}{\phi}\sum_{n=1}^{\infty}\sum_{s=1}^{l}sP_{s,1,n+1}=(l\rho-l-1)\left(\frac{1}{l}-P_{l,0}\right).$$

Partial differentiation of (53) shows

$$\frac{1}{y^2}H(x,y,z)+\left(1+\phi-\phi x-\frac{1}{y}\right)\sum_{n=1}^{\infty}\sum_{s=1}^{l}\sum_{m=1}^{k}mP_{s,m,n}x^s y^{m-1}z^{n-1}$$

$$=\frac{1}{y^2}\sum_{s=1}^{l}P_{s,0}x^s+\phi kxy^{k-1}P_{l,0}+\phi x(z-x^l)\sum_{n=1}^{\infty}\sum_{m=1}^{k}mP_{l,m,n}y^{m-1}z^{n-1}$$

$$+\frac{ky^{k-1}}{z}\sum_{n=2}^{\infty}\sum_{s=1}^{l}P_{s,1,n}x^s z^{n-1}.$$

Taking $y=1$ and $z=1$, we obtain

$$\sum_{n=1}^{\infty}\sum_{m=1}^{k}\sum_{s=1}^{l}P_{s,m,n}x^s+\phi(1-x)\sum_{n=1}^{\infty}\sum_{m=1}^{k}\sum_{s=1}^{l}mP_{s,m,n}x^s$$

$$=k\phi P_{l,0}x-k\sum_{s=1}^{l}P_{s,1,1}x^s+\phi x(1-x^l)\sum_{n=1}^{\infty}\sum_{m=1}^{k}mP_{l,m,n}+k\sum_{n=1}^{\infty}\sum_{s=1}^{l}P_{s,1,n}x^s$$

which turns to, by differentiation,

$$\sum_{n=1}^{\infty}\sum_{m=1}^{k}\sum_{s=1}^{l}sP_{s,m,n}x^{s-1}-\phi\sum_{n=1}^{\infty}\sum_{m=1}^{k}\sum_{s=1}^{l}mP_{s,m,n}x^s$$

$$+\phi(1-x)\sum_{n=1}^{\infty}\sum_{m=1}^{k}\sum_{s=1}^{l}msP_{s,m,n}x^{s-1}=-k\sum_{s=1}^{l}sP_{s,1,1}x^{s-1}+k\phi P_{l,0}$$

$$+\phi\{1-(l+1)x^l\}\sum_{n=1}^{\infty}\sum_{m=1}^{k}mP_{l,m,n}+k\sum_{n=1}^{\infty}\sum_{s=1}^{l}sP_{s,1,n}x^{s-1}.$$

Putting $x=1$ into both sides of this equation, we obtain

$$k\sum_{n=2}^{\infty}\sum_{s=1}^{l}sP_{s,1,n}-\phi l\sum_{n=1}^{\infty}\sum_{m=1}^{k}mP_{l,m,n}+k\phi P_{l,0}$$

$$=\sum_{s=1}^{l}s\sum_{n=1}^{\infty}\sum_{m=1}^{k}P_{s,m,n}-\phi\sum_{m=1}^{k}m\sum_{n=1}^{\infty}\sum_{s=1}^{l}P_{s,m,n},$$

which becomes, using (36) and (40),

$$k\sum_{n=2}^{\infty}\sum_{s=1}^{l}sP_{s,1,n}-\phi l\sum_{n=1}^{\infty}\sum_{m=1}^{k}mP_{l,m,n}+k\phi P_{l,0}$$

$$=\sum_{s=1}^{l}s\left(\frac{1}{l}-P_{s,0}\right)-\phi\frac{\rho}{k}\sum_{m=1}^{k}m.$$

Setting $S_0=\sum_{s=1}^{l}sP_{s,0}$, we have

$$(57)\qquad k\sum_{n=2}^{\infty}\sum_{s=1}^{l} sP_{s,1,n}-\phi l\sum_{n=1}^{\infty}\sum_{m=1}^{k} mP_{l,m,n}+k\phi P_{l,0}=\frac{l+1}{2}-S_0-\frac{l(k+1)}{2k}\rho^2.$$

To simplify the expression, we put

$$(58)\qquad S=\sum_{n=2}^{\infty}\sum_{s=1}^{l} sP_{s,1,n},\quad M=\sum_{n=1}^{\infty}\sum_{m=1}^{k} mP_{l,m,n},\quad N=\sum_{n=1}^{\infty}\sum_{m=1}^{k} nP_{l,m,n}.$$

Then, from (56) and (57), we get

$$(59)\qquad \phi M+(\phi k-l)N-\frac{S}{\phi}=(l\rho-l-1)\left(\frac{1}{l}-P_{l,0}\right)$$

and

$$(60)\qquad kS-\phi lM+k\phi P_{l,0}=\frac{l+1}{2}-S_0-\frac{l(k+1)}{2k}\rho^2.$$

From (55) we can derive the formula of the mean queue length L_q as follows:

$$(61)\qquad L_q=\frac{d}{dz}H(1,1,z)_{z=1}$$

$$=k\phi\sum_{n=1}^{\infty}\sum_{m=1}^{k} nP_{l,m,n+1}+\phi\sum_{n=0}^{\infty}\sum_{m=1}^{k} mP_{l,m,n+1}$$

$$=k\phi N+\phi M-l\rho\left(\frac{1}{l}-P_{l,0}\right).$$

By (59), we obtain

$$L_q=lN+\frac{S}{\phi}-(l+1)\left(\frac{1}{l}-P_{l,0}\right)$$

If we multiply this equation by ρ, and subtract the result from both sides of (61) respectively, we obtain

$$(1-\rho)L_q=\frac{1}{l}(l\phi M-kS)+\rho\left(\frac{1}{l}-P_{l,0}\right)=\frac{\rho}{l}-\frac{1}{l}(kS-l\phi M+k\phi P_{l,0}),$$

which lead us to

$$(1-\rho)L_q=\frac{\rho}{l}-\frac{1}{l}\left\{\frac{l+1}{2}-\frac{\phi\rho(k+1)}{2}-S_0\right\}.$$

Then we finally get

$$(61)\qquad L_q=\frac{1}{1-\rho}\left\{\frac{k+1}{2k}\rho^2-\frac{l-1}{2l}+\frac{S_0-(1-\rho)}{l}\right\},$$

where $S_0=\sum_{s=1}^{l} sP_{s,0}$ and $P_{s,0}\,(s=1,2,\cdots\cdots\cdots,l)$ were given by (45).

Since the number of units in the service channel is $\rho\,(\sum_{i=1}^{\infty}P_i=\rho)$, the average number L of units in the system is

(61′) $L = L_q + \rho.$

5. Waiting Time.

In this section we shall find the formulas for the waiting time distribution and the mean waiting time.

As is well-known, the probability that exactly i exponential services with mean $1/\gamma$ will sueccessively be finished in a service channel in time less than τ is

$$\frac{e^{-\gamma\tau}(\gamma\tau)^i}{i!}$$

and accordingly the probability that a unit should wait for the successive services of N preceeding units in queue more than time τ is

$$\sum_{i=0}^{N-1} \frac{e^{-\gamma\tau}(\gamma\tau)^i}{i!}.$$

When a unit which has been staying at l-th phase in the arrival timing channel enters into the queuing system, if it finds no units in the queuing system, then it will be immediately served (no wait), and, if it finds that $n\,(\geqq 1)$ units are in the queuing system and the unit being served is at m-th phase in the service channel, then the probability that it should wait more than time $\tau\,(\geqq 0)$ is

$$\sum_{i=0}^{(n-1)k+m-1} \frac{e^{-k\mu\tau}(k\mu\tau)^i}{i!}.$$

Therefore we can easily see that the probability $G_q(\tau)$ that anyone of the units which enter into the queuing system from l-th phase of the arrival timing channel should wait more than τ is

$$G_q(\tau) = l \sum_{n=1}^{\infty} \sum_{m=1}^{k} P_{l,m,n} \sum_{i=0}^{(n-1)k+m-1} \frac{e^{-k\mu\tau}(k\mu\tau)^i}{i!}.$$

Then it is easily seen that

$$(62) \qquad \begin{aligned} G_q(\tau) &= l \sum_{\nu=1}^{\infty} Q_{l,\nu} \sum_{i=0}^{\nu-1} \frac{e^{-k\mu\tau}(k\mu\tau)^i}{i!} \\ &= l \sum_{\nu=1}^{\infty} \sum_{j=1}^{k} C_j v_j^{l-1} u_j^{\nu-1} \sum_{i=0}^{\nu-1} \frac{e^{-k\mu\tau}(k\mu\tau)^i}{i!} \\ &= l \sum_{j=1}^{k} C_j v_j^{l-1} \sum_{h=0}^{\infty} u_j^h \sum_{i=0}^{h} \frac{e^{-k\mu\tau}(k\mu\tau)^i}{i!} \\ &= l \sum_{j=1}^{k} C_j v_j^{l-1} \frac{e^{-k\mu\tau(1-u_j)}}{1-u_j}, \end{aligned}$$

where $C_j\,(j=1, 2, \cdots\cdots, k)$ are given by (47).

From this result we can easily derive the formula for the mean waiting time in the queue

$$W_q=-\int_0^\infty \tau dG_q(\tau)=lk\mu\sum_{j=1}^{k}C_j v_j^{l-1}\int_0^\infty \tau e^{-k\mu\tau(1-u_j)}d\tau$$

$$=\frac{l}{k\mu}\sum_{j=1}^{k}C_j\frac{v_j^{l-1}}{(1-u_j)^2}=\frac{l}{k\mu}\sum_{j=1}^{k}C_j\sum_{\nu=1}^{\infty}\nu v_j^{l-1}u_j^{\nu-1}$$

$$=\frac{l}{k\mu}\sum_{\nu=1}^{\infty}\nu\sum_{j=1}^{k}C_j v_j^{l-1}u_j^{\nu-1}=\frac{l}{k\mu}\sum_{\nu=1}^{\infty}\nu Q_{l,\nu}.$$

From (51) we obtain

$$W_q=\frac{1}{\lambda(1-\rho)}\left\{\frac{k+1}{2k}\rho^2-\frac{l-1}{2l}+\frac{S_0-(1-\rho)}{l}\right\} \tag{63}$$

and, comparing this result with (61),

$$L_q=\lambda W_q. \tag{64}$$

Since the mean service time for one unit is $1/\mu$, the average delay time W in the system is

$$W=W_q+\frac{1}{\mu}. \tag{65}$$

Henee we finally get

$$L=\lambda W. \tag{66}$$

6. Numerical Tables.

The values of queue length L_q and waiting time W_q in the queue for some values of l, k and ρ are shown in the following table which we have computed by the formulas (61) and (63).

Table

$k=1,\ l=2$

ρ	L_q	W_q
0.1	0.00301	0.03006
0.2	0.02070	0.10348
0.3	0.06498	0.21661
0.4	0.15196	0.37989
0.5	0.30902	0.61803
0.6	0.59010	0.98350
0.7	1.12044	1.60062
0.8	2.27518	2.84398
0.9	5.92946	6.58828

$k=1,\ l=3$

ρ	L_q	W_q
0.1	0.00139	0.01393
0.2	0.01255	0.06273
0.3	0.04558	0.15192
0.4	0.11543	0.28857
0.5	0.24686	0.49372
0.6	0.48804	0.81339
0.7	0.95041	1.35772
0.8	1.96754	2.45942
0.9	5.20625	5.78472

$k=1,\ l=4$

ρ	L_q	W_q
0.1	0.00068	0.00684
0.2	0.00898	0.04491
0.3	0.03645	0.12149
0.4	0.09771	0.24429
0.5	0.21626	0.43252
0.6	0.43739	0.72899
0.7	0.86569	1.23669
0.8	1.81391	2.26738
0.9	4.84474	5.38305

$k=2,\ l=2$

ρ	L_q	W_q
0.1	0.00166	0.01663
0.2	0.01208	0.06041
0.3	0.03929	0.13095
0.4	0.09483	0.23546
0.5	0.19519	0.39039
0.6	0.37839	0.63065
0.7	0.72734	1.03906
0.8	1.49221	1.86527
0.9	3.92314	4.35905

$k=2,\ l=3$

ρ	L_q	W_q
0.1	0.00052	0.00523
0.2	0.00589	0.02943
0.3	0.02326	0.07754
0.4	0.06234	0.15585
0.5	0.13886	0.27772
0.6	0.28313	0.47189
0.7	0.56498	0.80711
0.8	1.19299	1.49124
0.9	3.20903	3.56558

$k=2,\ l=4$

ρ	L_q	W_q
0.1	0.00021	0.00208
0.2	0.00349	0.01743
0.3	0.01624	0.05415
0.4	0.04756	0.11889
0.5	0.11188	0.22376
0.6	0.23668	0.39447
0.7	0.48492	0.69275
0.8	1.04443	1.30553
0.9	2.85292	3.16991

$k=3,\ l=2$

ρ	L_q	W_q
0.1	0.00128	0.01277
0.2	0.00948	0.04741
0.3	0.03129	0.10429
0.4	0.07583	0.18958
0.5	0.15852	0.31704
0.6	0.30944	0.51574
0.7	0.59828	0.85469
0.8	1.23353	1.54192
0.9	3.25700	3.61889

$k=3,\ l=3$

ρ	L_q	W_q
0.1	0.00034	0.00341
0.2	0.00408	0.02041
0.3	0.01678	0.05592
0.4	0.04621	0.11553
0.5	0.10506	0.21012
0.6	0.21766	0.36277
0.7	0.43993	0.62847
0.8	0.93880	1.17349
0.9	2.54779	2.83088

$k=3,\ l=4$

ρ	L_q	W_q
0.1	0.00012	0.00115
0.2	0.00215	0.01074
0.3	0.01072	0.03574
0.4	0.03289	0.08222
0.5	0.07999	0.15997
0.6	0.17350	0.28917
0.7	0.36250	0.51785
0.8	0.79314	0.99142
0.9	2.19485	2.43872

$k=4,\ l=2$

ρ	L_q	W_q
0.1	0.00110	0.01098
0.2	0.00825	0.04123
0.3	0.02743	0.09144
0.4	0.06690	0.16724
0.5	0.14052	0.28105
0.6	0.27542	0.45903
0.7	0.53431	0.76329
0.8	1.10486	1.38107
0.9	2.92470	3.24966

$k=4,\ l=3$

ρ	L_q	W_q
0.1	0.00027	0.00265
0.2	0.00328	0.01639
0.3	0.01378	0.04592
0.4	0.03857	0.09643
0.5	0.08879	0.17758
0.6	0.18576	0.30960
0.7	0.37844	0.54062
0.8	0.81293	1.01616
0.9	2.21859	2.46510

$k=4,\ l=4$

ρ	L_q	W_q
0.1	0.00008	0.00080
0.2	0.00159	0.00795
0.3	0.00828	0.02759
0.4	0.02613	0.06531
0.5	0.06490	0.12979
0.6	0.14305	0.23842
0.7	0.30270	0.43243
0.8	0.66918	0.83647
0.9	1.86773	2.07526

7. Special Cases

1) The System $(M/E_k/1)$

D. P. Gaver [6], R. R. P. Jackson [5], P. M. Morse [1], [7] and others found the formulas on this system, some of which we can get from our results in the previous sections.

In the present case the equation (12) becomes

$$(67)\qquad u^k=\frac{\theta}{1+\theta-u}\qquad\qquad\left(\theta=\frac{\rho}{k}\right)$$

or

$$(68)\qquad (u-1)\{u^k-\theta(u^{k-1}+u^{k-2}+\cdots\cdots+u+1)\}=0,$$

which has the root $u=1$ and k roots $u_j\ (j=1,2,\cdots\cdots,k)$ inside the unit circle. Then it results

(69) $\quad u_j^k=\theta(u_j^{k-1}+u_j^{k-2}+\cdots\cdots+1).$

Let P_0 be the probability that the service channel is free, then from (46) we have

(70) $\quad P_0=1-\rho.$

Let $P_{m,n}$ $(1\leqq m\leqq k, n\geqq 1)$ be the probabilities that n units are in the system and a unit being served stays in m-th phase, and put $l=s=1$ into (48), then it follows

$$(71)\qquad P_{m,n}=\frac{-\theta(1-\rho)}{A}\begin{vmatrix} 1 & 1 & \cdots\cdots & 1 & 1 \\ u_1 & u_2 & \cdots\cdots & u_k & 1+\theta \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_1^{k-1} & u_2^{k-1} & \cdots\cdots & u_k^{k-1} & (1+\theta)^{k-1} \\ U_1^{(m,n)} & U_2^{(m,n)} & \cdots & U_k^{(m,n)} & 0 \end{vmatrix}$$

where $U_j^{(m,n)}=u_j^{k(n-1)+(m-1)}$

Then we have the formula of state probabilities

$$(71')\qquad P_n=\sum_{m=1}^{k}{}' P_{m,n}$$

$$=-\frac{1-\rho}{A}\begin{vmatrix} 1 & 1 & \cdots\cdots & 1 & 1 \\ u_1 & u_2 & \cdots\cdots & u_k & 1+\theta \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u_1^{k-1} & u_2^{k-1} & \cdots\cdots & u_k^{k-1} & (1+\theta)^{k-1} \\ u_1^{kn} & u_2^{kn} & \cdots\cdots & u_k^{kn} & 0 \end{vmatrix}.$$

Setting $Q_{k(m-1)+(m-1)}=P_{m,n}$, from (3) and (69) we obtain

$$Q_m=\theta(1+\theta)^m(1-\rho)\quad(0\leqq m\leqq k-1),$$

$$(72)\qquad P_0=1-\rho,\quad Q_0=\theta P_0,\quad Q_m=\theta(P_0+Q_0+\cdots\cdots+Q_{m-1})\quad(1\leqq m\leqq k-1)$$

$$Q_{j+k}=\theta(Q_j+Q_{j+1}+\cdots\cdots\cdots+Q_{j+k-1})\quad(j\geqq 0)$$

respectively. The last equation finally lead us to

$$(73)\qquad P_n=\sum_{m=1}^{k}{}' P_{m,n}=\frac{1}{\theta}P_{1,n+1}=\frac{1}{\theta}Q_{kn}\quad(n\geqq 1).$$

By the last three equalities we can successively compute the state probabilities P_n $(n\geqq 1)$.

If we put $l=1$ into (61) and (63), then, notifying that $S_0=P_0=1-\rho$ in the present case, we get the formulas on the mean queue length and waiting time

$$(74)\qquad L_q=\frac{(k+1)\rho^2}{2k(1-\rho)},\quad W_q=\frac{(k+1)\rho}{2k\mu(1-\rho)}.$$

2) The System $(E_l/M/1)$

R. R. P. Jackson, D. G. Nickols [4] and P. M. Morse [1] have dealt with this system. The results which they have found will be obtained from our formulas.

In this case, the number k of the phases in the service channel being 1, the equation (11) becomes

$$\left(\frac{\psi}{1+\psi-u}\right)^{l}=u \qquad (\psi=l\rho), \tag{75}$$

which has the root $u=1$ and a root $u=u_1$ inside the unit circle and other $l-1$ roots $u_2, u_3, \cdots\cdots, u_l$. Setting

$$v=\frac{\psi}{1+\psi-u} \quad \text{or} \quad u=\frac{(1+\psi)v-\psi}{v} \tag{76}$$

into the above equation, we have

$$v^{l}=1+\psi-\frac{\psi}{v} \tag{77}$$

or

$$\begin{aligned} &v^{l+1}-(1+\psi)v+\psi=0,\\ &(v-1)(v^{l}+v^{l-1}+\cdots\cdots\cdots+v-\psi)=0, \end{aligned} \tag{78}$$

which has the root $v=1$ and only one root v_1 inside the unit circle and $l-1$ other roots $v_2, v_3, \cdots\cdots, v_l$.

It can be easily seen that $v_1, v_2, \cdots, v_l$ satisfy the equation

$$v^{l}+v^{l-1}+\cdots\cdots\cdots+v-\psi=0, \tag{79}$$

and that v_l is real and satisfies the inequality $0<v_1<1$. Using v_1, we see this equation turns out to

$$(v-v_1)\{v^{l-1}+(1+v_1)v^{l-2}+(1+v_1+v_1^2)v^{l-3} + \cdots\cdots\cdots + (1+v_1+v_1^2+\cdots\cdots\cdots+v_1^{l-1})\}=0.$$

Therefore from (45') we have

$$\sigma_s=(-1)^s(1+v_1+v_1^2+\cdots\cdots\cdots+v_1^s). \tag{80}$$

Let $P_{s,n}$ $(1\leqq s\leqq l, n\geqq 0)$ be the probabilities that n units are in the system and the unit passing through the arriving channel stays in s-th phase, then, putting $k=m=1$ into (45) and (48), we obtain the formulas on the phase and state probabilities:

$$\begin{aligned} P_{s,0}&=\frac{(-1)^{s-1}(1-\rho)\sigma_{s-1}}{(1-v_2)(1-v_3)\cdots\cdots\cdots(1-v_l)}\\ &=\frac{(1-\rho)(1+v_1+\cdots\cdots\cdots+v_1^{s-1})}{1+(1+v_1)+(1+v_1+v_1^2)+\cdots\cdots+(1+v_1+v_1^2+\cdots\cdots+v_1^{l-1})}\\ &=\frac{(1-\rho)(1-v_1^s)}{(1-v_1)+(1-v_1^2)+\cdots\cdots\cdots+(1-v_1^l)}\\ &=\frac{(1-\rho)(1-v_1^s)}{l-\psi}=\frac{1-v_1^s}{l}, \end{aligned} \tag{81}$$

$$(82)\qquad P_{s,n}=\frac{-\psi(1-\rho)}{(1-v_2)(1-v_3)\cdots\cdots(1-v_l)}\begin{vmatrix}1 & 1\\ v_1^{l(n-1)+(s-1)} & 0\end{vmatrix}$$

$$=\rho(1-v_1)\,v_1^{l(n-1)+(s-1)}\qquad(n\geqq 1),$$

$$(83)\qquad P_0=1-\rho,$$

$$(84)\qquad P_n=\sum_{s=1}^{l}P_{s,n}=\rho(1-v_1^l)\,v_1^{l(n-1)}\qquad(n\geqq 1).$$

Calculating

$$(85)\qquad S_0=\sum_{s=1}^{l}sP_{s,0}=\frac{1}{l}\sum_{s=1}^{l}s(1-v_1^s)=\frac{l+1}{2}-\frac{\rho-v_1^{l+1}}{1-v_1}=\frac{l-1}{2}+\frac{1-\rho}{1-v_1}-l\rho,$$

we obtain the formulas on the average queue length and the waiting time:

$$(86)\qquad L_q=\frac{1}{1-\rho}\left\{\rho^2-\frac{1-\rho}{l}+\frac{1-\rho}{l(1-v_1)}-\rho\right\}=\frac{\rho v_1^l}{1-v_1^l}$$

and

$$(87)\qquad W_q=\frac{L_q}{\lambda}=\frac{v_1^l}{\mu(1-v_1^l)},$$

where v_1 is the root of the equation (79) which is real and satisfies the inequality $0<v_1<1$.

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