ON THE CONNECTIONS OF THE TANGENT BUNDLE OF A FINSLER SPACE

By

TANJIRO OKUBO

Introduction.

This paper is attributed its origin far back to the works of Y. Muto and K. Yano [11] made in 1935. They developed the connection theory on a differentiable manifold X_n with each point of which is associated another kind of manifold Y_m. At that time their work was observed as rebelling against setting conformity with those of formal generalized spaces then extant, until when K. Yano and E. T. Davies [18] raised their works up to the level of contact tensor calculus. The distributions tangent to these X_n and Y_m are generally not holonnmic and the classical theory of anholonomity introduced by T. C. Doyle [5] to homogeneous contact transformations served them to refine their original idea. These two kinds of complementary distributions with equal dimensions, namely the contact frames, were found later to be available for defining the horizontal and vertical subspaces of a tangent bundle. The horizontal distributions give a connection and consequently give the parallelism of tangent vector field attached to the base space M_n in the sense of C. Ehresmann [6]. It is remarkable that such a tangent bundle $T(M_n)$ with base space M_n of class C^r, $r \geqq 4$, admits an almost complex structure [17]. K. Yano and E. T. Davies [23] recently proved that $T(M_n)$ admits a complex structure with respect to these complementary distributions. He nominated them the adapted frames in order to distinguish them from the natural frames spanned by the tangent vectors defined on base space and on fibre. On the basis of the latter frames S. Sasaki [15] [16] developed his own theory of tangent bundles by taking Riemann manifold as its base some years ago. However, the notion of complex structure seems to us likely more favorable by its simplicity in handling the connection theory of such $T(M_n)$ as ours.

In Yano's connection theory of tangent bundles $T(M_n)$ in which the base is a Finsler space, what arises with him as question is the definition of the lift of a vector field, namely, we may well consider that the lift is the vector contained either in the horizontal or in the vertical distributions. Or also it may be their algebraic sum. In accordance with these heterogeneous ways of defining a lift it is of course natural to have the various formulations of the connections for $T(M_n)$.

According to the observation of the present author it would be plausible, or rather desirable to have the connection Γ on $T(M_n)$ such that when we consider the horizontal lift of a vector in M_n, the components of the connection parameter with respect to the horizontal distributions should satisfy those postulates of parallelisms due to C. Ehresmann or to K. Nomizn [12], since the usual connection of M_n is in short the choice of the horizontal subspace in $T(M_n)$. If then so, those components Γ^h_{ij} are, of course, classical and we have those ones such as due to E. Cartan [2] or to H. Rund [13] for Finsler spaces. On the other hand it was E. T. Davies [21] and A. Deicke [4] who pointed out that in order to have a Finsler connection on horizontal distributions it is necessary for $T(M_n)$ to have a connection Γ with torsion.

This paper intends to show as to the way how to introduce the torsion tensors to $T(M_n)$ in order to obtain the classical Finsler connections from the theory of tangent bundles, especially those two kinds stated above. And at the same time it tries determine the connection Γ of $T(M_n)$ in such a way that a theorem obtained by K. Yano on base Riemann spaces holds good as its direct generalization, that is to say, we prove that ***if the lift of a curve C defined on the base Finsler space M_n is an autoparallel curve in the tangent bundle $T(M_n)$, then the sectional curvature of M_n defined by the osculating plane associated to the curve C is always*** (-1).

§ 1. Structure of the tangent bundle of a Finsler space.*

Let M_n be a differentiable manifold of class C^r, $(r \geqq 4)$, and (ξ^h) be its local coordinate system. The tangent bundle $T(M_n)$ with local coordinate system $(\xi^A)=(\xi^h, \dot{\xi}^h)$, $\dot{\xi}^h=d\xi^h/dt$, is differentiable subject to the coordinate transformations:

$$(1.1)\qquad \begin{cases} \xi^{h'}=f^{h'}(\xi), \\ \dot{\xi}^{h'}=A^{h'}_h\,\dot{\xi}^h, \qquad A^{h'}_h=\partial\xi^{h'}/\partial\xi^h. \end{cases}$$

$\dot{\xi}^h \overset{\text{def}}{=} \xi^{n+h} \overset{\text{def}}{=} \xi^{h*}$ is a tangent vector at a point (ξ^h) of M_n. If we write (1.1) briefly as

$$\xi^{A'}=\xi^{A'}(\xi^B),$$

---

*. We adopt the following conventions for indices:

$A, B, C, D, \cdots=1, 2, \cdots, n, n+1, \cdots, 2n,$

$\alpha, \beta, \gamma, \delta, \cdots=1, 2, \cdots, n, 1^*, 2^*, \cdots n^*,$

a, b, c, h, i, j, k, l, m, $\cdots=1, 2, \cdots\cdots\cdots, n,$

a*, b*, c*, h*, i*, j*, k*, l*, m* $=1^*, 2^*, \cdots\cdots, n^*$

we have

$$(1.2)\qquad (\partial\xi^{A'}/\partial\xi^{B}) = \begin{pmatrix} A^{h'}_{h} & 0 \\ A^{h'}_{jh}\,\dot{\xi}^{j} & A^{h'}_{h} \end{pmatrix}$$

$$(1.3)\qquad A^{h'}_{jh} = \partial_h A^{h'}_{j}, \quad \partial_h = \partial/\xi^{h}.$$

where

(1.2) shows that the manifold $T(M_n)$ is orientable, as

$$(1.4)\qquad |\partial\xi^{A}/\partial\xi_{B}| = |A^{h'}_{h}|^{2} > 0.$$

Any fibre on a point (ξ_0) in M_n is a subspace of $T(M)$ and the distribution tangent to it has the compoments

$$(1.5)\qquad C^{A}_{i} = (0, \delta^{h}_{i})$$

We introduce a function $F(\xi, \dot{\xi})$ in $T(M)$ satisfying the postulates that

(i) $F(\xi, \dot{\xi})$ is positive if not all $\dot{\xi}^{i}$ vanish simultaneously.

(ii) $F(\dot{\xi}, \lambda\xi) = |\lambda| \cdot F(\xi, \dot{\xi})$ for some arbitrary number λ.

(iii) $g_{ji}(\xi, \dot{\xi})\,\eta^{j}\eta^{i}$ is positive definite for a non-vanishing vector η^{i}, where

$$g_{ji}(\xi, \dot{\xi}) = \frac{1}{2}\,\partial^{2} F(\xi, \dot{\xi}) \,/\, \partial\dot{\xi}^{j}\,\partial\dot{\xi}^{i}.$$

$g_{ji}(\xi, \dot{\xi})$ is a tensor defined over M_n and is called a Finsler metric, [2], [10], [18], [20], [21], [22]. The scalar function

$$(1.6)\qquad \omega = g_{ji}\,\dot{\xi}^{i}\,d\xi^{i}$$

is a 1-form defined globally on $T(M_n)$ and its exterior differential

$$(1.7)\qquad d\omega = \frac{1}{2}\,F_{CB}\,d\xi^{C} \wedge d\xi^{B}$$

is also defined on $T(M_n)$. F_{BA} has the components

$$(1.8)\qquad (F_{CB}) = \begin{pmatrix} (\partial_j g_{ia} - \partial_i g_{ja})\,\dot{\xi}^{a} & g_{ji} \\ -g_{ji} & 0 \end{pmatrix},$$

and the matrix (F_{CB}) is skew-symmetric and of rank 2n as $g_{ji} \neq 0$. Consequently we can always find a tensor G^{BA} such that [19]

$$(1.9)\qquad \begin{aligned} G^{BA}F_{CB} &= F^{A}_{C}, \\ F^{B}_{C}\,F^{A}_{B} &= -\delta^{A}_{C}, \end{aligned}$$

where δ denotes the Cronecker delta. The covariant vector G_{BA} defined by $G_{CA}G^{BA} = \delta^{B}_{C}$ satisfies

$$(1.10)\qquad F^{E}_{C}\,F^{D}_{B}\,G_{ED} = G_{CB},$$

and thus $T(M_n)$ is endowed with a Hermitian structure [6]. [7]. Furthermore, as we have $d(d\omega) = 0$, we get

(1.11) $$d(F_{CB}\,d\xi^{C}\wedge d\xi^{B})=0,$$
which shows that $T(M_n)$ admits an almost complex structure [6].

The distribution $B_i{}^A$ obtained by transforming the distribution C_{i*}^{A} by F_A^B has the direction orthogonal to C_{j*}^{A} and has the components

(1.12) $$B_j{}^A \overset{\mathrm{def}}{=} F_B^A\, C_{j*}^{B}=(g_{ja}\,G^{ah}, g_{ja}\,G^{ah^*}).$$

We call this couple of the complementary distributions $B_j{}^A$ and C_{j*}^{A} the horizontal and vertical distributions respectively. Since the metric G^{BA} are arbitrary to within the requirements (1.9), we can assign the following conditions on it. First the G_{BA} should satisfy

(1.13) $$C_{j*}^{C}\,C_{i*}^{B}\,G_{CB}=g_{ji}.$$

Then we have

(1.14) $$G_{j^*i^*}=g_{ji}$$

in virtue of (1.5). For G_{j^*i} we assume that B_i^A has the components

(1.15) $$B_j^A=(\delta_j^i, -\Gamma_j^i),$$

and hence by the orthogonality property

(1.16) $$G_{BA}\,B_j^B\,C_{i*}^{A}=0,$$

we get

(1.17) $$G_{j^*i}=\Gamma_{ij},$$

where we have put

(1.18) $$\Gamma_{ji}=g_{ia}\,\Gamma_j^a.$$

Finally we assume that

(1.19) $$G_{BA}\,B_j^B\,B_i^A=g_{ji},$$

and this yields

(1.20) $$G_{ji}=g_{ji}+g_{ba}\,\Gamma_j^b\,\Gamma_i^a,$$

as we have assumed (1.15). Thus the matrix (G_{BA}) has the form

(1.21) $$(G_{BA})=\begin{pmatrix} g_{ji}+g_{ba}\,\Gamma_j^b\,\Gamma_i^a & \Gamma_{ji} \\ \Gamma_{ji} & g_{ji} \end{pmatrix}.$$

We take the function Γ_{ji} in such a way it satisfy the relation

(1.22) $$\Gamma_{ji}-\Gamma_{ij}=(\partial_j\,g_{ia}-\partial_i\,g_{ja})\,\xi^a,$$

the matrix (F_B^A) whose components are given by (1.9), takes the form

(1.23) $$(F_B^A)=\begin{pmatrix} \Gamma_j^h & \delta_j^h \\ -\delta_j^h-\Gamma_j^a\,\Gamma_a^h & -\Gamma_j^h \end{pmatrix}$$

because of (1.18) and (1.21), and this shows that for any of such kinds of the functions Γ_{ji} it holds

$$FF=-E.$$

A tangent space of $T(M_n)$ is spanned by

$$\frac{\partial}{\partial\xi^j}d\xi^j+\frac{\partial}{\partial\dot{\xi}^j}d\dot{\xi}^j$$

and we say that if the displacement $d\xi^J$ satisfies

$$d\dot{\xi}^j+\Gamma^j_a d\xi^a=0 \tag{1.24}$$

the tangent space is called ***horizontal.*** This is the reason why we call B_i^A the horizontal distribution of $T(M_n)$. If we consider the distributions $B^i_{.A}$ and C^{i*}_A each of which is dual to B_i^A and C_{i*}^A in this order, they have the components

$$B^i_{.A}=(\delta^i_j,\ \ 0), \tag{1.25}$$

$$C^{i*}_{.A}=(\Gamma^i_j,\ \delta^i_j), \tag{1.26}$$

as we have

$$B_i^A B^j_{.A}=\delta^j_i,\ B_i^A C^{j*}_{.A}=0,\ C_{i*}^A B^j_{.A}=0,\ C_{i*}^A C^{j*}_{.A}=\delta^j_i, \tag{1.27}$$

$$B_i^A B^i_{.B}+C_{i*}^A C^{i*}_{.B}=\delta^A_B: \tag{1.28}$$

Then (1.24) is written as

$$C^{i*}_A d\xi^A=0, \tag{1.29}$$

which shows that a horizontal tangent vector has no component in the subspace tangent to the fibre, and this is the reason why we call C_i^A the vertical distribution, Therefore if we call the distributions spanned by $\partial/\partial\xi^i$ and $\dfrac{\partial}{\partial\dot{\xi}^i}$, or in short, by $\partial/\partial\xi^A$ the natural frames, the distributions

$$A_\alpha^A=(B_i^A, C_{i*}^A) \tag{1.29}$$

can be called the adapted frames of $T(M_n)$. On the other hand those B_i^A and C_{i*}^A are the projective quantities in the sense of loc al subspace theory. The displacement dx^A is projected by A_α^A so as to span the sub-displacement

$$(dx)^\alpha=A^\alpha_A dx^A. \tag{1.30}$$

where we have put

$$A^\alpha_A=(B^i_{.A}, C^{i*}_{.A}). \tag{1.31}$$

Each part of (1.30) has the form

$$(d\xi)^i=B^i_{.A}d\xi^A=d\xi^i, \tag{1.32}$$

$$(d\xi)^{i*}=C^{i*}_{.A}d\xi^A=\Gamma^i_j d\dot{\xi}^j \tag{1.33}$$

and their dual operators are

$$X_i f=\partial_i f-\Gamma^a_i\partial'_a f, \qquad \partial'_a=\partial/\partial\dot{\xi}^a, \tag{1.34}$$

$$X_{i*}f=\partial'_i f. \tag{1.35}$$

The subspaces spanned by these distributions are not in general holonomic and the condition of integrability

$$\Omega_{\beta\alpha}{}^\gamma=A^\gamma_A(X_\beta A^A_\alpha-X^A_\alpha A^A_\beta) \tag{1.36}$$

are not all zero. It has the components

$$
(1.37)\qquad \begin{cases} \Omega_{ji}{}^{h}=\Omega_{j^*i}{}^{h}=\Omega_{i^*i^*}{}^{h}=\Omega_{j^*i^*}{}^{h^*}=0, \\ \Omega_{ji}{}^{h^*}=X_i\Gamma^h_j-X_j\Gamma^h_i, \\ \Omega^{h^*}{}_{j^*i}=-\partial'_j\Gamma^h_i : \end{cases}
$$

The metric $G_{\beta\alpha}$ and the almost complex structure F^{β}_{α} with respect to the adapted frames have the components

$$
(1.38)\qquad (G_{\beta\alpha})=(A^B_\beta A^A_\alpha G_{BA})=\begin{pmatrix} g_{ji} & 0 \\ 0 & g_{ji} \end{pmatrix}
$$

$$
(1.39)\qquad (F^{\beta}_{\alpha})=(A^\beta_B A^A_\alpha F^B_A)=\begin{pmatrix} 0 & \delta^j_i \\ -\delta^j_i & 0 \end{pmatrix}
$$

This shows that the $T(M_n)$ is Kaehlerian with respect to the adapted frame.

§ 2. Infinitesimal connections of $T(M_n)$.

It is always possible to introduce an affine connection in a differentiable manifold and let Γ^A_{CB} and $S_{CB}{}^A$ the parameter of affine connection and torsion tensor introduced to natural frames of $T(M_n)$ respectively, i. e.

$$
(2.1)\qquad S_{CB}{}^A=\Gamma^A_{CB}-\Gamma^A_{BC},
$$

Then for those that are transformed into the adapted frames we have

$$
(2.2)\qquad \Gamma^\alpha_{\gamma\beta}=A^\alpha_A(A^C_\gamma A^B_\beta\Gamma^A_{CB}+X_\gamma A^A_\beta),
$$

$$
(2.3)\qquad \Gamma^\alpha_{\gamma\beta}-\Gamma^\alpha_{\beta\gamma}=S_{\gamma\beta}{}^\alpha+\Omega_{\gamma\beta}{}^\alpha,
$$

where

$$
S_{\gamma\beta}{}^\alpha=A^\alpha_A A^C_\gamma A^B_\beta S_{CB}{}^A.
$$

We assume that the $\Gamma^\alpha_{\gamma\beta}$ satisfies the Euclidean connection with respect to $g_{\beta\alpha}$, namely

$$
(2.4)\qquad \nabla_\alpha g_{\beta\gamma}=X_\alpha g_{\beta\gamma}-\Gamma^\varepsilon_{\alpha\beta}g_{\varepsilon\gamma}-\Gamma^\varepsilon_{\alpha\beta}g_{\beta\gamma}=0.
$$

Then from the identity

$$
\frac{1}{2}(-\nabla_\gamma g_{\beta\alpha}+\nabla_\beta g_{\alpha\gamma}+\nabla_\alpha g^{\gamma\beta})\,g^{\gamma\delta}=0,
$$

We deduce

$$
(2.5)\qquad \begin{aligned} \Gamma^\alpha_{\gamma\beta}=\left\{{\alpha \atop \gamma\beta}\right\}&+\frac{1}{2}(\Omega_{\gamma\beta}{}^\alpha+\Omega^\alpha_{\cdot\gamma\beta}+\Omega^\alpha_{\cdot\beta\gamma}) \\ &+\frac{1}{2}(S_{\gamma\beta}{}^\alpha+S^\alpha_{\cdot\gamma\beta}+S^\alpha_{\beta\gamma}), \end{aligned}
$$

where

$$
(2.6)\qquad \left\{{\alpha \atop \gamma\beta}\right\}=\frac{1}{2}g^{\delta\alpha}(X_\gamma g_{\delta\alpha}+X_\beta g_{\gamma\delta}-X_\delta g_{\gamma\beta}),
$$

(2.7) $$\Omega_{\gamma\beta}{}^{\alpha}=g^{\alpha\varepsilon}g_{\beta\delta}\,\Omega_{\varepsilon\gamma}{}^{\delta},$$

(2.8) $$S^{\alpha}_{.\gamma\beta}=g^{\alpha\varepsilon}g_{\beta\delta}\,S_{\varepsilon\gamma}{}^{\delta}.$$

We now assume that

(2.9) $$X_j\,\xi^h+\Gamma'^h_{ji}\,\dot{\xi}^i=0,$$

and we get the relation

(2.10) $$\Gamma'^h_j=\Gamma'^h_{ji}\,\dot{\xi}^i,$$

because of (1.35). Then we have

(2.11) $$\delta\dot{\xi}^h=(d\dot{\xi})^i,$$

Originally a vector field v^h in a Finsler space is the function of both ξ's and $\dot{\xi}$'s, and consequently its lift is well defined in $T(M_n)$. And if the lift belongs to the horizontal subspace in $T(M_n)$ and consequently it is of the form $B_i^A v^i$, the connection of it is of course defined along the horizontal subspace in the usual sense such as of *C*. Ehresmann and the parallelisms serving to define it are classical. we shall show that our connection parameter $\Gamma'^{\alpha}_{\gamma\beta}$ can be determined in such a way that the parallelism defined on a horizontal lift agrees with the classical ones, for example, that of *E*. Cartan [2] or of *H*. Rund [13]. Those $\Gamma^{\alpha}_{\gamma\beta}$ are, of course, determined as they should be even if otherwise we consider the lift of a vector field $v^i(\xi,\dot{\xi})$ as a vertical one, or the sum of these two.

(1). Γ'^h_{ji}

By virtue of (1.37) and (1.38) the functions $\Gamma'^{\alpha}_{\gamma\beta}$ endowed with the form (2.6) takes the form

$$\Gamma'^h_{ji}=\frac{1}{2}g^{ha}(X_j g_{ai}+X_i g_{ja}-X_a g_{ji})+\frac{1}{2}S_{ji}{}^h+\frac{1}{2}S^h_{.ji}+\frac{1}{2}S^h_{.ij}.$$

We set up the assumption that $S_{ji}{}^h$ satisfies

(2.12). $$S_{ji}{}^h=\frac{1}{2}g^{ha}(\Gamma'^m_i\partial'_m g_{ja}-\Gamma'^m_j\partial'_m g_{ia}).$$

Then we get

(2.13) $$\Gamma'^h_{ji}=\{{}^h_{ji}\}-\frac{1}{2}g^{ha}(\partial'_m g_{ai})\,\Gamma'^m_{jb}\,\dot{\xi}^b$$

by the use of the relations (1.39) where $\{{}^h_{ji}\}$ is the Christoffel symbol:

(2.14) $$\{{}^h_{ji}\}=\frac{1}{2}g^{ha}(\partial_j g_{ai}+\partial_i g_{ja}-\partial_a g_{ji}).$$

But as we have had (2.10), we have from (2.13)

(2.15) $$\Gamma'^h_{ji}\,\dot{\xi}^i=\Gamma'^h_j=\{{}^h_{ji}\}\,\dot{\xi}^i,$$

Since the function $g_{ji}(\xi,\dot{\xi})$ is homogeneous of degree zero in the $\dot{\xi}$'s, that is,

$$C_{jia}\,\dot{\xi}^a=0,$$

where we have put

$$(2.16)\qquad C_{jia}=\frac{1}{2}\,\partial'_a g_{ji},$$

and C_{jik} is symmetric in all the three indices. Substitution of (2.15) and (2.16) in to (2.16) yields

$$(2.17)\qquad \Gamma'^h_{ji}=\{^h_{ji}\}-C^h_{mi}\{^m_{ja}\}\dot{\xi}^a,$$

where

$$(2.18)\qquad C_{ji}{}^h=g^{hb}C_{jia},$$

and the Γ'^h_{ji} thus obtained is ***nothing but the connection parameter of a Finsler space introduced by H. Rund*** [13], [14] ***in*** 1951 ***on the postulate that the parallelism of a vector*** $v^i(\xi,\dot{\xi})$ ***dragged along a curve*** $(\xi^i(t))$ ***is not only Euclidean but also is one that preserves the scalar function*** $g_{ji}v^i v^i$ ***as a direct generalization of Levi Civita's parallelism in a Riemann space aloug a curve.*** It is easily shown that the function Γ_{hi} defined by (1.18) satisfies the requirement (1.22) for this Γ'^h_{ji}.

We now put another assumption that [21]

$$(2.19)\qquad S_{ji}{}^h=0.$$

Then (2.5) takes the form

$$(2.20)\qquad \Gamma'^h_{ji}=\{^h_{ji}\}-\frac{1}{2}\Gamma'^k_j C_{ki}{}^h-\frac{1}{2}\Gamma'^k_i C_{kj}{}^h+\frac{1}{2}g^{ha}\Gamma'^k_a C_{kji}.$$

Multiplying $\dot{\xi}^i$ and again using (2.14) we have

$$(2.21)\qquad \Gamma'^h_j=\{^h_{ja}\}\dot{\xi}^a-\Gamma'^k_i\dot{\xi}^i C^h_{kj},$$

or

$$\Gamma'_{jh}=g_{hb}\{^b_{ja}\}\dot{\xi}^a-\Gamma'^k_i\dot{\xi}^i C_{khj},$$

from which we see that the requirement (1.22) holds too.

Contracting (2.20) by $\dot{\xi}^j$ we get

$$\Gamma'^h_j\dot{x}^j=\{^h_{00}\}$$

where we put

$$\{^h_{0i}\}=\{^h_{ji}\}\dot{\xi}^j,\ \{^h_{00}\}=\{^h_{ji}\}\dot{\xi}^j\dot{\xi}^i.$$

Then (2.21) can be written as

$$\Gamma'^h_j=\{^h_{j0}\}-\{^k_{00}\}C^h_{kj}.,$$

Now, for

$$L=\frac{1}{2}g_{ji}\dot{\xi}^j\dot{\xi}^i,$$

we have

$$\nabla'_k L=X'_k L=\partial'_k L=\dot{\xi}^a g_{ka},$$

$$\nabla'_j\nabla'_i L=\nabla'_j X'_i L=\nabla'_j\partial'_i L=\partial'_j\partial'_i L=g_{ji}.$$

and thus we get

$$\left\{ {m \atop 00} \right\} g_{me} = \dot{\xi}^h \partial_h \partial'_e L - \partial_e L.$$

If we differentiate this equality partially by $\dot{\xi}^j$, we have

$$\partial'_j \left(\left\{ {m \atop 00} \right\} g_{mi} \right) = 2 \left\{ {m \atop j0} \right\} g_{mi}.$$

Hence if we introduce the function G such that

$$G^m = \frac{1}{2} \left\{ {m \atop 00} \right\},$$

$$G_e = \frac{1}{2} g_{me} \left\{ {m \atop 00} \right\}$$

we find that

$$\partial'_j G_i = \left\{ {m \atop j0} \right\} g_{mi},$$

$$\partial'_j G_h = \left\{ {h \atop j0} \right\} - \frac{1}{2} \left\{ {h \atop 00} \right\} C_{jk}{}^m.$$

Using the latter, (2.20) is now expressed to

(2.22) $$\Gamma'^h_{ji} = \left\{ {h \atop ji} \right\} - (\partial'_j G^m) C_{mi}{}^h - (\partial'_i G^m) C_{mj}{}^h + g^{kh} (\partial'_k G^m) C_{mji}$$

which coincides with ***the connection parameter of a Finsler space due to E. Cartan*** [2], [20], [21], [22].

(ii). $\Gamma'^{h^*}_{j^*i^*}$.

By virute of (1.37) and (1.38) the $\Gamma'^{\gamma}_{\beta\alpha}$ takes the form

$$\Gamma'^{h^*}_{j^*i^*} = \frac{1}{2} g^{ha} (X_{j^*} g_{ai} + X_{i^*} g_{ja} - X_{a^*} g_{ji}) + S_{j^*i^*}{}^{h^*} + g^{ha} g_{kl} S_{a^*j^*}{}^{k^*} + g^{ha} g_{kj} S_{a^*i^*}{}^{k^*}.$$

We set up the assumption that

(2.23) $$S_{j^*i^*}{}^{h^*} = 0.$$

Then by using the tensor C_{hji} defined by (2.16) we have

(2.24) $$\Gamma'^{h^*}_{j^*i^*} = C^h_{ji}.$$

(iii). $\Gamma'^h_{j^*i}$ and $\Gamma'^h_{ji^*}$.

First $\Gamma'^h_{j^*i}$ has the form

$$\Gamma'^h_{j^*i} = \frac{1}{2} g^{ha} \partial'_i g_{ai} - \frac{1}{2} g^{ha} g_{jk} (X_a \Gamma'^k_i - X_i \Gamma'^k_a) + \frac{1}{2} S_{j^*i}{}^h + \frac{1}{2} (S^h_{.j^*i} + S^h_{.ji^*})$$

We now assume that

(2.25) $$\Gamma'^h_{j^*i} = \Gamma'^{h^*}_{j^*i^*},$$

(2.26) $$S^h_{.j^*i} + S^h_{.ji^*} = 0.$$

By the use of which and together with (2.24) we deduce

(2.27) $$S_{j}{}_{i^*}{}^{h}=g^{ha}\,g_{kt}\,(X_a\,\Gamma^{k}_{i}-X_j\,\Gamma^{k}_{a}):$$

From the equality (2.3) we have

$$\Gamma^{h}_{j^*i}-\Gamma^{h}_{ji^*}=S_{j^*i}{}^{h}+\Omega_{j^*i}{}^{h},$$

and we have had

(2.28) $$\Gamma^{h}_{j^*i}=C^{h}_{ji}$$

in consequence of (2.24) and (2.25), while $\Omega_{j^*i}{}^{h}$ vanishes because of (1.37). Hence the above equality yields

(2.29) $$\Gamma^{h}_{ji}=C^{h}_{ji^*}-g^{ha}\,g_{kj}\,(X_a\,\Gamma^{k}_{i}-X_i\,\Gamma^{k}_{a}).$$

Also if we multiply (2.26) by G_{ha} and $G^{m^*l^*}$ which are nothing but g_{ha} and g^{ml} respectively as we have proved (1.38) and take (2.27) into account, we obtain

(2.30) $$S_{ji}{}^{h^*}=X_i\,\Gamma^{h}_{j}-X_j\,\Gamma^{h}_{i}.$$

(iv). $\boldsymbol{\Gamma}^{h^*}_{ji}$.

In virtue of (1.37) and (1.38) the $\Gamma^{\gamma}_{\beta\alpha}$ takes the form

$$\Gamma^{h^*}_{ji}=-\frac{1}{2}g^{ha}\,X'_a\,g_{jt}+\frac{1}{2}\,\Omega_{ji}{}^{h^*}+\frac{1}{2}\,S_{jt}{}^{h^*}+\frac{1}{2}\,(S^{h^*}_{.ii}+S^{h^*}_{.ij}).$$

But as $G^{cb}=G^{c^*b^*}=g^{cb}$ and $G^{c^*b}=G^{cb^*}=0$, we have

$$S^{h^*}_{.ji}+S^{h^*}_{.ij}=g^{ha}\,g_{kt}\,S_{a^*j}{}^{k}+g^{ha}\,g_{kj}\,S_{a^*i}{}^{k},$$

and the right member vanishes identically because of (2.27). Thus we get

(2.31) $$\Gamma^{h^*}_{ji}=-C^{i}_{ji}-X_j\,\Gamma^{h}_{i}+X_t\,\Gamma^{h}_{j}.$$

(v) $\boldsymbol{\Gamma}^{h^*}_{ji^*}$ and $\boldsymbol{\Gamma}^{h^*}_{j^*i}$.

We set up the assumptions

(2.32) $$\Gamma^{h^*}_{ji^*}=\Gamma^{h}_{ji},$$

(2.33) $$\Gamma^{h^*}_{j^*i}=0.$$

Then from

$$\Gamma^{h^*}_{ji^*}=\frac{1}{2}\,(g^{ha}\,X_j\,g_{ai}+\Gamma^{h}_{ji}-g^{ha}\,g_{ik}\,\Gamma^{k}_{ja}+g^{ha}\,g_{jk}\,S_{a^*i^*}{}^{k})=\Gamma^{h}_{ij}+\frac{1}{2}\,g^{ha}\,g_{ki}\,S_{a^*j^*}{}^{h},$$

we have

(2.34) $$S_{a^*j^*}{}^{h}=0$$

by the first of these two assumptions.

Also from the equality

$$\Gamma^{h^*}_{ji^*}-\Gamma^{h^*}_{i^*j}=S_{ji^*}{}^{h^*}+\Omega_{ji^*}{}^{h^*},$$

we get

(2.35) $$S^{h^*}_{ji^*}=-(\partial'_i\,\Gamma^{h}_{ja})\,\dot{\xi}^{a},$$

by virtue of (1.38), and (2.33).

(vi). $\boldsymbol{\Gamma}^{h}_{j^*i^*}$.

If we take account of the relation (2.34) and also of a fact resulting from the anholonomity of the distribtions, namely the relation

$$\Omega_{j^* i}{}^{k^*} = -\partial'_j \Gamma'^k_i,$$

we see that the component $\Gamma'^h_{j^* i^*}$ of (2.6) is reduced to $-\frac{1}{2} g^{ha} \nabla_a g_{ji}$, and consequently we have

$$\Gamma'^h_{j^* i^*} = 0. \tag{2.36}$$

Summarizing the results we have had

(2.37)
- (1) Γ'^i_{ji} (E. Cartan's or H. Rund's connection parameters given by (2.22) or (2.17) respectively).
- (2) $\Gamma'^h_{ji^*} = C_{ji}{}^h$, ((2.28)).
- (3) $\Gamma'^h_{j^* i} = C_{ji}{}^h - g^{ha} g_{ki} (X_a \Gamma'^k_j - K_j \Gamma'^k_a), = C^h_{ji} - K^h_{.jai} \dot{\xi}^a$, ((2.29)).
- (4) $\Gamma'^h_{j^* i^*} = 0$, ((2.36)).
- (5) $\Gamma'^{h^*}_{ji} = -C_{ji}{}^h - X_j \Gamma'^h_i + X_i \Gamma'^h_j, \; = -C_{ji}{}^h - K_{jia}{}^h \dot{\xi}^a$, ((2.31)).
- (6) $\Gamma'^{h^*}_{ji^*} = \Gamma'^h_{ji}$, ((2.32)).
- (7) $\Gamma'^{h^*}_{j^* i} = 0$, ((2.33)).
- (8) $\Gamma'^{h^*}_{j^* i^*} = C_{ji}{}^h$, ((2.24)).

where

$$K^h_{kji} = X_k \Gamma'^h_{ji} - X_j \Gamma'^h_{ki} + \Gamma'^h_{ka} \Gamma'^a_{ji} - \Gamma'^h_{ja} \Gamma'^a_{ki}, \tag{2.38}$$

and

$$K^{\,h}_{.kji} = g^{bh} g_{i\lambda} K_{bkj}{}^a.$$

(2.42)
- (1) $S_{ji}{}^h = 0$, ((2.19)),
  $= \frac{1}{2} g^{ha} (\Gamma'^m_i \partial'_m g_{ia} - \Gamma'^m_j \partial'_m g_{ai})$, ((2.12)).
  which serves to derive E. Cartan's and H. Rund's connection parameter Γ'^h_{ji} respectively in this order.
- (2) $S_{ji^*}{}^h = g^{ha} g_{ki} (X_a \Gamma'^k_j - X_j \Gamma'^k_a), = K^h_{.jai} \dot{\xi}^a$ ((2.27)),
- (3) $S_{j^* i^*}{}^h = 0$, ((2.34)),
- (4) $S_{ji}{}^{h^*} = X_i \Gamma'^h_j - X_j \Gamma'^h_i, = -K_{ija}{}^h \dot{\xi}^a$, ((2.30)),
- (5) $S_{ji^*}{}^{h^*} = -(\partial'_i \Gamma'^h_{ja}) \dot{\xi}^a$, ((2.35)),
- (6) $S_{j^* i^*}{}^{h^*} = 0$. ((2.23)).

For F^β_α we have derived its components at the end of § 1, from which we find that

$$\delta F^{i}_{j^*} = \delta F^{i^*}_{j} = 0$$

holds good subject to the connection parameters derived above. Hence from (1.39) we have

$$\delta F^\beta_\alpha = 0. \tag{2.40}$$

In determination of $\Gamma^{\alpha}_{\beta\gamma}$ K. Yano and E. T. Davies [21] adopted such assumptions that

a) geodesics and autoparallel curves coincide, so that $S_{\beta\gamma}{}^{\alpha}+S_{\gamma\beta}{}^{\alpha}=0$.

b) $\Gamma^{h}_{ji}=\Gamma^{h*}_{ji*}$, $\Gamma^{h}_{j*i}=\Gamma^{h*}_{j*i*}$,

c) $S_{ji}{}^{h}=S_{j*i*}{}^{h*}=0$.

As was seen above we get E. Cartan's connection parameters from (c), but Γ^{h}_{j*i*} thus obtained involves Berwald's curvature $\dot{\partial}_j\Gamma^{h}_{a}$ explicitly and we feel it hard to avail these $\Gamma^{\gamma}_{\beta\alpha}$ for the discussions in the following paragraph especially when we speak of the relation between the arc length of a curve in the base space and that of an autoparallel curve corresponding as its natural lift in $T(M_n)$.

§ 3. Auto-parallel curves in $T(M_n)$.

Let ω^{α} denote the components of a displacement $d\xi^{A}$ in $T(M_n)$ on the adapted frames:

$$\omega^{\alpha}=A^{\alpha}_{A}\,d\xi^{A}, \tag{3.1}$$

To each frames we have

$$\omega^{i}=B^{i}_{.A}\,d\xi^{A}=d\xi^{i}, \tag{3.2}$$

$$\omega^{i*}=C^{i*}_{.A}\,d\xi^{A}=(d\dot{\xi})^{i}=\delta\dot{\xi}^{i}, \tag{3.3}$$

becauc of (1.25), (1.26) and (2.14).

Given a curve $\xi^{i}=\xi^{i}(s)$ be a curve in the base spaee, where we take its arc length s as its parameter. Its lift in $T(M_n)$ has the form $(\xi^{i}, \dot{\xi}^{i}(s))$ where

$$\dot{\xi}^{i}=\frac{d\xi^{i}}{ds}.$$

We assume that the lift draws an autoparallel curve in $T(M)$. Then if we denote its arc length by t, we have for it the equations

$$\frac{d}{dt}\,\omega^{\gamma}(t)+\Gamma^{\gamma}_{\beta\alpha}\frac{\omega^{\beta}(t)}{dt}\frac{\omega^{\alpha}(t)}{dt}=0. \tag{3.4}$$

For $\gamma=h$ we have

$$\frac{d}{dt}\left(\frac{d\xi^{h}}{dt}\right)+\frac{d\xi^{j}}{dt}\Gamma^{h}_{ji}\frac{d\xi^{i}}{dt}+\frac{d\xi^{j}}{dt}\Gamma^{h}_{ji*}\frac{\delta\dot{\xi}^{i}}{dt}+\frac{\delta\dot{\xi}^{j}}{dt}\Gamma^{h}_{j*i}\frac{d\xi^{i}}{dt}+\frac{\delta\dot{\xi}^{j}}{dt}\Gamma^{h*}_{j*i*}\frac{\delta\dot{\xi}^{i}}{dt}=0,$$

in virtue of (3.2) and (3.3). Using the list (2.37) of each component for $\Gamma^{\gamma}_{\beta\alpha}$, we have for (3.4) the equations

$$\frac{\delta^2\xi^{h}}{dt^2}+K_{kji}{}^{h}\,\dot{\xi}^{k}\frac{\delta\dot{\xi}^{j}}{dt}\frac{d\xi^{i}}{dt}=0, \tag{3.5}$$

where

(3.6) $$\frac{\delta^2\xi^h}{dt^2}=\frac{d^2\xi^h}{dt^2}+\frac{d\xi^j}{dt}\Gamma^{h}_{ji}\frac{d\xi^i}{dt}.$$

For $\gamma=h^*$ we have

$$\frac{d}{dt}\frac{\delta\dot\xi^h}{dt}+\frac{d\xi^j}{dt}\Gamma^{h^*}_{ji}\frac{d\xi^i}{dt}+\frac{d\xi^j}{dt}\Gamma^{h^*}_{ji^*}\frac{\delta\dot\xi^i}{dt}+\frac{\delta\dot\xi^j}{dt}\Gamma^{h^*}_{j^*i}\frac{d\xi^i}{dt}+\frac{\delta\dot\xi^j}{dt}\Gamma^{h^*}_{j^*i^*}\frac{\delta\dot\xi^i}{dt}=0$$

in virtue of (3.2) and (3.3). Using (2.37) as above we have then the equations

(3.8) $$\frac{\delta^2\dot\xi^h}{dt^2}=\frac{d}{dt}\left(\frac{\delta\dot\xi}{dt}\right)^h+\frac{d\xi^j}{dt}\Gamma^{h}_{ji}\frac{\delta\dot\xi^j}{dt}+\frac{\delta\dot\xi^j}{dt}C^{h}_{ji}\frac{\delta\dot\xi^i}{dt}=0.$$

Now, since t was taken to the arc length of the autoparallel curve, we should have

$$g_{\beta\alpha}\frac{\omega^\beta}{dt}\frac{\omega^\alpha}{dt}=1,$$

or

(3.9) $$g_{ji}\frac{d\xi^j}{dt}\frac{d\xi^i}{dt}+g_{ji}\frac{\delta\dot\xi^j}{dt}\frac{\delta\dot\xi^i}{dt}=1$$

But as

$$\frac{d}{dt}\left(g_{ji}\frac{\delta\dot\xi^i}{dt}\frac{\delta\dot\xi^i}{dt}\right)=\frac{\delta}{dt}\left(g_{ji}\frac{\delta\dot\xi^j}{dt}\frac{\delta\dot\xi^j}{dt}\right)$$

vanishes because of (3.8) and of $\delta g_{ji}=0$. Hence $g_{ji}\dfrac{\delta\dot\xi^j}{dt}\dfrac{\delta\dot\xi^i}{dt}=$const., and accordingly we see from (3.9) that $g_{ji}\dfrac{d\xi^j}{dt}\dfrac{d\xi^i}{dt}=$const. Then as it holds $g_{ji}\dfrac{d\xi^j}{ds}\dfrac{d\xi^i}{ds}=1$ for the projection, we have $\dfrac{ds}{dt}=$const. and observe that the arc lengths t and s are linearly related.

Therefore the equations (3.5) and (3.8) can be written as

(3.10) $$\frac{\delta}{ds}\left(\frac{d\xi^h}{ds}\right)+K_{kji}{}^{h}\frac{d\xi^k}{ds}\frac{\delta}{ds}\left(\frac{d\xi^j}{ds}\right)\frac{d\xi^i}{ds}=0$$

(3.11) $$\frac{\delta^2}{ds^2}\left(\frac{d\xi^h}{ds}\right)=0.$$

The unit tangent vector $\dfrac{d\xi^i}{ds}$ together with its normal vector $\dfrac{\delta}{ds}\left(\dfrac{d\xi^i}{ds}\right)$ spans the osculating plane along the curve. Then let us write

(3.12) $$\frac{d\xi^h}{ds}=\lambda^h,$$

(3.13) $$\frac{\delta}{ds}\left(\frac{d\xi^h}{ds}\right)=\rho\mu^h,$$

and we have $g_{ji}\lambda^i\lambda^i=1, g_{ji}\mu^j\mu^i=1, g_{ji}\mu^i\lambda^i=0$. Substituting (3.12) and (3.13) into

(3.10) we have

$$K_{khij}\, \lambda^k\, \eta^h\, \lambda^j\, \mu^i = -1. \tag{3.14}$$

Hence we have the

Theorem. *If a lift of a curve C of a Finsler space is an autoparallel curve in the tangent bundle $T(M_n)$, then the sectional eurvature of the Finsler space determined by osculating planes along C is always* (-1).

From (3.10) and (3.11) we see easily that the lift of any geodesic is an autoparallel curve in $T(M_n)$.

University of California, Los Angeles.

REFERENCES

[1] Berwald, L., Untersuchung der Krümmung allgemeiner metrischen Raüme auf Grund des ihnen herrschenden Parallelismus. Math. Zeit., 25 (1925), 40-73.

[2] Cartan, E., Les espaces de Finsler, Actual Sci. et Ind. (1933), 79.

[3] Chern, S. S., Topics in differential geometry, Princeton (1951).

[4] Deicke, A., Finsler spaces as non-holonomic subspaces of Riemannian spaces. Journ. London Math. Soc. 30 (1955), 53 58.

[5] Doyle, T. C., Tensor decomposition with applications to the contact and complex spaces. Annals of Math., 42 (1941), 698-721.

[6] Eckmann, B., Sur les structure complexes et presque complexes. Colloques Internationaux de C. N. R. S. Geométrie différentielle. Strasbourg (1953), 151-159.

[7] Eckmann, B. and A. Frölicher, Sur l'intégrabilité des structures presque complexes. C. R. Acad. Sci. Peris, 232 (1951), 2284-2286.

[8] Ehresmann, C., Les connexion infinitésimales dans un espace fibré différentiable.

[9] Ehresmann, C., Sur les variétés presque complexes Proc. Int. Congr. Math., (1952), 412-419.

[10] Finsler, P., Über Kurven und Flächen in allgemeiner Raümen. Dissert, Göttingen (1918).

[11] Muto, Y. and K. Yano, On the connections in X_n associated with the points of Ym. Proc. Phys.-Math. Soc. Japan, 17 (1935), 397-398; 18 (1936), 1-9.

[12] Namizu, K., Lie groups and differential geometry, (1956). Jap, Math. Soc.

[13] Rund, H., Über die Parallelvershiebung in Finslerschen Räume. Math. Zeit., 54 (1951), 115-128.

[14] Rund, H., Differential geometry of Finsler spaces, Berlin (1959).

[15] Sasaki. S., On the differential geometry of tangent bundles of Riemannian manifolds. Tôhôhu Math. Journ. 10 (1958), 338-354.

[16] _______, On the differential geometry of tangent bundles of Riemannian manifolds (II). Tôhôku Math. Journ. 14 (1962), 146-155.

[17] Tachibana, S. and M. Okumura, On the almost complex structure of tangent bundles of Riemannian spaces. Tôhôku Math. Journ. 14 (1962), 156-161.

[18] Yano, K. and E. T. Davies, Contact tensor calculus, Ann. di Mat. Series 3, 37 (1954), 1-36.

[19] Yano, K. and I. Mogi, On real representations of Kaehlerian manifolds. Annals of Math., 61 (1955), 170-189.

[20] Yano, K., Quelques remarques sur les variétés a' structure presque comlexe. Bull. Soc. Math. Soc. France, 33 (1955), 57-58.

[21] Yano, K. and E. T. Davies, On some local properties of fibred spaces. Kôdai Math. Sem. 11 (1959), 158-177.

[22] Yano, K. and T. Okubo, Fibred spaces and nonlinear connections, Annali di Math. 55 (1961) 203-244.

[23] Yano, K. and E. T. Davies, On the tangent bundles of Finsler and Riemannian manifolds, Rendi. Circo. Math., Palermo. vol. 12 (1963) 1—18

(Received May 1, 1964)